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On the H-space structures of finite complexes[1])

by C. R. CURJEL

1. The results

The purpose of this paper is to study the isomorphism classes of H-space structures on one and the same underlying topological space. The notion of isomorphism used here is the one appropriate to homotopy theory, viz. that of isomorphism of multiplicative objects in the homotopy category. Our results answer questions raised in Massey's list of problems in algebraic topology [6].

Throughout we work in the homotopy category of countable CW-complexes which is denoted by $\mathbf{T}_h$. The objects of $\mathbf{T}_h$ are pathwise connected topological spaces, with a basepoint and of the homotopy type of a countable CW-complex, and the morphisms are homotopy classes of basepoint preserving maps. The set of morphisms $A \to B$ is denoted by $[A, B]$. Following [4] we consider H-spaces consistently as objects in $\mathbf{T}_h$. Thus an *H-space structure* or a *multiplication* on a space X is an element $m \in [X \times X, X]$ such that

$$m j_1 = m j_2 = 1_X \in [X, X],$$

where j_1, j_2 are the homotopy classes of the two injections $X \to X \times X$, and an *H-space* is then a couple (X, m) of a space X together with a specific multiplication m. A multiplication m is *associative* if the relation

$$m(m \times 1_X) = m(1_X \times m)$$

holds in $[X \times X \times X, X]$. We recall from [5] that under our hypotheses on the spaces involved every associative multiplication on X has an inverse $\iota: X \to X$. Therefore an associative multiplication m on X will also be called a *group-like structure* on X, and in this case (X, m) is a *group in* $\mathbf{T}_h$.

Let (Y, n) be another H-space. Then $f \in [X, Y]$ is a *homomorphism* of (X, m) into (Y, n) if $fm = n(f \times f)$:

$$\begin{array}{ccc} X \times X & \xrightarrow{m} & X \\ \downarrow{\scriptstyle f \times f} & & \downarrow{\scriptstyle f} \\ Y \times Y & \xrightarrow{n} & Y \end{array}$$

A homomorphism f is an *isomorphism* if f is an equivalence in $\mathbf{T}_h$, i.e. if f is the

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homotopy class of a homotopy equivalence. Two H-spaces which are isomorphic in this sense are equivalent in the homotopy category in exactly the same way as two isomorphic groups are equivalent for all purposes of group theory. The proper categorical classification problem for H-spaces is therefore the problem of describing and enumerating the different possible *isomorphism classes* of H-spaces. This was suggested already in [6, p. 350].

The underlying spaces of two isomorphic H-spaces are homotopy equivalent. Thus one is led to consider the H-space structures on homotopy equivalent spaces or, equivalently, the H-space structures on one and the same CW-complex. Now it was shown in [1] that practically all finite complexes which are known to carry an H-space structure admit not only one such structure, but infinitely many of them. What can one say about the number of *isomorphism classes* of H-space structures on a finite complex?

THEOREM I. *A connected finite* CW-*complex admits at most a finite number of mutually non-isomorphic structures as a group in* $\mathbf{T}_h$.

Some restriction on the underlying space is necessary, as is shown in

PROPOSITION Ia: $S^1 \times K(\mathbf{Z}, 2)$ *admits infinitely many mutually non-isomorphic structures as a group in* $\mathbf{T}_h$.

For non-associative multiplications the situation is different. Let $\beta_i(X)$ be the i-th Betti number of X and $\gamma_i(X)$ the rank of $\pi_i(X)$.

THEOREM II. *Let* (G, m) *be a group in* $\mathbf{T}_h$ *with* G *a connected finite* CW-*complex. Then* G *admits infinitely many mutually nonisomorphic structures as a non-associative* H-*space if and only if* $\beta_n(G \# G)\, \gamma_n(G) \neq 0$ *for some* n.

Theorems I and II solve Problem 43 of [6] in that they describe the situation collectively for all H-spaces whose underlying spaces are of the homotopy type of a finite complex. We remark to Theorem I that S^3 is known to admit four isomorphism classes of group-like structures. Theorem II shows that a group-like space admits infinitely many non-isomorphic H-space structures as soon as it admits infinitely many multiplications (which is, as pointed out, practically always the case).

The proofs yield additional information. Let $n_1, \ldots, n_r$ be the integers k for which $\gamma_k(G) \neq 0$.

THEOREM I (continued). *If* $\beta_{n_i}(G) = \gamma_{n_i}(G)$ *for all* i *then* G *admits only a finite number of group-like structures. If* $\beta_{n_i}(G) > \gamma_{n_i}(G)$ *for some* i *then each of the finitely many isomorphism classes of group-like structures contains infinitely many of them.*

Let $\mathbf{Q}$ be the rationals.

THEOREM II (continued). *If* $\beta_n(G \# G)\, \gamma_n(G) \neq 0$ *for some* n *then infinitely many non-isomorphic* H-*space structures* $m_1, m_2, \ldots$ *can be chosen in such a way that all Pon-*

tryagin algebras $H_*((G, m_i); \mathbf{Q})$ are non-associative. At least N of these algebras are mutually non-isomorphic as algebras over $\mathbf{Q}$, where N is the number of integers k such that $\beta_k(G \# G)\,\gamma_k(G) \neq 0$.

The latter statement answers the question raised in Problem 45 of [6]. For instance the different H-space structures on $SU(n)$, $n \geqslant 6$, give rise to at least $n-4$ mutually non-isomorphic Pontryagin algebras.

