Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces (Part II: Method based on a Maximum Principle)

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Introduction

This is the continuation of the previous paper [1] of which we assume at least the "Introduction" as well known to the present reader. In [1] it has been proved: If $W, \bar{W}$ are closed hypersurfaces in an $(m+1)$-dimensional Riemann space with $\bar{W} = T_\tau W$, where the transformations T_τ (depending on the parameter τ) are *properly conformal* (and even a bit more generally so) and if

$$\bar{H}(\bar{p}) = \tilde{H}_p(\bar{p}) \qquad \text{for each } p \in W \tag{1}$$

holds (for the notation compare the quoted "Introduction"), then W, $\bar{W}$ *are congruent modulo* G (where G is the group of the transformations T_τ). In the present paper, we shall cancel the assumption that the transformations T_τ are properly conformal; in fact, they are essentially arbitrary; however we will assume that *no orbit of a transformation* T_τ *is tangent to the surface* W (and even a weaker assumption on the orbits will be sufficient). Then we shall prove: W *and* $\bar{W}$ *are congruent* (that is, $\bar{W} = T_\tau W$). – Here we like to call the reader's attention to the fact that neither the assumption made in part I (that the T_τ are properly conformal) nor the assumption made in part II (on the orbits of the T_τ) covers the other.

As said in the introduction of [1], the method of proof of our theorem in the present paper is based on the maximum principle of the solution of an elliptic differential equation. The kernel of this principle is contained in a theorem of E. HOPF [2] which we treat together with two rather easy consequences in § 1. Then § 2 contains the proof of our congruence theorem.

§ 1. Some auxiliary theorems on a linear partial differential expression of elliptic type

In an m-dimensional coordinate neighbourhood U we consider a linear partial differential expression of the second order of elliptic type

$$L(\Phi) = g^{\alpha\beta} \frac{\partial^2 \Phi}{\partial u^\alpha \partial u^\beta} + h^\gamma \frac{\partial \Phi}{\partial u^\gamma}$$

where $g^{\alpha\beta}(u)$ and $h^\gamma(u)$ are continuous functions of a point $p(u)$ in U and where the quadratic form $g^{\alpha\beta}\lambda_\alpha\lambda_\beta$ is supposed to be positive definite everywhere in U. Throughout this paper repeated lower case Greek indices call for summation from 1 to m. Then the following theorem has been proved by E. HOPF:

THEOREM 1.1. *If in a coordinate neighbourhood U a function $\Phi(p)$ of class C^2 satisfies the inequality $L(\Phi)\geqq 0$ and if there exists a fixed point p_0 in U such that $\Phi(p)\leqq\Phi(p_0)$ everywhere in U, then we have $\Phi(p)=\Phi(p_0)$ everywhere in U. If $L(\Phi)\leqq 0$ and $\Phi(p)\geqq\Phi(p_0)$ everywhere in U, then we have $\Phi(p)=\Phi(p_0)$ everywhere in U* ([2], *p.* 147).

We prove easily

THEOREM 1.2. *Let $g^{\alpha\beta}(u,t)$ be a continuous function of a point $u\in U$ and of a parameter t, $0\leqq t\leqq 1$, and let the quadratic form $g^{\alpha\beta}(u,t)\lambda_\alpha\lambda_\beta$ be positive definite everywhere, then $\int_0^1 g^{\alpha\beta}(u,t)\,dt\cdot\lambda_\alpha\lambda_\beta$ is positive definite everywhere in U.*

Proof. If we integrate the quantity $g^{\alpha\beta}(u,t)\lambda_\alpha\lambda_\beta$ over the interval $0\leqq t\leqq 1$, then we have

$$\int_0^1 g^{\alpha\beta}(u,t)\,\lambda_\alpha\lambda_\beta\,dt=\left\{\int_0^1 g^{\alpha\beta}(u,t)\,dt\right\}\lambda_\alpha\lambda_\beta .$$

