On the Homotopy Suspension.

On the Homotopy Suspension [1)]

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Introduction

The present paper is devoted to a study of the following process: take an arbitrary map $d: A \to X$, use it to attach the cone CA to X and obtain a space $B = X \cup CA$, and then lift d in a natural way to a map $e: A \to F$, where F is the "fibre" of the inclusion map $X \to B$. An important particular case arises when X is a point; the resulting map e is then readily identified to the natural embedding $A \to \Omega \Sigma A$ of A in the loops of its suspension. This case, which is crucial for computing homotopy groups of spheres, has been thoroughly investigated in [21], and most of the results therein extend to the general situation considered here [4]. Next, as shown in [4] and [6], study of the above process yields a satisfactory theory of the dual of Lusternik-Schnirelmann category; in particular, it elucidates the relationship between this dual and the homotopical nilpotency of loop spaces. Finally, the analysis of low-dimensional cases presented in the last section of this paper reveals that some results of homological algebra may also be derived from the general theorems pertinent to the process described.

The main problem in this context is to study the map e; specifically, one seeks convenient descriptions of the homotopy types of the fibre E and of the cofibre K of e so as to obtain generalizations of the *EHP* sequence [21]. Duality and the result in [4; 1.1] suggest that the homotopy type of K only depends on those of A and B; however, examples (see 1.4 below) disprove this conjecture. Nevertheless, the homotopy type of the suspension ΣK is determined by those of A and B, and this enables us to express the homotopy type of K itself in terms of A and B in a limited range of dimensions. There are several ways of doing that, and the maps which relate K to various functors of the two arguments A and B appear as generalizations of various forms of the Hopf invariant. Description of the homotopy type of E brings the generalized Whitehead product into the picture.

Some of those questions were already studied in [4]. The present paper improves and simplifies the results in [4; § 3 and § 4], and is independent of them. In [4], a sequence

$$A \to \cdots \to F_{p+1} \to F_p \to \cdots \to F_1 \to F_0 = CA$$

was associated with any space A; F_{p+1} is the fibre of the inclusion $F_p \to F_p \cup CA$, and $A \to F_{p+1}$ is the natural lifting of $A \to F_p$. The exact sequences

$$\cdots \to \pi_{p+q}(F_{p+1}, A) \to \pi_{p+q}(F_p, A) \to \pi_{p+q}(F_p \cup CA) \to \pi_{p+q-1}(F_{p+1}, A) \to \cdots$$

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yield an exact couple and the resulting spectral sequence is dual to that defined in [7; § 1]. Most of the results in [7; § 1 and § 2] readily dualize to the present situation. The dual of [7; 2.1] asserts that $d^r=0$ if $r>\operatorname{cocat} A$, the latter being the least integer $k\geqslant 0$ such that A is a retract of F_k; the proof follows the pattern described in [7; § 2] and its geometric part is obtained by straightforward duality from the results in [6; § 4], so that no details seem to be necessary.

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1. The homotopy type of a certain cofibre

Consider a cofibration

$$A \xrightarrow{d} X \xrightarrow{f} B = X \cup CA \tag{1}$$

where d and f are inclusion maps, and B results by erecting a reduced cone over the subset A of X; a point in the cone CA is denoted by $sa(s\in I,\, a\in A)$, the map $a\to 1a$ embeds A as base in the cone, and $0A\cup I*$ is identified to the vertex ($*$ stands for the base-point in any space). Let $\pi:PB\to B$ be the "end-point" fibre map on the space of paths in B, starting at $*$, and let F be the fibre space over X induced by f from π; since f is an inclusion, $F=\pi^{-1}(X)$. The triple $F\to X\to B$ may be considered as a fibration and we will call F the fibre of f; this process, which may be applied to any map, does not lead to any ambiguity since F has the homotopy type of $f^{-1}(*)$ in case f is a fibre map. The loop space ΩB operates on F through the map $\varrho\colon \Omega B\times F\to F$ given by $\varrho(\omega,\beta)=\omega+\beta$, where $+$ and $-$ denote path addition and subtraction. A *natural* lifting $e:A\to F$ of d is defined by $e(a)(s)=sa\in B$. Let $K=F\cup CA$ be the cofibre of e, let $k:F\to K$ be the inclusion, and introduce the composite

$$H:\Omega B*A\xrightarrow{1*e}\Omega B*F\xrightarrow{V}\Sigma(\Omega B\times F)\xrightarrow{\Sigma\varrho}\Sigma F\xrightarrow{\Sigma k}\Sigma K$$

where $*$ and Σ are the reduced join and suspension functors, and V collapses to a point the two ends of the join so that $V((1-s)\,\omega\oplus s\beta)=\langle s,(\omega,\beta)\rangle$; thus, $\Sigma\varrho\circ V$ results by the Hopf construction associated with ϱ. The following result was obtained independently in [5] and [10; 2.1].