The paper is organized as follows. In section 2 we set up the machinery for the proofs which are outlined at the end of the same section. Sections 3 and 4 are purely algebraic and deal with graded symmetric algebras over the integers. In section 5 we collect facts about maps and their induced cohomology homomorphisms. These facts are needed for the proofs of Theorems I, II which are given in § 6, 7 respectively. Finally we prove Proposition Ia in § 8.

Last but not least the author wishes to acknowledge that some of the ideas incorporated in this paper developed out of joint work with M. ARKOWITZ. It is a pleasure to thank him for his contributions.

2. Structures on an object

To avoid repetitive reasoning it is convenient to introduce the following definition of a "collection of structures on an object."

Let $\mathbf{C}$ be a category and T_1, T_2 functors $\mathbf{C} \to \mathbf{C}$. For an object X of $\mathbf{C}$ we denote by $E(X)$ the group of equivalences $X \to X$. Then we define *a collection $\Sigma = \Sigma(X; T_1, T_2)$ of structures on X* to be a subset of $\mathrm{Hom}(T_1(X), T_2(X))$ such that $\sigma \in \Sigma$ implies $T_2(\theta)^{-1} \sigma T_1(\theta) \in \Sigma$ for any $\theta \in E$:

$$T_1(X) \xrightarrow{T_1(\theta)} T_1(X) \xrightarrow{\sigma} T_2(X) \xrightarrow{T_2(\theta)^{-1}} T_2(X).$$

Now let E' be a subgroup of E. Then two structures $\sigma, \sigma' \in \Sigma$ are *isomorphic relative to E'*, written as $\sigma \cong \sigma'$ rel E', if $\sigma T_1(\theta) = T_2(\theta)\,\sigma'$ for some $\theta \in E'$:

$$\begin{array}{ccc} T_1(X) & \xrightarrow{\sigma} & T_2(X) \\ {\scriptstyle T_1(\theta)}\uparrow & & \uparrow{\scriptstyle T_2(\theta)} \\ T_1(X) & \xrightarrow{\sigma'} & T_2(X) \end{array}$$

Isomorphism rel E' is an equivalence relation on Σ. To study the isomorphism classes rel E' of structures on X it is convenient to define an operation of E' on Σ by

$$\theta * \sigma = T_2(\theta)^{-1} \sigma T_1(\theta) \tag{*}$$

for $\theta \in E'$ and $\sigma \in \Sigma$. Thus $\sigma \cong \sigma'$ rel E' if and only if σ and σ' lie on the same orbit of the operation (*) of E' on Σ. Obviously E' operates as a group on the set Σ, i.e.

$$\theta' * (\theta * \sigma) = (\theta' \theta) * \sigma$$

and

$$1_X * \sigma = \sigma$$

for any θ, $\theta' \in E'$ and $\sigma \in \Sigma$. The isomorphism classes relE' of structures Σ on X are therefore in $1-1$-correspondence with the collection $\Sigma // E'$ of orbits of the operation (*). In the following Lemmas 2.1, 2.2, 2.3, we state some obvious facts which will be used in the proof of Theorems I and II.

LEMMA 2.1. *Let E'' be a subgroup of E'.*
(i) *If $\Sigma // E''$ is finite, so is $\Sigma // E'$.*
(ii) *Assume that E'' is of finite index in E'. If $\Sigma // E'$ is finite, so is $\Sigma // E''$.*
A subset Σ' of Σ is called *invariant under E'* if $\theta * \sigma \in \Sigma'$ for any $\sigma \in \Sigma'$ and $\theta \in E'$.

LEMMA 2.2. *Let $\Sigma' \subset \Sigma$ be invariant under E'. If $\Sigma // E'$ is finite, so is $\Sigma' // E'$.*
Now let σ be a fixed element of Σ. The subset $E'(\sigma)$ of E' of all $\theta \in E'$ with $\theta * \sigma = \sigma$:

$$\begin{array}{ccc} T_1(X) & \xrightarrow{\sigma} & T_2(X) \\ \uparrow {\scriptstyle T_1(\theta)} & & \uparrow {\scriptstyle T_2(\theta)} \\ T_1(X) & \xrightarrow{\sigma} & T_2(X) \end{array}$$

is clearly a subgroup of E'.

LEMMA 2.3. *Let R be a system of representatives of the left cosets of E' modulo $E'(\sigma)$. Then*

$$E' * \sigma = R * \sigma \cdot$$

The following two kinds of structures are considered in this paper:

(1) $\mathbf{C} = \mathbf{T}_h$, $T_1(X) = X \times X$, $T_2(X) = X$, Σ the collection $M(X)$ of all $m \in [X \times X, X]$ such that (X, m) is an H-space. Here $E(X)$ is the group $\mathbf{E}(X)$ of homotopy classes of homotopy equivalences $X \to X$. Clearly $m \cong m'$ rel$\mathbf{E}(X)$ if and only if the H-spaces (X, m) and (X, m') are isomorphic.

(2) $\mathbf{C}$ the category of graded connected algebras over the rationals $\mathbf{Q}$, $T_1(X) = X$, $T_2(X) = X \otimes X$, Σ the collection $\Delta(X)$ of all algebra maps $\delta : X \to X \otimes X$ such that (X, δ) is a Hopf algebra. Here $E(X)$ is the group $\mathit{Aut}_A X$ of algebra automorphisms of X.