Since $g^{\alpha\beta}(u,t)\lambda_\alpha\lambda_\beta$ is positive definite everywhere in U and in the interval $0\leqq t\leqq 1$, its integral over the interval $0\leqq t\leqq 1$ is also positive. Therefore $\{\int_0^1 g^{\alpha\beta}(u,t)\,dt\}\,\lambda_\alpha\lambda_\beta$ must be positive definite everywhere in U. –

Now we consider in U a linear partial differential expression of the second order

$$l(\Phi)=\int_0^1 g^{\alpha\beta}(u,t)\,dt\,\frac{\partial^2\Phi}{\partial u^\alpha\,\partial u^\beta}+\int_0^1 h^\gamma(u,t)\,dt\,\frac{\partial\Phi}{\partial u^\gamma}$$

where $g^{\alpha\beta}(u,t)$ and $h(u,t)$ are continuous functions of the point $u\in U$ and of the point t in the interval $0\leqq t\leqq 1$; the quadratic form $g^{\alpha\beta}(u,t)\lambda_\alpha\lambda_\beta$ is supposed to be positive definite everywhere in U and in the interval $0\leqq t\leqq 1$. Then from Theorem 1.1 and Theorem 1.2 we get the following

THEOREM 1.3. *If in a coordinate neighbourhood U a function $\Phi(p)$ of class C^2 satisfies the inequality $l(p)\geqq 0$ and if there exists a fixed point p_0 in U such that $\Phi(p)\leqq\Phi(p_0)$ everywhere in U, then we have $\Phi(p)=\Phi(p_0)$ everywhere in U. If $l(\Phi)\leqq 0$ and $\Phi(p)\geqq\Phi(p_0)$ everywhere in U, then we have $\Phi(p)=\Phi(p_0)$ everywhere in U.* –

Especially in the case that $g^{\alpha\beta}(u,t)$ and $h^\gamma(u,t)$ are constant with respect to the parameter t, Theorem 1.3 becomes E. HOPF's theorem.

§ 2. A congruence theorem for closed hypersurfaces

We suppose an $(m+1)$-dimensional Riemann space R^{m+1} of class C^ν $(\nu\geqq 3)$ which admits an infinitesimal transformation

$$\hat{x}^i = x^i + \xi^i(x)\,\delta\tau \tag{2.1}$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a contravariant vector). We assume that the orbits of the transformations generated by ξ cover R^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. Let us choose a coordinate system such that the orbits of the transformations generated by ξ are new x^1-coordinate curves, that is a coordinate system in which the vector ξ has components $\xi^i = \delta^i_1$, where the symbol δ^i_j denotes the Kronecker delta; then (2.1) becomes

$$\hat{x}^i = x^i + \delta^i_1\,\delta\tau\,. \tag{2.1'}$$

Thus R^{m+1} admits a one-parameter continuous group G of transformations which are (1–1)-mappings of R^{m+1} onto itself and are given by

$$\hat{x}^i = x^i + \delta^i_1\,\tau \tag{2.2}$$

in the new special coordinate system [3].

We consider now two hypersurfaces W^m and $\bar{W}^m$ of class C^ν imbedded in R^{m+1} which do not pass through a singular point of the vector field ξ. Let points on the two hypersurfaces correspond along the orbits of the transformations. Then the two hypersurfaces W^m and $\bar{W}^m$ are given by

$$\left.\begin{array}{ll} W^m\colon\ x^i = x^i(u^\alpha), & i = 1,\dots,m+1 \\ \bar{W}^m\colon\ \bar{x}^i = x^i(u^\alpha) + \delta^i_1\,\tau(u^\alpha), & \alpha = 1,\dots,m \end{array}\right\} \tag{2.3}$$

where u^α are local coordinates of W^m and $\tau(u^\alpha)$ is a fonction of class C^ν defined on W^m. We shall henceforth confine ourselves to Latin indices running from 1 to $m+1$ and Greek indices from 1 to m.

