On the Uniform Approximation of Analytic Functions by Means of Interpolation Polynomials.

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1. Introduction

Let D be a domain of the complex plane bounded by a smooth closed Jordan curve Γ.

Let $\{z_k^{(n)}\}$, $1 \leqq k \leqq n$, $n=1, 2, 3, \ldots$ be a triangular matrix of points of Γ ($z_i^{(n)} \neq z_k^{(n)}$ for $i \neq k$), and consider the fundamental polynomials of the Lagrange interpolation:

$$l_j^{(n)}(z) = \prod_{k \neq j} \frac{z - z_k}{z_j - z_k} \tag{1.1}$$

which have the property that $l_j(z_j)=1$ and $l_j(z_k)=0$ for $k \neq j$. We will say that $\{z_k^{(n)}\}$ is a *regular point system* if the polynomials $l_j^{(n)}(z)$ are uniformly bounded in $\bar{D}$, i.e. if

$$|l_j^{(n)}(z)| \leqq M \quad \text{for} \quad z \in \bar{D}, \quad 1 \leqq j \leqq n, \quad n = 1, 2, \ldots \tag{1.2}$$

for some M. In particular a system of *Fekete points*[1] is a regular point system; in fact in this case (1.2) holds with $M=1$.

If $f(z)$ is a function regular in D, and continuous in $\bar{D}$, it can be uniformly approximated in $\bar{D}$ by polynomials. In this paper I shall construct a sequence of polynomials which converges uniformly to $f(z)$ and, at the same time interpolates $f(z)$ at a regular point system.

In 1942 Erdös proved the following result about real interpolation [3]. Let $\{x_k^{(n)}\}$ be a regular point system in $[-1, +1]$. Then to every continuous function $f(x)$ and $\eta > 0$ there exists a sequence of polynomials $p_n(x)$ such that, 1) the degree of $p_n(x)$ is $\leqq n(1+\eta)$, 2) $p_n(x_i^{(n)}) = f(x_i^{(n)})$, $1 \leqq i \leqq n$, $n=1, 2, \ldots$, 3) $p_n(x) \to f(x)$ uniformly in $[-1, +1]$.

Adapting Erdös's proof, I shall prove the following

Theorem: *Let D be a domain bounded by a closed Jordan curve Γ which satisfies Alper's smoothness condition* (2.1). *Let $\{z_k^{(n)}\} \in \Gamma$ be a regular point system for $\bar{D}$. Then, to every function $f(z)$ regular in D and continuous in $\bar{D}$ and every $\eta > 0$, there exists a sequence of polynomials $p_n(z)$ such that*

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[1]) The Fekete points $\{w^{(n)}_k\}$ are defined by the property that for each n, they maximise the discriminant $\prod_{\substack{1 \leqq k, j \leqq n \\ k \neq j}} |w_k - w_j|$.

1) *the degree of* $p_n(z) \leqq n(1+\eta)$
2) $p_n(z_k^{(n)}) = f(z_k^{(n)})$, $1 \leqq k \leqq n$, $n = 1, 2, \dots$
3) $p_n(z) \to f(z)$ *uniformly in* $\bar{D}$
4) $p_n = T_n(f)$ *is a linear operator, given explicitly.*

2. Preliminary results

Let $\vartheta(s)$ denote the angle between the tangent to Γ and the positive real axis (as a function of the arc length parameter s). Let $\omega(h)$ denote the modulus of continuity of the function $\vartheta(s)$. In the papers [1,2] S. Y. ALPER introduced the class of domains whose boundary Γ satisfies the condition

$$\int_0^h \frac{\omega(x)}{x} |\log x| \, dx < +\infty \tag{2.1}$$

Assuming that condition (2.1) is satisfied, we are able to estimate certain fundamental polynomials.