The cases (1) and (2) are related by the functor $\mathbf{h}$ which assigns to a space X its rational cohomology algebra $H^* = H^*(X; \mathbf{Q})$. For, if $m \in M(X)$, then $\mathbf{h}(m) \in \Delta(H^*)$. Obviously

$$m \cong m' \operatorname{rel} \mathbf{E}(X)$$

implies

$$\mathbf{h}(m) \cong \mathbf{h}(m') \operatorname{rel} A'(H^*)$$

where $A'(H^*)$ is the image of the antihomomorphism $\mathbf{E}(X) \to \mathrm{Aut}_A H^*$ induced by $\mathbf{h}$. Let $\Delta'(H^*)$ be the image of the map $M(X) \to \Delta(H^*)$ induced by $\mathbf{h}$. Clearly $\Delta'(H^*)$ is invariant under $A'(H^*)$, and thus $\mathbf{h}$ induces a map

$$M(X)//\mathbf{E}(X) \to \Delta'(H^*)//A'(H^*).$$

This observation also explains why one has to consider structures isomorphic not only relative to the full group $E(X)$ of equivalences, but also relative to a subgroup E'.

Now the method of our proofs can be described roughly as follows. Enough is known on the way the functor $\mathbf{h}$ acts on maps to conclude that $M(X)//\mathbf{E}(X)$ is finite if and only if the same is true for $\Delta'(H^*)//A'(H^*)$. Then one uses the same kind of information on $\mathbf{h}$ to relate $\Delta'(H^*)//A'(H^*)$ to $\Delta(H^*)//\mathrm{Aut}_A H^*$ via the Lemmas 2.1–2.3. Thus the problem has become a purely algebraic one. If m is associative, then the diagonal $\mathbf{h}(m)$ is associative, and the stepping stone for the proof of Theorem I is Samelson's theorem on the Hopf algebra structures on a certain class of algebras. To prove Theorem II we construct sufficiently many non-associative diagonals and observe that they all are elements of $\Delta'(H^*)$, i.e. induced by multiplications of X. The proof of Proposition Ia proceeds similarly with associative diagonals.

3. Automorphisms of a symmetric algebra

We consider a finite graded set

$$X = \{x_{11}, \ldots x_{1p_1}, x_{21}, \ldots x_{2p_2}, \ldots x_{l1}, \ldots x_{lp_l}\}$$

with $\deg x_{ik} = i$, and we write $X_n = \{x_{n1}, \ldots x_{np_n}\}$ for its component of degree n. Let $B = S(X)$ be the symmetric algebra of X over the integers $\mathbf{Z}$. Let $Y \subset B$ be the collection of the monomials in at least two elements of X. For each $n \leqslant l$ we order Y_n, the component of Y of degree n, in an arbitrary but fixed way. Then $X_n \cup Y_n = \{x_{n1}, \ldots x_{np_n}, y_{n1}, \ldots y_{nd_n}\}$ is a basis for the free abelian group B_n. In terms of this basis any algebra map $\varphi: B \to B$ is represented in degree $n \leqslant l$ by an integral $((p_n + d_n) \times (p_n + d_n))$-matrix φ_n of the form

$$\varphi_n = \begin{pmatrix} \Phi'_n & \Phi_n \\ 0 & \Phi''_n \end{pmatrix}$$

where Φ'_n is a $(p_n \times p_n)$-matrix corresponding to $x_{n1}, \ldots x_{np_n}$, and Φ''_n corresponds to $y_{n1}, \ldots y_{nd_n}$. Since B is the free commutative algebra on X the matrices Φ'_n and Φ_n are not subject to any restrictions, whereas Φ''_n is uniquely determined by Φ'_i and Φ_i for $i < n$. Thus any algebra map $\varphi: B \to B$ can be considered as a sequence $\varphi = (\varphi_1, \ldots \varphi_l)$ of such matrices φ_n.

If Φ'_n is nonsingular for all $n \leqslant l$ then $X \subset \mathrm{Im}\,\varphi$, i.e. φ is surjective. Since any surjective endomorphism of a finitely generated abelian group is an automorphism we see that det $\Phi'_n = \pm 1$ for all $n \leqslant l$ implies that φ is an automorphism. Conversely if φ is an automorphism then det $\Phi'_n = \pm 1$ because φ induces an automorphism of the free abelian group generated by X_n. Thus φ is an automorphism of B if and only if Φ'_n is nonsingular for all $n \leqslant l$. We denote by $\mathrm{Aut}_A B$ the group of algebra automorphisms of B.

Let $\varphi = (\Phi_1, \dots \Phi_l)$ be an element of the graded group $\mathrm{GL}(p_*, \mathbf{Z}) = \{\mathrm{GL}(p_1, \mathbf{Z}), \dots \mathrm{GL}(p_l, \mathbf{Z})\}$. We define a map $j: \mathrm{GL}(p_*, \mathbf{Z}) \to \mathrm{Aut}_A B$ by

$$j(\varphi)_n = \begin{pmatrix} \Phi_n & 0 \\ 0 & \Phi''_n \end{pmatrix}$$

where Φ''_n is determined by Φ_i for $i < n$. Clearly j is a homomorphism. Denote by I the identity matrix.

LEMMA 3.1. *The set* $\mathbf{A}$ *of all algebra automorphisms* $\varphi = (\varphi_1, \dots \varphi_l)$ *such that*

$$\varphi_n = \begin{pmatrix} I & \Phi_n \\ 0 & \Phi''_n \end{pmatrix}$$

for all n is a normal subgroup of $\mathrm{Aut}_A B$, *and* $\mathrm{Aut}_A B$ *is the semidirect product of* $\mathbf{A}$ *by* $\mathrm{GL}(p_*, \mathbf{Z})$. *In particular* $\mathbf{A}$ *is a system of representatives of the left cosets of* $\mathrm{Aut}_A B$ *modulo* $j(\mathrm{GL}(p_*, \mathbf{Z}))$.