Besides the surfaces (2.3) we now consider, to each point $p_0 \in \bar{W}$, the surface

$$\tilde{W}^m_{p_0}\colon \tilde{x}^i = x^i(u^\alpha) + \delta^i_1\,\tau(u^\alpha_0),$$

where u^α_0 are the local coordinates of p_0. Then the corresponding point $\bar{p}_0$ lies on $\bar{W}^m$ and on $\tilde{W}^m_{p_0}$. We can consider the additional hypersurfaces $\tilde{W}^m_p = T_{\tau(p)}(W^m)$ to each point $p \in W^m$ and the mean curvatures H, $\bar{H}$, $\tilde{H}_p$ of W^m, $\bar{W}^m$, $\tilde{W}^m_p$ ([4], p. 250), and we claim that the following theorem holds:

THEOREM 2.1. *Let W^m and $\bar{W}^m$ given by* (2.3) *be two closed hypersurfaces in R^{m+1}. Suppose that no orbit of the transformations generated by ξ ever contacts W^m at the maximum point $p_0 \in W^m$ so that $\tau(p) \leqq \tau(p_0)$ everywhere in W^m. If the relation*

$$\bar{H}(\bar{p}) = \tilde{H}_p(\bar{p}) \tag{1}$$

holds for each point $p \in W^m$, then W^m and $\bar{W}^m$ are congruent mod. G. (W^m and $\bar{W}^m$ are congruent mod. G means that $\bar{W}^m = T_\tau W^m$ for a certain $T_\tau \in G$).

Proof. We consider the family of the hypersurfaces

$$W^m(t) = (1-t)\, W^m + t\, \bar{W}^m, \quad 0 \leqq t \leqq 1,$$

generated by W^m and $\bar{W}^m$ whose points correspond along the orbits of the transformations T_τ where W^m and $\bar{W}^m$ mean $W^m(0)$ and $W^m(1)$ respectively.

Then according to (2.3), $W^m(t)$ is given by the expression

$$W^m(t)\colon\ x^i(u^\alpha, t) = (1-t)\, x^i(u^\alpha) + t\bar{x}^i(u^\alpha), \quad 0 \leqq t \leqq 1. \tag{2.4}$$

(2.4) may be rewritten as follows:

$$W^m(t)\colon\ x^i(u^\alpha, t) = x^i(u^\alpha) + \delta^i_1\, t \cdot \tau(u^\alpha), \quad 0 \leqq t \leqq 1. \tag{2.5}$$

The relation between $\bar{W}^m$ and $W(t)$ becomes as follows:

$$\bar{x}^i(u^\alpha) = x^i(u^\alpha, t) + \delta^i_1 (1-t)\, \tau(u^\alpha).$$

If we take the hypersurface $W^m(t_0)$ defined by a fixed value t_0 in $0 \leqq t \leqq 1$, then we have the transformation $T_{(1-t_0)\tau(p_0)} \in G$ attached to the point on $W^m(t_0)$ corresponding to $p_0 \in W^m$, given by

$$T_{(1-t_0)\,\tau(p_0)}\colon\ \hat{x}^i = x^i + \delta^i_1 (1-t_0)\, \tau(u^\alpha_0), \quad (1-t_0)\, \tau(u^\alpha_0) = \text{const}.$$

Thus we get the additional hypersurface

$$\tilde{W}^m_{p_0}(t_0) \overset{\text{def}}{=} T_{(1-t_0)\,\tau(p_0)} \cdot W^m(t_0)$$

which passes through the corresponding point $\bar{p}$ on $\bar{W}^m$, and is given by

$$\tilde{W}^m_{p_0}(t_0)\colon \tilde{x}^i_{p_0}(u^\alpha, t_0) = x^i(u^\alpha, t_0) + \delta^i_1 (1-t_0)\, \tau(u^\alpha_0), \quad (1-t_0)\, \tau(u^\alpha_0) = \text{const}. \tag{2.6}$$

Therefore we have the hypersurfaces

$$\tilde{W}^m_{p_0}(t) = T_{(1-t)\,\tau(p_0)}\, W^m(t), \quad 0 \leqq t \leqq 1,$$

for all hypersurfaces in the family which pass through the corresponding point $\bar{p}_0$ on $\bar{W}^m$. Thus we can consider $\tilde{W}^m_p(t) = T_{(1-t)\tau(p)} W^m(t)$ for each $p \in W^m$.