THEOREM 1.1. *If (X, A) has the homotopy type of a* CW-*pair, and if A and X are connected, then H is a homotopy equivalence.*

Proof. Let $E=\pi^{-1}(A)$ and $G=\pi^{-1}(CA)$ so that $PB=F\cup G$ and $E=F\cap G$. Let the maps

$$G\cup CE\xrightarrow{\varphi}PB\cup CF\xrightarrow{\sigma}\Sigma F$$

be given by inclusion and by $\sigma(PB)=*$, $\sigma(s\beta)=\langle 1-s,\beta\rangle$, respectively. Since

$(PB; F, G, E)$ has the homotopy type of a CW-*ad* [12] and since PB is contractible, φ and σ induce isomorphisms of homology groups. The triples $E \to A \to B$ and $G \to CA \to B$ may be regarded as fibrations, and the fibre of a nullhomotopic fibre map is homotopy equivalent to the Cartesian product of the total space with the loop space of the base. Hence, the maps $\Omega B \times A \to E$ and $\Omega B \times CA \to G$, given by $(\omega, sa) \to \omega + e(sa)$ with $e(sa)(t) = tsa \in CA$ and $s=1$ for the first, are homotopy equivalences. So is then [14; p. 314] the induced map ψ in the diagram

$$\Sigma A \vee \Omega B * A \xrightarrow{\lambda} (\Omega B \times CA) \cup C(\Omega B \times A) \xrightarrow{\psi} G \cup CE,$$

where λ is the homotopy equivalence given by

$$\lambda\langle s, a\rangle = (*, 2sa) \quad \text{or} \quad (2-2s)(*, a),$$
$$\lambda((1-s)\omega \oplus sa) = (\omega, 2sa) \quad \text{or} \quad (2-2s)(\omega, a),$$

according as $0 \leqslant 2s \leqslant 1$ or $1 \leqslant 2s \leqslant 2$. Since A and X are 0-connected, the domain and range of $\sigma \circ \varphi \circ \psi \circ \lambda$ are 1-connected so that the latter is a homotopy equivalence. Since $\sigma \circ \varphi \circ \psi \circ \lambda$ coincides with Σe on ΣA, $\Sigma k \circ \sigma \circ \varphi \circ \psi \circ \lambda$ restricted to $\Omega B * A$ is also a homotopy equivalence which, as direct inspection reveals, coincides with H.

REMARK 1.2. Σe and Σk have a left and a right homotopy inverse, respectively; if $\sigma: B \to \Sigma A$ collapses X to a point, a left homotopy inverse r of Σe is given by $r\langle s, \beta\rangle = \sigma \circ \beta(s)$.

EXAMPLE 1.3. Take $X = CA$ and $B = \Sigma A$ in (1); $e: A \to F$ is then readily seen to be equivalent to the homotopy suspension $A \to \Omega \Sigma A$. If A is a CW-complex, $\Omega \Sigma A * A$ has the homotopy type of the collapsed product $\Sigma \Omega \Sigma A \circledast A$. Hence, by 1.1, $\Sigma \Omega \Sigma A$ has the homotopy type of $\Sigma A \vee \Sigma \Omega \Sigma A \circledast A$. By iterated substitution of this expression in place of $\Sigma \Omega \Sigma A$, and noting that $A \circledast \cdots \circledast A$ is $(m\text{-}1)$-connected if it contains m connected factors, we obtain the known [11] result

$$\Sigma \Omega \Sigma A \simeq \Sigma A \vee \Sigma(A \circledast A) \vee \cdots \vee \Sigma(A \circledast \cdots \circledast A) \vee \cdots$$