Proof. Define $\varrho: \mathrm{Aut}_A B \to \mathrm{GL}(p_*, \mathbf{Z})$ as follows. If $\varphi = (\varphi_1, \dots \varphi_l) \in \mathrm{Aut}_A B$,

$$\varphi_n = \begin{pmatrix} \Phi'_n & \Phi_n \\ 0 & \Phi''_n \end{pmatrix},$$

then $\varrho(\varphi) = (\varrho(\varphi)_1, \dots \varrho(\varphi)_l)$, where

$$\varrho(\varphi)_n = \begin{pmatrix} \Phi'_n & 0 \\ 0 & \Phi'''_n \end{pmatrix}.$$

Then the sequence

$$1 \to \mathbf{A} \to \mathrm{Aut}_A B \to \mathrm{GL}(p_*, \mathbf{Z}) \to 1$$

is exact. Let $j: \mathrm{GL}(p_*, \mathbf{Z}) \to \mathrm{Aut}_A B$ be the map defined above. Clearly ϱj is the identity of $\mathrm{GL}(p_*, \mathbf{Z})$. Therefore $\mathrm{Aut}_A B$ is the semidirect product of $\mathbf{A}$ by $\mathrm{GL}(p_*, \mathbf{Z})$. Since every element φ of $\mathrm{Aut}_A B$ has a unique representation $\varphi = \alpha \cdot j(\gamma)$ with $\alpha \in \mathbf{A}$ and $\gamma \in \mathrm{GL}(p_*, \mathbf{Z})$ the group $\mathbf{A}$ is a system of representatives of the left cosets modulo $j(\mathrm{GL}(p_*, \mathbf{Z}))$.

LEMMA 3.2. *Let N be a positive integer. Then the set* $\mathbf{A}(N) = \{\varphi \in \mathbf{A}, \Phi_n \equiv 0(N)\}$ *is a normal subgroup of finite index in* $\mathbf{A}$.

Proof. We first show that $\varphi = (\varphi_1, \dots \varphi_l) \in \mathbf{A}$ is an element of $\mathbf{A}(N)$ if and only if

$\varphi_n \equiv I(N)$ for all n. Let $\varphi \in \mathbf{A}(N)$ and consider a monomial $x_1 x_2 \dots x_r$ where $x_1, \dots x_r$ are some elements of X. Then

$$\varphi(x_1 x_2 \dots x_r) = \varphi(x_1)\varphi(x_2)\dots\varphi(x_r)$$

because φ is an algebra map. On the other hand $\Phi_n \equiv 0(N)$ for all n implies

$$\varphi(x_i) = x_i + N y_i$$

where y_i is a linear combination of monomials. Therefore

$$\begin{aligned}\varphi(x_1 x_2 \dots x_r) &= (x_1 + N y_1)(x_2 + N y_2)\dots(x_r + N y_r)\\ &= x_1 x_2 \dots x_r + N z\end{aligned}$$

for some decomposable z. Thus $\varphi \in \mathbf{A}(N)$ means for

$$\varphi_n = \begin{pmatrix} I & \Phi_n \\ 0 & \Phi''_n \end{pmatrix}$$

that $\Phi''_n \equiv I(N)$, i.e. $\varphi \in \mathbf{A}(N)$ implies $\varphi_n \equiv I(N)$ for all n. Conversely, $\varphi_n \equiv I(N)$ implies in particular $\Phi_n \equiv 0(N)$. This shows that $\varphi \in \mathbf{A}(N)$ if and only if $\varphi_n \equiv I(N)$.

Now let $\Gamma(s, N)$ be the kernel of the homomorphism $\mathrm{GL}(s, \mathbf{Z}) \to \mathrm{GL}(s, \mathbf{Z}_N)$, and write $\mathbf{G}(p_* + d_*, K)$ for the graded object $\{\mathbf{G}(p_1 + d_1, K), \dots \mathbf{G}(p_l + d_l, K)\}$ where $\mathbf{G}$ is either Γ or GL and K is one of the symbols N, $\mathbf{Z}$, $\mathbf{Z}_N$. An element γ of $\Gamma(p_* + d_*, N)$ is a sequence $\gamma = (\gamma_1, \dots \gamma_l)$ of nonsingular integral matrices such that $\gamma_n \equiv I(N)$. It follows from the characterization of $\mathbf{A}(N)$ in the preceding paragraph that

$$\mathbf{A}(N) = \mathbf{A} \cap \Gamma(p_* + d_*, N).$$

Therefore $\mathbf{A}(N)$ is a normal subgroup of $\mathbf{A}$ because $\Gamma(p_* + d_*, N)$ is normal in $\mathrm{GL}(p_* + d_*, \mathbf{Z})$. Furthermore the map $\mathbf{A}/\mathbf{A}(N) \to \mathrm{GL}(p_* + d_*, \mathbf{Z}_N)$ induced by the inclusion $\mathbf{A} \to \mathrm{GL}(p_* + d_*, \mathbf{Z})$ is a monomorphism. Since $\mathrm{GL}(p_* + d_*, \mathbf{Z}_N)$ is a finite group the same is true for $\mathbf{A}/\mathbf{A}(N)$, i.e. $\mathbf{A}(N)$ is of finite index in $\mathbf{A}$.