Let $\tilde{H}_{p_0}(t_0)$, $\tilde{n}_{p_0}(t_0)$, $\tilde{g}^{*\alpha\beta}_{p_0}(t_0)$ be the mean curvature, the normal unit vector and the metric tensor of $\tilde{W}^m_{p_0}(t_0)$ at $\bar{p}_0$ respectively. Then we can consider the mean curvature $\tilde{H}_p(t)$, the normal unit vector $\tilde{n}_p(t)$ and the metric tensor $\tilde{g}^{*\alpha\beta}_p(t)$ of $\tilde{W}^m_p(t)$, $0 \leqq t \leqq 1$, at the corresponding point $\bar{p}$ to each point $p \in W^m$.

From the definition of the mean curvature of a hypersurface we have

$$\tilde{H}_p(t) = \frac{1}{m}\, \tilde{n}_{p\,i}(t)\, \frac{\delta^2 \tilde{x}^i_p(u, t)}{\partial u^\alpha\, \partial u^\beta}\, \tilde{g}^{*\alpha\beta}_p(t) \tag{2.7}$$

where it is understood that

$$\frac{\delta^2 \tilde{x}^i_p(u, t)}{\partial u^\alpha\, \partial u^\beta} = \frac{\partial^2 \tilde{x}_p(u, t)}{\partial u^\alpha\, \partial u^\beta} + \bar{\Gamma}^i_{jk}\, \frac{\partial \tilde{x}^j_p(u, t)}{\partial u^\alpha}\, \frac{\partial \tilde{x}^k_p(u, t)}{\partial u^\beta} - \tilde{\Gamma}^\gamma_{p\,\alpha\beta}(t)\, \frac{\partial \tilde{x}^i_p(u, t)}{\partial u^\gamma} \tag{2.8}$$

Γ^i_{jk} and $\tilde{\Gamma}^\gamma_{p\alpha\beta}(t)$ are the Christoffel symbols with respect to the metric tensor g_{ij} of R^{m+1} and $\tilde{g}^*_{p\alpha\beta}(t)$ respectively, at the corresponding point $\bar{p}$ to $p \in W^m$. (Throughout this paper repeated lower case Latin indices call for summation from 1 to $m+1$; but p is not a summation index!)

From the definition of the normal unit vector of a hypersurface we have

$$\tilde{n}_{p\tau_1}(t) = \frac{\bar{\varepsilon}_{i_2 \dots i_{m+1} i_1}}{\sqrt{\tilde{g}^*_p(t)}} \cdot \frac{\partial \tilde{x}^{i_2}_p(u,t)}{\partial u^{[1}} \cdots \frac{\partial \tilde{x}^{i_{m+1}}_p(u,t)}{\partial u^{m]}} \tag{2.9}$$

where

$$\bar{\varepsilon}_{i_1 \dots i_{m+1}} = \sqrt{\bar{g}'}\, e_{i_1, \dots i_{m+1}}$$

$\bar{g}$ being determinant of the metric tensor g_{ij} of R^{m+1} at the corresponding point $\bar{p}$, and the symbol $e_{i_1 \dots i_{m+1}}$ means plus one or minus one depending on whether the indices $i_1 \dots i_{m+1}$ denote an even or an odd permutation, of $1, 2, \dots m+1$, and zero when at least two indices have the same value ([4], p. 25). The symbol [...] means alternating in m ([4], p. 14); $\tilde{g}^*_p(t)$ is the determinant of the metric tensor $\tilde{g}^*_{p\alpha\beta}(t)$ on the hypersurface $\tilde{W}_p(t)$ at the corresponding point $\bar{p}$.