EXAMPLE 1.4. We now show that the homotopy type of K is not determined by those of A and B in (1). Consider the cofibration $S^n \to CS^n \to S^{n+1}$, where S^n is the n-sphere, $n \geqslant 2$. The resulting K is $\Omega S^{n+1} \cup CS^n$ which is well known to have non-trivial cup products. Therefore, it cannot have the homotopy type of a suspension. On the other hand, the K corresponding to the cofibration $S^n \to S^n \vee S^{n+1} \to S^{n+1}$ has the homotopy type of a suspension, as follows from

PROPOSITION 1.5. *If, in the cofibration* $A \xrightarrow{d} X \xrightarrow{f} B$, *the map* f *has a right homotopy inverse or, more generally, if the inclusion* $\partial: \Omega B \to F$ *is nullhomotopic, then* K *has the homotopy type of* $\Omega B \circledast A$ *provided* A *is* 0-*connected and* (X, A) *is* 1-*connected.*

Proof. Define p in the diagram

$$\begin{array}{ccccc} \Omega B \vee A & \xrightarrow{j} & \Omega B \times A & \xrightarrow{q} & \Omega B \circledast A \\ \Big\downarrow p & & \downarrow 1\times e & & \Big\downarrow \mu \\ & & \Omega B \times F & & \\ & & \downarrow \varrho & & \\ A & \xrightarrow{e} & F & \xrightarrow{k} & K \end{array}$$

by $p|\Omega B = 0$ and $p|A = 1$; let j be the inclusion, and let q denote the collapsing map. Since $\partial \simeq 0$, the first square homotopy commutes and there results a map μ yielding homotopy commutativity in the second square. Therefore, inspection of the definition of H reveals that $\Sigma\mu \circ \Sigma q \circ V \simeq H$, where $V: \Omega B * A \to \Sigma(\Omega B \times A)$ is defined as before. Since $\Sigma q \circ V$ is a homotopy equivalence, 1.1 implies that so is also $\Sigma\mu$. The connectivity assumptions imply that both $\Omega B \circledast A$ and K are 1-connected, and μ is a homotopy equivalence since it induces isomorphisms of homology groups.

2. The Hopf invariants

Introduce the diagram

$$\begin{array}{ccccc} F & \xrightarrow{k} & K & \xrightarrow{J} & \Omega(\Omega B * A) \\ \uparrow \partial & & \downarrow T & & \downarrow \Omega\psi \\ \Omega B & \xrightarrow{W} & \Omega(\Sigma A \flat B) & \xrightarrow{\Omega\varphi} & \Omega^2(\Sigma A \circledast B) \end{array} \tag{2}$$

where k is as before, ∂ is the inclusion, $Y \flat Z$ is the fibre $(PY \times \Omega Z) \cup (\Omega Y \times PZ)$ of the inclusion $j: Y \vee Z \to Y \times Z$, and φ, ψ are natural maps given by

$$\varphi(\eta, \zeta)(t) = \eta(t) \circledast \zeta(t), \quad \psi((1-s)\omega \oplus s a)(t) = \langle t, a \rangle \circledast \omega(1-s);$$

thus, φ is induced by the collapsing map $q: Y \times Z \to Y \circledast Z$, and ψ is a composite of standard maps.

To obtain J, compose the inclusion $K \to \Omega\Sigma K$ with ΩG, where G is a homotopy inverse of H. It follows from 1.3 that J may be considered as a generalization of the James invariants as described in [1]. Next, recall that ΣA co-operates on B through the map $\tau: B \to \Sigma A \vee B$ which pinches together all points halfway up the cone CA; let $\tau = (\sigma, \theta)$ and note that

$$(\sigma(ta), \theta(sa)) \in \Sigma A \vee B \quad \text{for } 0 \leqslant s \leqslant t \leqslant 1, \text{ and } \theta \simeq 1. \tag{3}$$

In the diagram

$$\Omega(Y \vee Z) \xrightarrow{R} \Omega(Y \flat Z) \xrightarrow{\Omega l} \Omega(Y \vee Z) \xrightarrow{\Omega j} \Omega(Y \times Z) \xrightarrow{M} \Omega(Y \vee Z),$$

where l is the projection and $M(\eta, \zeta) = (\eta, *) + (*, \zeta)$, one has $\Omega j \circ M \simeq 1$ and there results a map R such that

$$\Omega l \circ R + M \circ \Omega j \simeq 1 \quad \text{and} \quad R \circ \Omega l \simeq 1; \tag{4}$$