4. Hopf algebra structures on a symmetric algebra

Let $B = S(X)$ as above and denote by $\eta: \mathbf{Z} \to B$ the unit of B. Since B is connected there exists a unique augmentation $\varepsilon: B \to \mathbf{Z}$ of B. By a *diagonal of B* we mean a map $\delta: B \to B \otimes B$ such that (B, δ) is a Hopf algebra with unit η and counit ε. It is not assumed that a diagonal is associative. Clearly δ is a diagonal of B only if it is an algebra map $B \to B \otimes B$. Any such δ is uniquely determined by its values on the set X. The diagonal δ_0 defined by

$$\delta_0(x) = x \otimes 1 + 1 \otimes x$$

for $x \in X$ is associative and commutative. The space of primitive elements of the

Hopf algebra (B, δ_0) is the graded free abelian group $F(X)$ generated by X (this follows, for instance, from Prop. 3.7 of [3, 2–08] and the fact that the projection of $F(X)$ onto the indecomposables of B is an isomorphism). Since (B, δ_0) is primitively generated an algebra map $\varphi: B \to B$ is a map of Hopf algebras $(B, \delta_0) \to (B, \delta_0)$ if and only if φ maps X into $F(X)$. In the setup of § 3, a Hopf algebra map $\varphi: (B, \delta_0) \to (B, \delta_0)$ is therefore represented as a sequence $\varphi = (\varphi_1, \dots \varphi_l)$ of integral matrices of the form

$$\varphi_n = \begin{pmatrix} \Phi_n' & 0 \\ 0 & \Phi_n'' \end{pmatrix}.$$

Let $\mathrm{Aut}_{HA}(B, \delta_0)$ be the group of Hopf algebra automorphisms of (B, δ_0). Then

$$\mathrm{Aut}_{HA}(B, \delta_0) = j(\mathrm{GL}(p_*, \mathbf{Z}))$$

where j is the monomorphism defined in connection with Lemma 3.1.

Now let $\Delta(B)$ be the collection of all *associative* diagonals of B. Clearly $\Delta(B)$ is a collection of structures on B in the sense of § 2, and the set $\Delta(B)//\mathrm{Aut}_A B$ of orbits (of the operation (*) of $\mathrm{Aut}_A B$ on $\Delta(B)$) represents the isomorphism classes of Hopf algebra structures with associative diagonals on the algebra B. A relation between the standard diagonal δ_0 and the other associative diagonals is established in the

THEOREM OF SAMELSON-LERAY. *Let X be odd (i.e., $X_n = \emptyset$ for n even). Then $\Delta(B) = \mathrm{Aut}_A B * \delta_0$. In other words: $\Delta(B)//\mathrm{Aut}_A B$ consists of one element.*

This theorem is usually stated for algebras over a field. For algebras over $\mathbf{Z}$ it is discussed in [3, Exp. 2].

5. Maps into group-like spaces

In this section we collect the facts needed to reduce the discussion of H-space structures on a space to the discussion of Hopf algebra structures on the cohomology algebra of the space. From now on G will always stand for a connected finite CW-complex equipped with a fixed grouplike structure m_0.

The subspace of primitive elements of the Hopf algebra $H^*((G, m_0); \mathbf{Q})$ is denoted by $P^*(G)$. For any space A we consider $[A, G]$ as an additively written group with the group structure induced by m_0. The functor $H^*(\ ; \mathbf{Q})$ induces a homomorphism

$$h: [A, G] \to \mathrm{Hom}_{\mathbf{Q}}(P^*(G), H^*(A; \mathbf{Q})).$$

LEMMA 5.1. *Let A be a finite connected CW-complex. Then the collection T of all elements of finite order in $[A, G]$ is a finite subgroup of $[A, G]$, and T is the kernel of h. Furthermore the cokernel of h is periodic.*

This Lemma follows from known properties of $[A, G]$ and $H^*((G, m_0); \mathbf{Q})$. For a detailed discussion see e.g. [2].

Let (G', m_0') be another group-like space like G. We consider an ordered basis X (resp. X') of $P^*(G)$ (resp. $P^*(G')$) and the corresponding additive basis $X \cup Y$ of $H^*(G; \mathbf{Q})$ (resp. $X' \cup Y'$ of $H^*(G'; \mathbf{Q})$) as in § 3. A map $f: G' \to G$ is called *integral with respect to* (X, X') if $f^*: H^*(G; \mathbf{Q}) \to H^*(G'; \mathbf{Q})$ is represented in terms of the bases $X \cup Y$ and $X' \cup Y'$ by matrices with integral coefficients. Under our assumptions on G and G' the group $[G', G]$ is finitely generated (see e.g. [2]). Let $f_1 \dots f_R$ be a system of generators of $[G', G]$. Obviously every element of $[G', G]$ is integral with respect to (X, X') if and only if this holds for $f_1, \dots f_R$.

LEMMA 5.2. *There exists a basis* $X = \{x_{ik}\}$ *of* $P^*(G)$ *such that*

(i) *every map* $G \to G$ *is integral with respect to* (X, X);

(ii) *every map* $G \times G \to G$ *is integral with respect to* (X, X_0) *where* X_0 *is the basis* $\{x_{ik} \otimes 1, 1 \otimes x_{ik}\}$ *of* $P^*(G \times G)$.

Proof. In a) we show that a basis satisfying (i) also satisfies (ii). For the proof of the remaining assertion in c) we condiser in b) the integral cohomology groups modulo their torsion.

a) Let X' be a basis of $P^*(G)$ satisfying (i). Consider a map $f: G \times G \to G$ and an element $x \in X_n'$. Then

$$f^*(x) = 1 \otimes a_1 + a_2 \otimes 1 + b$$

where $1 \otimes a_1 \in H^0(G; \mathbf{Q}) \otimes H^n(G; \mathbf{Q})$ and $a_2 \otimes 1 \in H^n(G; \mathbf{Q}) \otimes H^0(G; \mathbf{Q})$. Denote by i_1, i_2 the two inclusions $G \to G \times G$. Then $(i_1 f)^*(x) = a_1$ and $(i_2 f)^*(x) = a_2$. By hypothesis a_1 and a_2 are integral linear combinations of elements of $X' \cup Y'$. Hence $1 \otimes a_1$ and $a_2 \otimes 1$ are integral linear combinations of elements of $X_0' \cup Y_0'$. Now we change X_n' inductively to make the crossterm b integral for the system $f_1, \dots f_R$ of generators of $[G \times G, G]$.