Since from (2.5) and (2.6) we obtain

$$\left.\begin{aligned} \frac{\partial \tilde{x}^i_p(u,t)}{\partial u^\alpha} &= \frac{\partial x^i(u,t)}{\partial u^\alpha} = \frac{\partial x^i(u)}{\partial u^\alpha} + \delta^i_1\, t\, \frac{\partial \tau(u)}{\partial u^\alpha} \\ \frac{\partial^2 \tilde{x}^i_p(u,t)}{\partial u^\alpha \partial u^\beta} &= \frac{\partial^2 x^i(u,t)}{\partial u^\alpha \partial u^\beta} = \frac{\partial^2 x^i(u)}{\partial u^\alpha \partial u^\beta} + \delta^i_1\, t\, \frac{\partial^2 \tau(u)}{\partial u^\alpha \partial u^\beta} \end{aligned}\right\} \tag{2.10}$$

and since

$$\tilde{n}_{pi}(t) \frac{\partial \tilde{x}^i_p(u,t)}{\partial u^\gamma} = 0,$$

(2.7) becomes

$$\tilde{H}_p(t) = \frac{1}{m} \tilde{n}_{pi}(t)\, \tilde{g}^{*\alpha\beta}_p(t) \left(\frac{\partial^2 x^i(u,t)}{\partial u^\alpha \partial u^\beta} + \bar{\Gamma}^i_{jk} \frac{\partial x^j(u,t)}{\partial u^\alpha} \frac{\partial x^k(u,t)}{\partial u^\beta} \right)$$

and we have

$$\tilde{g}^*_{p\alpha\beta}(t) = \bar{g}_{ij} \frac{\partial x^i(u,t)}{\partial u^\alpha} \frac{\partial x^j(u,t)}{\partial u^\beta}$$

where $\bar{g}_{ij}$ is the metric tensor of R^{m+1} at the point $\bar{p}$.

As seen from the above results, only $\partial x^1(u,t)/\partial u^\gamma$ and $\partial^2 x^1(u,t)/\partial u^\alpha \partial u^\beta$ contained in $\tilde{H}_p(t)$ depend on the parameter t. Therefore if we now differentiate the mean curvatures $\tilde{H}_p(t)$ of $\tilde{W}_p(t)$, $0 \leqq t \leqq 1$, at the point $\bar{p}$ corresponding to $p \in W$ with respect to t, we have

$$\frac{d\tilde{H}_p(t)}{dt} = \frac{\partial \tilde{H}(t)}{\partial \left(\dfrac{\partial^2 x^1(u,t)}{\partial u^\alpha \partial u^\beta} \right)} \cdot \frac{\partial^2 \tau(u)}{\partial u^\alpha \partial u^\beta} + \frac{\partial \tilde{H}_p(t)}{\partial \left(\dfrac{\partial x^1(u,t)}{\partial u^\gamma} \right)} \cdot \frac{\partial \tau(u)}{\partial u^\gamma}. \tag{2.11}$$

Integrating both members of (2.11) over the interval $0 \leqq t \leqq 1$, we get

$$\tilde{H}_p(u^\alpha, 1) - \tilde{H}_p(u^\alpha, 0) = \int_0^1 \frac{\partial \tilde{H}_p(t)}{\partial\left(\dfrac{\partial^2 x^1(u, t)}{\partial u^\alpha \partial u^\beta}\right)} dt \frac{\partial^2 \tau(u)}{\partial u^\alpha \partial u^\beta} + \int_0^1 \frac{\partial \tilde{H}_p(t)}{\partial\left(\dfrac{\partial x^1(u, t)}{\partial u^\gamma}\right)} dt \frac{\partial \tau(u)}{\partial u^\gamma},$$

where we see easily that $\tilde{H}_p(u, 1) = \bar{H}(\bar{p})$ and $\tilde{H}_p(u, 0) = \bar{H}_p(\bar{p})$. Now we make use of the hypothesis $\bar{H}(\bar{p}) = \bar{H}_p(\bar{p})$; then we have

$$\int_0^1 \frac{\partial \tilde{H}_p(t)}{\partial\left(\dfrac{\partial^2 x^1(u, t)}{\partial u^\alpha \partial u^\beta}\right)} dt \frac{\partial^2 \tau(u)}{\partial u^\alpha \partial u^\beta} + \int_0^1 \frac{\partial \tilde{H}_p(t)}{\partial\left(\dfrac{\partial x^1(u, t)}{\partial u^\gamma}\right)} dt \frac{\partial \tau(u)}{\partial u^\gamma} = 0. \tag{2.12}$$