the values of $R(\eta,\zeta)(s)$ on the thirds of $0\leqslant s\leqslant 1$ are

$$(\eta_{3s},\zeta_{3s}),\ (\eta,\zeta_{2-3s}),\ (\eta_{3-3s},*)\text{ with }\xi_u(v)=\xi(uv)$$

for any path ξ. Then, W is defined as $R\circ\Omega\tau$; it obviously generalizes the delicate Hopf invariant of G. W. WHITEHEAD as modified by HILTON [8], whereas $\Omega\varphi\circ W$ generalizes the crude Hopf invariant. To define T, let G be the fibre of the inclusion $i:B\to\Sigma A\vee B$; the map $h:G\to\Omega(\Sigma A\vee B)$, given by $h(\gamma)=\gamma-i\circ r\circ\gamma$, satisfies

$$\partial\circ h\simeq 1 \quad\text{and}\quad h\circ\partial+\Omega i\circ\Omega r\simeq 1,$$

where $r:\Sigma A\vee B\to B$ is the retraction and ∂ the inclusion. Define $S=R\circ h\circ g:F\to\Omega(\Sigma A\flat B)$, where $g(\beta)=\tau\circ\beta$. This construction is due to TODA and $\Omega\varphi\circ S$ coincides with his relative Hopf invariant [20]. Using (4) and the fact that $\sigma(X)=*$, the map $\Omega l\circ S$ is easily seen to coincide with $\Omega l\circ Q$ where, for $\beta\in F$, the values of $Q(\beta)(s)\in\Sigma A\flat B$ on the thirds of $0\leqslant s\leqslant 1$ are

$$(\sigma\circ\beta_{3s},\theta\circ\beta_{3s}),\qquad(\sigma\circ\beta,\theta\circ\beta_{2-3s}),\qquad(\sigma\circ\beta_{3-3s},*);\tag{5}$$

hence, by (4), we may assume $S(\beta)(s)$ to be given by (5). Using (3), it is easily seen now that $S\circ e\simeq 0$, and there results a map T with $T\circ k\simeq S$; its homotopy class is uniquely determined since, by 1.2, Σk has a right homotopy inverse. Next, let the value of $h_u:\Omega B*A\to\Omega(\Sigma A\circledast B)$ at $(1-s)\omega\oplus sa$ and t be

$$\begin{aligned}&\sigma\circ\omega(2t)\circledast\theta\circ\omega((1-s)2tu) && \text{if}\quad 0\leqslant 2t\leqslant 1,\\ &\sigma\circ e(a)(2t-1)\circledast\theta\circ\omega\circ\mathrm{Min}((1-s)(2tu+1-u),1) && \text{if}\quad 1\leqslant 2t\leqslant 2.\end{aligned}$$

Using primes to denote adjoints, $\varphi\circ S':\Sigma F\to\Omega(\Sigma A\circledast B)$ is homotopic to ϕ given by $\phi\langle s,\beta\rangle(t)=\sigma\circ\beta(t)\circledast\theta\circ\beta((1-s)t)$ so that, by (3), $h_1\simeq\varphi\circ T'\circ H$; also, since $\theta\simeq 1$, $h_0\simeq\psi$. This yields the first part of our next result; the low-dimensional cases of the second part are obtained using [16].

THEOREM 2.1. *Diagram* (2) *homotopy commutes. If A is $(n-1)$-connected and (X,A) is m-connected $(n\geqslant 1, m\geqslant 0)$, then e is $(m+n-1)$-connected, J is $(2m+2n-1)$-connected, ψ is $(2m+n+1)$-connected, φ is $(N+1)$-connected, and T is $(N-1)$-connected, where $N=m+n+\mathrm{Min}(m,n)$.*

REMARK 2.2. In the relative Hopf invariant $\Omega\varphi\circ T\circ k$, the map $\Omega\varphi\circ T$ is $(2m+n)$-connected, and T is monomorphic on homology and homotopy groups in dimensions $\leqslant 2m+n-1$.