Let

$$z = \psi(\zeta) = \beta\zeta + a_0 + \frac{a_1}{\zeta} + \cdots$$

map $|\zeta| > 1$ conformally onto the exterior of Γ. We can assume without loss of generality that $\beta = 1$. By a classical result $\psi(\zeta)$ is continuous for $|\zeta| \geqq 1$. Thus, the regular point system $\{z_k^{(n)}\}$ can be written in the form: $z_k^{(n)} = \psi(e^{i\vartheta_k^{(n)}})$, where we can assume that $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_n$, $\vartheta_n - \vartheta_1 < 2\pi$. In another paper [4, Theorem 3] POMMERENKE and I proved the following result:

LEMMA 2.1. *If* Γ *satisfies the condition* (2.1) *and* $\psi(e^{i\vartheta_k^{(n)}})$ is a regular point system, then:

$$\vartheta_{k+\nu}^{(n)} - \vartheta_k^{(n)} > \frac{c\nu}{n}, \quad k = 1, 2, \dots n, \quad \nu = 1, 2, \dots n-1 \quad (\vartheta_{n+j}^{(n)} = \vartheta_j^{(n)} + 2\pi) \tag{2.2}$$

where the constant $c > 0$ does not depend on n, k or ν. To avoid an excess of indices, we shall now drop the index: (n).

Let $w_{k,j}^{(m)} = \psi(e^{i(\vartheta_k + 2\pi j/m)})$, $1 \leqq k \leqq n$, $0 \leqq j < m$, $w_{k,0}^{(m)} = z_k^{(n)}$ and let $\tilde{l}_k^{(m)}(z)$ denote the fundamental polynomials:

$$\tilde{l}_k^{(m)}(z) = \prod_{j=1}^{m-1} \frac{z - w_{k,j}^{(m)}}{z_k - w_{k,j}^{(m)}}.$$

In the above quoted paper [4, (4.3), (4.6)] it was shown, that if (2.1) is satisfied, one has the following estimates

LEMMA 2.2

(i)
$$|\tilde{l}_k^{(m)}(\psi(e^{i\vartheta}))| < B \tag{2.3}$$

(ii)
$$|\tilde{l}_k^{(m)}(\psi(e^{i\vartheta}))| < \frac{\pi B}{m|\vartheta - \vartheta_k|} \tag{2.4}$$

for $-\pi \leqq \vartheta - \vartheta_k \leqq +\pi$, *where the constant B depends only on the domain D.*

From lemmas 2.1 and 2.2 we deduce

LEMMA 2.3
$$\sum_{k=1}^{n} |\tilde{l}_k^{(m)}(z)|^2 \leqq C_1 + C_2 \frac{n^2}{m^2}, \quad \text{for} \quad z \in \bar{D} \tag{2.5}$$

where C_1 *and* C_2 *does not depend on n or m.*

Proof: Since the sum on the left hand side is subharmonic, it is sufficient to prove (2.5) for $z \in \Gamma$, i.e. $z = \psi(e^{i\vartheta})$. Without loss of generality we can also assume that $\vartheta_n - 2\pi < \vartheta \leqq \vartheta_1$. Then, applying Lemma 2.2. and Lemma 2.1,

$$\begin{aligned}
\sum_{k=1}^{n} |\tilde{l}_k^{(m)}(\psi(e^{i\vartheta}))|^2 = {} & |\tilde{l}_1^{(m)}(\psi(e^{i\vartheta}))|^2 \\
& + \sum_{j=1}^{[(n+1)/2]-1} |\tilde{l}_{1+j}^{(m)}(\psi(e^{i\vartheta}))|^2 + \sum_{j=1}^{[n/2]-1} |\tilde{l}_{n-j}^{(m)}(\psi(e^{i\vartheta}))|^2 \\
& + |\tilde{l}_n^{(m)}(\psi(e^{i\vartheta}))|^2 \\
& \leqq B^2 + \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[(n+1)/2]-1} \frac{1}{(\vartheta - \vartheta_{1+j})^2} \\
& + \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[n/2]-1} \frac{1}{(\vartheta - \vartheta_{n-j} + 2\pi)^2} + B^2 \\
& \leqq 2B^2 + \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[(n+1)/2]-1} \frac{1}{(\vartheta_1 - \vartheta_{1+j})^2} \\
& + \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[n/2]-1} \frac{1}{(\vartheta_n - \vartheta_{n-j})^2} \\
& \leqq 2B^2 + \frac{2\pi^2 B^2 n^2}{c^2 m^2} \sum_{1 \leqq j \leqq n/2} \frac{1}{j^2} \\
& \leqq 2B^2 + \frac{\pi^4 B^2}{c^2} \frac{n^2}{m^2}.
\end{aligned}$$