Define $X_1 = X_1'$. Then $f^*(x)$ is integral for every f and every $x \in X_1$. Therefore we can assume for induction that we already have constructed $X_1, \dots, X_{n-1}$ such that $f_j^*(x)$ is integral for every f_j and every $x \in X_i$, $i \leqslant n-1$. Let $x_{nk}' \in X_n'$ and write

$$f_j^*(x_{n\,k}') = a(j, k) + b(j, k)$$

where $a(j, k)$ corresponds to $1 \otimes a_1 + a_2 \otimes 1$ above. There exists an integer $N(j, k)$ such that $N(j, k)\, b(j, k)$ is an integral linear combination (of elements of the form $y \otimes y'$ with $\dim y < n$, $\dim y' < n$). Let N be the product of all $N(j, k)$, and define $X_n = (N x_{n1}', \dots N x_{n p_n}')$ as basis for $P^n(G)$. Then $f_j^*(x)$ is integral for all f_j and all $x \in X_i$, $i \leqslant n$.

b) For any space A define $\mathbf{H}^q(A)$ to be the factor group of $H^q(A)$ modulo its torsion subgroup. Let $\pi: H^q(A) \to \mathbf{H}^q(A)$ be the projection and $\varrho: H^q(A) \to H^q(A; \mathbf{Q})$ the natural map. Clearly ϱ induces a monomorphism

$$\tau: \mathbf{H}^q(A) \to H^q(A; \mathbf{Q}).$$

Note that the cokernel of τ is periodic because the same holds for ϱ. For $A=G$ we consider $\mathbf{H}^q(G)$ as a subgroup of $H^q(G; \mathbf{Q})$ via the map τ and define

$$\mathbf{P}^q(G) = P^q(G) \cap \mathbf{H}^q(G).$$

In (1)–(3) below we list some properties needed for c).

(1) *$\mathbf{P}^q(G)$ is a direct summand of $P^q(G)$.* For there exists a basis $x_1, \ldots x_p, x_{p+1}, \ldots x_N$ of the free abelian group $\mathbf{H}^q(G)$ such that $\mathbf{P}^q(G) = \langle \alpha_1 x_1, \ldots \alpha_p x_p \rangle$ for some integers $\alpha_i \geqslant 1$. Clearly $\tau(x_i) \in P^q(G)$. Hence $x_i \in \mathbf{P}^q(G)$ for all i. Therefore x_i is a linear combination of $\alpha_1 x_1, \ldots \alpha_p x_p$. Since the x_i are linearly independent we conclude that $\alpha_i = \pm 1$ for all i. Therefore $\mathbf{P}^q(G) = \langle x_1, \ldots x_p \rangle$.

(2) *If $\{x_i\}$ is a basis of $\mathbf{P}^q(G)$, then $\{\tau(x_i)\}$ is a basis of $P^q(G)$.* The linear independence of the $\tau(x_i)$ follows from that of the x_i. The $\tau(x_i)$ are seen to span $P^q(G)$ because the cokernel of τ is periodic.

We extend a basis $\{x_i\}$ of $\mathbf{P}^q(G)$ to a basis X of $\mathbf{H}^q(G)$. Then $\tau(X)$ is a basis of $H^q(G; \mathbf{Q})$. Let i be the inclusions of $\mathbf{P}^q(G)$ into $\mathbf{H}^q(G)$ and of $P^q(G)$ into $H^q(G; \mathbf{Q})$ as direct summands in terms of the bases X and $\tau(X)$. Similarly let k be the projections $\mathbf{H}^q(G) \to \mathbf{P}^q(G)$ and $H^q(G; \mathbf{Q}) \to P^q(G)$. Finally we observe that any $f: G \to G$ induces an endomorphism f^0 of $\mathbf{H}^q(G)$. The following statement (3) is an immediate consequence of the definition of the maps involved:

(3) *The diagram*

$$\begin{array}{ccccccc}
\mathbf{P}^q(G) & \xrightarrow{i} & \mathbf{H}^q(G) & \xrightarrow{f^0} & \mathbf{H}^q(G) & \xrightarrow{k} & \mathbf{P}^q(G) \\
\downarrow \tau & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
P^q(G) & \xrightarrow{i} & H^q(G; \mathbf{Q}) & \xrightarrow{f^*} & H^q(G; \mathbf{Q}) & \xrightarrow{k} & P^q(G)
\end{array}$$

is commutative.

c) Let $f_1 \ldots f_R$ be a system of generators of $[G, G]$. Pick a basis X_1 of $\mathbf{P}^1(G) \subset P^1(G)$. Since $\mathbf{P}^1(G) = \mathbf{H}^1(G)$ and $P^1(G) = H^1(G; \mathbf{Q})$ we conclude from the diagram (3) that $f_j^*(x)$ is integral for all f_j and all $x \in \tau(X_1)$. Therefore we can assume for induction the existence of a basis X_i for $P^i(G)$, $i \leqslant n-1$, such that $f_j^*(x)$ is integral for all f_j and all $x \in X_i$, $i \leqslant n-1$. Let $x_1, \ldots x_{p_n}$ be an arbitrary basis of $\mathbf{P}^n(G)$ and $X'_n = \{\tau(x_i)\}$ the corresponding basis of $P^n(G)$. In terms of $X_1, \ldots X_{n-1}, X'_n$ the map f_j^* is represented in dimension n by a matrix

$$\begin{pmatrix} {}^j\Phi'_n & {}^j\Phi_n \\ 0 & {}^j\Phi''_n \end{pmatrix}$$

For dimension reasons ${}^j\Phi''_n$ is integral. It follows from (3) in b) that ${}^j\Phi'_n$ is integral. Let N be a positive integer such that $N\,{}^j\Phi_n$ is integral for all j. Then $X_n = \{N\tau(x_i)\}$ is a basis of $P^n(G)$ such that $f_j^*(x)$ is integral for all f_j and all $x \in X_n$. The proof of Lemma 5.2 is complete.