Resume consideration of (1). Suppose that X has a comultiplication $\xi:X\to X\vee X$ and that A has a suspension structure given by some homotopy equivalence $\alpha:\Sigma Y\to A$ with inverse η. Using primes to denote adjoints, define the crude Hopf (α,ξ)-invariant of d by

$$H(d)=(\phi'\circ\eta)':\Sigma A\to X\circledast X$$

where, with R as in (4) and φ as in (2), ϕ is the composite

$$Y \xrightarrow{\alpha'} \Omega A \xrightarrow{\Omega d} \Omega X \xrightarrow{\Omega \xi} \Omega(X \vee X) \xrightarrow{R} \Omega(X \flat X) \xrightarrow{\Omega \varphi} \Omega^2(X \divideontimes X);$$

this definition is consistent with that in [2; 2.11] where the suspension structure on A is unique. Let $q: B \times B \to B \divideontimes B$ and $\sigma: B \to \Sigma A$ be the collapsing maps, and let Δ be the diagonal map. By arguments similar to those in [2; 3.7], one obtains

PROPOSITION 2.3. *If X has a comultiplication and if A is a suspension, then* $q \circ \Delta \simeq (f \divideontimes f) \circ H(d) \circ \sigma : B \to B \divideontimes B$.

This result may be dualized and admits various generalizations. Note that $q \circ \Delta$ induces the cup product in any multiplicative cohomology theory on B. Taking $A = S^{2k-1}$ and $X = S^k$, 2.3 readily yields the classical result [19] relating $H(d)$ to the cup product in $S^k \cup e^{2k}$.

3. The Whitehead product

The preceding results enable us to simplify and improve slightly the results in [4; 3.2 and 4.1]. Introduce the diagram

$$\begin{array}{ccccccc}
D \flat A & \xrightarrow{\iota} & D \vee A & \xrightarrow{j} & D \times A & \xrightarrow{q} & D \divideontimes A \\
\downarrow \lambda & & \downarrow p & & \downarrow r & & \downarrow \mu \\
E & \xrightarrow{h} & A & \xrightarrow{e} & F & \xrightarrow{k} & K \\
\downarrow i & & \| & & \downarrow \pi & & \\
\Omega B \xleftarrow{\varphi} D & \xrightarrow{g} & A & \xrightarrow{d} & X & \xrightarrow{f} & B
\end{array} \tag{6}$$

where D and E, with projections g and h, are the fibres of the inclusions d and e, respectively. The map i is induced by the projection π so that $i(\varepsilon) = \pi \circ \varepsilon$, and the bottom squares obviously commute. Next, φ is the natural map from the fibre of d to the loops of its cofibre followed by loop inversion; regarding ΩB as a subset of F under the inclusion ∂, one has $\varphi(\xi) = e(\xi(1)) - f \circ \xi$ and

$$\partial \circ \varphi \simeq e \circ g \quad \text{via} \quad h_t(\xi) = e(\xi(1)) + (-f \circ \xi)_t \in F, \tag{7}$$

where $\xi \in D$ and $\beta_t(s) = \beta(ts)$ for any path β. The map p is given by $p|D = g$ and $p|A = 1$, whereas $r = \varrho \circ (\varphi \times e)$ so that, by (7), $e \circ p \simeq r \circ j$. Since $D \divideontimes A$ has the homotopy type of the cofibre of j when A and D have the homotopy type of CW-complexes, p, r, and h_t induce maps λ and μ such that the remaining squares homotopy commute.

THEOREM 3.1. *If (X, A) has the homotopy type of a* CW*-pair, then* (6) *homotopy commutes. If A is $(n-1)$-connected and (X, A) is m-connected, then φ is $(m+n-1)$-connected and μ is $(m+2n-1)$-connected when $c \geq 1$, whereas λ is $(m+n+c-2)$-connected when $c \geq 2$; here* $c = \mathrm{Min}(m, n)$.

Proof. The connectivity of φ follows easily. The diagram

$$\begin{array}{ccc} D*A \xrightarrow{V} \Sigma(D\times A) \xrightarrow{\Sigma q} & \Sigma(D \divideontimes A) \\ \downarrow{\scriptstyle \varphi*1} & \downarrow{\scriptstyle \Sigma\mu} \\ \Omega B*A \xrightarrow{\qquad H \qquad} & \Sigma K \end{array}$$

where V and H are as in § 1, homotopy commutes. Since the horizontals are homotopy equivalences, $\Sigma\mu$ has the connectivity of $\varphi*1$, which is $m+2n$, and the connectivity of μ follows since its domain and range are 1-connected. When $c \geqslant 2$, the "relative Whitehead theorem" [13] may be invoked to derive the connectivity of λ from that of μ noting that p is c-connected.