which proves (2.5).

Remark. It follows from (2.3), that if condition (2.1) is satisfied, the system:
$$z_k^{(n)} = \psi(e^{i(\alpha_n + 2\pi i k/n)}) \tag{2.6}$$

provides another example for *regular* point systems. Here the real numbers $\alpha_1, \alpha_2, \alpha_3, \ldots$ are completely arbitrary. We call the system (2.6) a system of *Fejèr-points*.

3. Proof of the theorem

Let $\{q_n(z)\}$ be a sequence of polynomials with the property that

(i) $$q_n(z) \quad \text{is of degree} \quad n$$

(ii) $$q_n(z) \to f(z) \quad \text{uniformly in} \quad \bar{D}.$$

By Walsh's classical result, such a sequence certainly exists. We will specify the choice of $\{q_n(z)\}$ later. We have that

$$\varepsilon_n = \max_{z \in \bar{D}} |f(z) - q_n(z)| \to 0 \tag{3.1}$$

We now write $m = [\frac{1}{2}\eta n]$, and

$$p_{n-1}(z) = q_{n-1}(z) + \sum_{k=1}^{n} \{f(z_k) - q_{n-1}(z_k)\}\, l_k(z)\, [\tilde{l}_k^{(m)}(z)]^2 \tag{3.2}$$

Clearly:

$$p_{n-1}(z_\nu) = q_{n-1}(z_\nu) + f(z_\nu) - q_{n-1}(z_\nu) = f(z_\nu)$$

for $\nu = 1, 2, \dots n$. Further

$$|f(z) - p_{n-1}(z)| \leqq |f(z) - q_{n-1}(z)| + \sum_{k=1}^{n} |f(z_k) - q_{n-1}(z_k)|\, |l_k(z)|\, |\tilde{l}_k^{(m)}(z)|^2 .$$

Applying (3.1), (1.2), and (2.5):

$$|f(z) - p_{n-1}(z)| \leqq \varepsilon_n + M\varepsilon_n \sum_{k=1}^{n} |\tilde{l}_k^{(m)}(z)|^2$$

$$\leqq M\varepsilon_n\left(1 + C_1 + C_2 \frac{n^2}{m^2}\right) \leqq M\varepsilon_n\left(1 + C_1 + C_2 \frac{4}{\eta^2}\right) \quad \text{for} \quad n \geqq \frac{2}{\eta}.$$

Hence

$$\max_{z \in \bar{D}} |f(z) - p_{n-1}(z)| \to 0 \quad \text{as} \quad n \to \infty .$$

Thus we have proved Theorem 3 with the exception of the last assertion. To complete the proof it only remains to specify the choice of the polynomials $q_n(z)$.

The polynomial

$$s_{n-1}(z) = \frac{1}{2\pi i} \int_{|t|=1} f(\psi(t)) \left[\sum_{m=0}^{n-1} \left(1 - \frac{m}{n}\right) \frac{F_m(z)}{t^{m+1}}\right] dt$$

(where $F_m(z)$ is the m-th Faber-polynomial of the domain D) represents the arithmetic mean of the partial sums of the Faber-expansion of $f(z)$. It is known that for every $f(z)$ regular in D and continuous in $\bar{D}$,

$$s_n(z) \to f(z)$$

uniformly in $\bar{D}$ (cf. [1]). Thus we can choose

$$q_n(z) = s_n(z).$$

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