We combine Lemma 5.1 and 5.2 in

COROLLARY 5.3. *There exists a basis X of $P^*(G)$ such that the homomorphism h of Lemma* 5.1 *gives rise to homomorphisms*

$$h_1:[G, G] \to \mathrm{Hom}(F(X), S(X))$$

and

$$h_2:[G\times G, G] \to \mathrm{Hom}(F(X), S(X)\otimes S(X)).$$

The kernel and the cokernel of h_i are finite, $i=1, 2$.

LEMMA 5.4. *The restriction of the map h_1 of Corollary* 5.3 *to $\mathbf{E}(G)$ defines an antihommomorphism*

$$h':\mathbf{E}(G)\to \mathrm{Aut}_A S(X)$$

with the following property: There exists an integer $N>0$ such that the image of h' contains the group $\mathbf{A}(N)$ of Lemma 3.2.

Proof. a) We show first that for any given $x_{ik}\in X$ and any decomposable y_{ij} there exists a map $f=f(i;k,j):G\to G$ such that

(i) $f^*(x_{ik})=\lambda y_{ij}$ for some integer $\lambda=\lambda(i;k,j)>0$;

(ii) $f^*(x_{rs})=0$ for $(r,s)\neq(i,k)$;

(iii) $f_\#=0:\pi_k(G)\to\pi_k(G)$ for $k\leqslant \dim G$.

It follows from Corollary 5.3 that there exists a map $f':G\to G$ and an integer $\lambda'>0$ satisfying (i) and (ii). Let $Q^*(G)$ be the space of indecomposables of $H^*(G;\mathbf{Q})$. Conditions (i) and (ii) express that the map $Q^*(G)\to Q^*(G)$ induced by f' is trivial. Hence $f'_*:H_*(G;\mathbf{Q})\to H_*(G;\mathbf{Q})$ restricted to the homology primitives is also trivial. Now it follows from a theorem of Cartan-Serre (see e.g. [7, p. 263]) that $f_\#\otimes 1=0$: $\pi_*(G)\otimes\mathbf{Q}\to\pi_*(G)\otimes\mathbf{Q}$. Therefore there exists an integer $\lambda''>0$ such that $(\lambda'' f)_\#=0:\pi_k(G)\to\pi_k(G)$ for $k\leqslant\dim G$. The map $f=\lambda'' f'$ and the integer $\lambda=\lambda'\lambda''$ satisfy (i), (ii) and (iii). Note that $h_1(f(i;k,j))$ is an elementary endomorphism of $B=S(X)$ in the sense that the matrices φ_n representing $h_1(f(i;k,j))$ are of the form

$$\varphi_n=0 \quad \text{for} \quad n\neq i$$
$$\varphi_i=\begin{pmatrix}0 & \Phi\\ 0 & 0\end{pmatrix}$$

where $\Phi_{kj}=\lambda(i;k,j)$ and $\Phi_{rs}=0$ otherwise.

b) Denote by N the product of the finitely many integers $\lambda(i;k,j)$ for all i,k,j. Let $\varphi\in\mathbf{A}(N)$ be given, and let $\varphi'=(\varphi'_1,\dots\varphi'_l)$ be the endomorphism of B obtained from φ by removing the identity matrices in the upper left corners:

$$\varphi'_n=\begin{pmatrix}0 & \Phi_n\\ 0 & \Phi'''_n\end{pmatrix},\qquad \varphi_n=\begin{pmatrix}I & \Phi_n\\ 0 & \Phi''_n\end{pmatrix}.$$

This φ' is a sum of elementary endomorphisms B in the sense of a). Therefore one can use repeatedly the result of a) to construct a map $f:G\to G$ such that $f^*=\varphi'$ and

$f_{\#}=0:\pi_k(G)\to\pi_k(G)$ for $k\leqslant\dim G$. The map $1+f\in[G, G]$ is (the homotopy class of) a homotopy equivalence of G because $(1+f)_{\#}=1_{\#}+f_{\#}=1_{\#}:\pi_k(G)\to\pi_k(G)$ for $k\leqslant\dim G$. Furthermore $(1+f)^*=\varphi$. Thus the given φ has been realized by a homotopy equivalence of G. The proof of Lemma 5.4 is complete.

6. Proof of Theorem I

Sections a) and b) contain the reduction of the problem to an algebraic problem as indicated at the end of § 2. In c) the proof of the first part of Theorem I is completed. The remaining assertions are proved in d) and e).