To complete the picture, recall [4; 5.1] that, with no restrictions on m or n, there is a homotopy equivalence ϕ yielding homotopy commutativity in the diagram

$$\begin{array}{ccc} \Omega D*\Omega A \xrightarrow{V} \Sigma(\Omega D\times\Omega A) \xrightarrow{\Sigma q} & & \Sigma(\Omega D \divideontimes \Omega A) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle [g\circ S,\, S]} \\ D\flat A \xrightarrow{\quad \iota \quad} D\vee A & \xrightarrow{\quad p \quad} & A \end{array} \tag{8}$$

where the bracket on the right denotes the Whitehead product of

$$\Sigma\Omega D \xrightarrow{S} D \xrightarrow{g} A \quad \text{and} \quad \Sigma\Omega A \xrightarrow{S} A;$$

here, $S\langle s, \alpha\rangle = \alpha(s)$. Then, 3.1 and (8) enable us to replace, in a certain range of dimensions, the map h in (6) by a Whitehead product, and various generalizations of the *EHP* sequence [21] are available (cf. [4; § 5]).

4. Low dimensions

We study the extent to which the last statement in 3.1 survives when $c=1$. The notation is as in 3.1 and (6). We use square brackets to denote Whitehead products and subgroups generated by them; the Whitehead product of 1-dimensional elements is their commutator, and the operation of π_1 on π_m may be expressed in terms of Whitehead products. Suppose first that $n=1$ and $m\geqslant 1$. Let $\Pi=\pi_1(A)$, $G=\pi_m(A)$, and let $M=\operatorname{Im} g_m=\operatorname{Ker} d_m\subset G$, where the subscript m denotes induced homomorphisms of homotopy groups.

THEOREM 4.1. *Suppose that A is connected and that (X, A) has the homotopy type of an m-connected* CW*-pair with $m\geqslant 1$. Then*

$$\operatorname{Im} h_m = \operatorname{Im}(h\circ\lambda)_m = [M, \Pi] \quad \textit{and} \quad \pi_m(F) \simeq G/[M, \Pi]$$

under the epimorphism e_m. Furthermore, there is an exact sequence $\pi_{m+1}(X)\to\pi_{m+1}(B)\to M/[M, \Pi]\to 0$, where the first homomorphism is induced by f.

Proof. Obvious identifications enable us to insert h_m and g_m in the commutative diagram

$$\begin{array}{ccccc} & & \pi_{m+1}(F, A) & \overset{h_m}{\rightarrow} & \pi_m(A) \\ & & \downarrow \pi_\# & & \| \\ H_{m+1}(X, A) & \overset{\varrho}{\leftarrow} & \pi_{m+1}(X, A) & \overset{g_m}{\rightarrow} & \pi_m(A) \\ \downarrow f_* & & \downarrow f_\# & & \\ H_{m+1}(B, CA) & \overset{\varrho_0}{\leftarrow} & \pi_{m+1}(B, CA) & & \end{array}$$

in which the ϱ's are Hurewicz homomorphisms. One has

$$\operatorname{Im} h_m = g_m(\operatorname{Im} \pi_\#) = g_m(\operatorname{Ker} f_\#), \tag{9}$$

the latter equality being valid since the sequence $F \to X \to B$ is (essentially) a fibration. Since B is m-connected, ϱ_0 is isomorphic; also, f_* is isomorphic by excision. Therefore,

$$\operatorname{Ker} f_\# = \operatorname{Ker} \varrho .$$

Since (X, A) is m-connected, the structure of Ker ϱ is given by the relative Hurewicz theorem [15; Th. 4, p. 397], and we obtain

$$g_m(\operatorname{Ker} f_\#) = [M, \Pi] . \tag{10}$$

Next, the first inclusion in

$$[M, \Pi] \subset \operatorname{Im}(p \circ l)_m = \operatorname{Im}(h \circ \lambda)_m \subset \operatorname{Im} h_m \tag{11}$$

follows easily from (8); the equality is given by (6), and the second inclusion is obvious. The first part of 4.1 follows now by inspection of (9), (10), and (11). The connectivity of e is $m+n-1=m$ by 2.1. The exact sequence results upon noting that, in the homotopy sequence of the fibration $F \to X \to B$, $\pi_m(F) \to \pi_m(X)$ is equivalent to the natural homomorphism $G/[M, \Pi] \to G/M$ which has $M/[M, \Pi]$ as kernel.