From (2.7), (2.8) and (2.10) we have

$$\frac{\partial \tilde{H}_p(t)}{\partial\left(\dfrac{\partial^2 x^1(u, t)}{\partial u^\alpha \partial u^\beta}\right)} = \frac{1}{m} \tilde{n}_{p1}(t) \tilde{g}_p^{*\alpha\beta}(t)$$

and from (2.9) and (2.10)

$$\tilde{n}_{p1}(t) = \frac{(-1)^m m! \sqrt{g}}{\sqrt{\tilde{g}_p^*(t)}} \cdot \frac{\partial x^2(u)}{\partial u^{[1}} \cdots \frac{\partial x^{m+1}(u)}{\partial u^{m]}}.$$

Therefore setting

$$\left.\begin{aligned} n_1^* &= \frac{\partial x^2(u)}{\partial u^{[1}} \cdots \frac{\partial x^{m+1}(u)}{\partial u^{m]}}, \\ G_p^{*\alpha\beta}(t) &= \frac{\tilde{g}_p^{*\alpha\beta}(t)}{\sqrt{\tilde{g}_p^*(t)}}, \end{aligned}\right\} \tag{2.13}$$

we have

$$\int_0^1 \frac{\partial \tilde{H}_p(t)}{\partial\left(\dfrac{\partial^2 x^1(u, t)}{\partial u^\alpha \partial u^\beta}\right)} dt = (-1)^m (m-1)! \sqrt{g}\, n_1^* \int_0^1 G_p^{*\alpha\beta}(t)\, dt. \tag{2.14}$$

Since our closed hypersurface W^m is compact and the function τ is continuous, there is a point p_0 such that $\tau(p) \leqslant \tau(p_0)$ everywhere in W^m, and also the orbits of the transformations never are tangent to W^m at such a maximum point p_0 (that these orbits are tangent to W^m would mean that at this point $n_1 = n_1^* = 0$). Consequently we can take a neighbourhood U of p_0 in which $n_1^* \neq 0$. In U, it follows from (2.12) and (2.14)

$$\int_0^1 G_p^{*\alpha\beta}(t)\,dt\,\frac{\partial^2 \tau(u)}{\partial u^\alpha \partial u^\beta} + \frac{1}{(-1)^m (m-1)!\,\sqrt{\bar{g}}\,n_1^*} \int_0^1 \frac{\partial \tilde{H}_p(t)}{\partial\left(\dfrac{\partial x^1(u,t)}{\partial u^\gamma}\right)}\,dt\,\frac{\partial \tau(u)}{\partial u^\gamma} = 0.$$

As known from (2.13) the quadratic form $G_p^{*\alpha\beta}(t)\lambda_\alpha \lambda_\beta$ is positive definite everywhere in U and in the interval $0 \leqq t \leqq 1$; $G_p^{*\alpha\beta}(t)$and the factor before the second integral as well as the integrand are continuous in U and in $0 \leqq t \leqq 1$; and τ is a function of class C^ν, $\nu \geqq 2$, on U.

Consequently, it follows from Theorem 1.3 that we have $\tau(p) = \tau(p_0)$ for all $p \in U$; as one sees easily this is true for all $p \in W^m$. Thus $\tau(p) = \text{const.}$ and $\bar{W}^m = T_{\tau(p_0)} W^m$, q.e.d.

REFERENCES

[1] Y. KATSURADA, *Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces (*Part I: *Method based on Stokes' theorem)*, Comment. Math. Helv. *43* (1968), 176–194.

[2] E. HOPF, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitzungsber. Preuss. Akademie Wiss. *19* (1927), 147–152.

[3] L. P. EISENHART, *Continuous groups of transformations*, Princeton – London 1934.

[4] J. A. SCHOUTEN, *Ricci-Calculus* (2nd edition), Berlin 1954.

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