As a by-product of the proof we have

PROPOSITION 4.2. $\operatorname{Im}(p \circ l)_m = [M, \Pi]$.

REMARK 4.3. If $m=1$ and $n \geqslant 2$ so that $m+n-1=n$, then 4.1 is trivially true in the sense that

$$\operatorname{Im}(h \circ \lambda)_n \subset \operatorname{Im} h_n = \operatorname{Ker} e_n = 0;$$

this follows since now Ker e_n may be computed by passing to homology where 1.2 applies. The exact sequence becomes

$$\pi_{n+1}(X) \to \pi_{n+1}(B) \to N \to 0, \quad \text{where} \quad N = \operatorname{Im} g_n = \operatorname{Ker} d_n .$$

Thus, since $D\flat A$ is $(m+n-2)$-connected, we may say that $h \circ \lambda$ in (6) behaves as if λ were $(m+n+c-2)$-connected even when $c=1$.

EXAMPLE 4.4. Let A be an m-dimensional CW-complex with $\pi_q(A)=0$ for $1<q<m$, and let X result by attaching cells to A so as to kill $\pi_q(A)$ for all $q>1$. Thus, X is aspherical and $H_q(X)\simeq H_q(\Pi)$ for all q, where $\Pi=\pi_1(A)$. Consider the diagram

$$\begin{array}{ccccccccc} & & 0=\pi_{m+1}(X) & \to & \pi_{m+1}(B) & \to & \pi_{m+1}(B,X) & \to & \pi_m(X)=0 \\ & & & & \downarrow \varrho_1 & & \downarrow \varrho_2 & & \\ 0=H_{m+1}(A) & \to & H_{m+1}(X) & \to & H_{m+1}(B) & \to & H_{m+1}(B,X) & & \end{array}$$

where the ϱ's denote Hurewicz homomorphisms. Since B is m-connected, ϱ_1 is isomorphic and, by exactness of the rows, we obtain $H_{m+1}(X)\simeq \operatorname{Ker}\varrho_2$. To compute the latter, introduce the diagram

$$\begin{array}{ccccccc} & & \Gamma_m(A) & \longrightarrow & \Gamma_{m+1}(CA,A) & \longrightarrow & \Gamma_{m+1}(B,X) \\ & & \downarrow & & \downarrow & & \downarrow \\ [\pi_m(A),\Pi] & \to & \pi_m(A) & \xrightarrow{\partial^{-1}} & \pi_{m+1}(CA,A) & \xrightarrow{d_{m+1}} & \pi_{m+1}(B,X) \\ & & \downarrow \varrho_1 & & \downarrow \varrho & & \downarrow \varrho_2 \\ & & H_m(A) & \xrightarrow{\partial^{-1}} & H_{m+1}(CA,A) & \xrightarrow{d_*} & H_{m+1}(B,X) \end{array}$$

where ∂^{-1} is inverse to the appropriate boundary isomorphism, d_{m+1} and d_* are induced by the inclusion d, the Γ's are the kernels of the ϱ's, and the top horizontals are induced by the bottom squares. Since $d_{m+1}\circ\partial^{-1}$ coincides with e_m followed by a natural isomorphism $\pi_m(F)\to\pi_{m+1}(B,X)$, it is epimorphic and has $[\pi_m(A),\Pi]$ as kernel by 4.1; also, the bottom composite is clearly isomorphic. Therefore,

$$H_{m+1}(\Pi)\simeq \Gamma_{m+1}(B,X)\simeq \Gamma_m(A)/[\pi_m(A),\Pi],$$

a well known result due to HOPF [9].

EXAMPLE 4.5. Let A and X be aspherical spaces with fundamental groups Π and Π/M, respectively, where M is a normal subgroup of Π; let d induce $\Pi\to\Pi/M$. Since B is 1-connected, $H_2(B)\simeq\pi_2(B)$ by the Hurewicz theorem; since $\pi_2(X)=0$, $\pi_2(B)\simeq M/[M,\Pi]$ by the exact sequence in 4.1. Therefore, part of the homology sequence of the cofibration $A\to X\to B$ may be rewritten as

$$H_2(\Pi)\to H_2(\Pi/M)\to M/[M,\Pi]\to H_1(\Pi)\to H_1(\Pi/M)\to 0.$$

This exact sequence was recently obtained by methods of homological algebra in [3], [17], [18].

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