The cardinality of a set S is denoted by $|S|$. We use freely the notation of the preceding sections.

a) We let $M_0(G)$ stand for the collection of all associative multiplications of G. The set $M_0(G)$ is a subset of $[G\times G, G]$, and is obviously a collection of structures on G in the sense of § 2. In the notation of § 2 the first part of Theorem I then asserts

$$|M_0(G)//\mathbf{E}(G)|<\infty .$$

Let $M(G)\subset[G\times G, G]$ be the set of all multiplications of G which induce an associative diagonal in the algebra $H^*(G; \mathbf{Q})$. The set $M(G)$ is also a collection of structures on G because the same holds for the collection of associative diagonals of $H^*(G; \mathbf{Q})$. Furthermore $M_0(G)$ is invariant under $\mathbf{E}(G)$ in $M(G)$. Since $|M(G)//\mathbf{E}(G)|<\infty$ implies $|M_0(G)//\mathbf{E}(G)|<\infty$ by Lemma 2.2 we see that it suffices to show that $M(G)//\mathbf{E}(G)$ is finite.

b) Let X be a basis of $P^*(G)$ with the properties described in Corollary 5.3. Let

$$h':\mathbf{E}(G)\to \mathrm{Aut}_A B$$

be the antihomomorphism of Lemma 5.4, and

$$h'':M(G)\to \Delta(B)$$

the function obtained from restricting h_2 to $M(G)\subset[G\times G, G]$. Clearly $\mathrm{Im}\,h''\subset\Delta(B)$ is invariant under $\mathrm{Im}\,h'$. Now we show that $|\mathrm{Im}\,h''//\mathrm{Im}\,h'|<\infty$ implies $|M(G)//\mathbf{E}(G)|<\infty$.

Let $\mathrm{Im}\,h''//\mathrm{Im}\,h'$ consist of the orbits $[\delta_1],\dots [\delta_r]$ of r elements $\delta_1,\dots \delta_r$ of $\mathrm{Im}\,h''$. Denote by M'' the set of all $m\in M(G)$ with $h''(m)=\delta_j$ for some j. By Corollary 5.3 the kernel of h_2 is finite. Therefore M'' is a finite set $M''=\{m_1,\dots m_s\}$. Now let m be a given element of $M(G)$. Then for some j we have

$$[h''(m)]=[\delta_j]$$

in $\mathrm{Im}\,h''//\mathrm{Im}\,h'$, i.e.

$$\begin{aligned}\delta_j &= h'(\theta^{-1})*h''(m)\\ &= h''(\theta*m)\end{aligned}$$

for some $\theta \in \mathbf{E}(G)$. This means that $\theta * m \in M''$, i.e. $\theta * m = m_k$ for some $k = 1, \dots s$. Thus any given m is isomorphic rel $\mathbf{E}(G)$ to one of the finitely many elements of M''. Therefore it suffices to prove that $\operatorname{Im} h'' // \operatorname{Im} h'$ is finite.

c) By Lemma 3.1 the subgroup $\mathbf{A}$ of $\operatorname{Aut}_A B$ is a system of representatives of the left cosets of $\operatorname{Aut}_A B$ modulo $j(\mathrm{GL}(p_*, \mathbf{Z}))$. As noticed in § 4 the latter group is the group $\operatorname{Aut}_{HA}(B, \delta_0)$ of Hopf algebra automorphisms of (B, δ_0). Therefore by Lemma 2.3

$$\operatorname{Aut}_A B * \delta_0 = \mathbf{A} * \delta_0 .$$

On the other hand the Samelson-Leray theorem shows that $\operatorname{Aut}_A B * \delta_0 = \Delta(B)$. Therefore we obtain $\Delta(B) = \mathbf{A} * \delta_0$, i.e.

$$|\Delta(B) // \mathbf{A}| = 1 < \infty .$$

By Lemma 5.4 there exists an integer $N > 0$ such that $\mathbf{A}(N) \subset \operatorname{Im} h'$. By Lemma 3.2 the index of $\mathbf{A}(N)$ in $\mathbf{A}$ is finite. Hence

$$|\Delta(B) // \mathbf{A}(N)| < \infty$$

by Lemma 2.1 (ii). Since $\mathbf{A}(N) \subset \operatorname{Im} h'$ we infer

$$|\Delta(B) // \operatorname{Im} h'| < \infty$$

from Lemma 2.1 (i). Since $\operatorname{Im} h'' \subset \Delta(B)$ is invariant under $\operatorname{Im} h'$ we use Lemma 2.2 to conclude

$$|\operatorname{Im} h'' // \operatorname{Im} h'| < \infty .$$

The proof of the first part of Theorem I is complete.

d) To prove the statement on the finite number of associative multiplications (Theorem I (continued)) we assume $\beta_i(G) = \gamma_i(G)$ for $i = n_1, \dots n_r$, where $n_1, \dots n_r$ are the integers k for which $\gamma_k(G) \neq 0$. If m is a multiplication of G we denote by $P^*(m)$ the primitives of the Hopf algebra $H^*((G, m); \mathbf{Q})$. In particular $P^*(G)$ will be written as $P^*(m_0)$.

Let m_1 any group like structure on G. By Samelson's theorem $H^* = H^*((G, m_1); \mathbf{Q})$ is the exterior algebra on $P^*(m_1)$, and $P^*(m_1)$ is isomorphic to the indecomposables of the algebra $H^*(G; \mathbf{Q})$ because both the multiplication and the diagonal of H^* are commutative. Passing to homology and arguing as in a) of the proof of Lemma 5.4 it is seen that $\dim P^i(m_1) = \gamma_i(G)$. We combine this with the hypothesis $\beta_i(G) = \gamma_i(G)$ for $i = n_1, \dots n_r$, and conclude

$$P^*(m_0) = P^*(m_1)$$

for any group like structure m_1. The element $m_1 \in [G \times G, G]$ is of the form $m_1 = m_0 + qm$ for some $m \in [G \# G, G]$, where $q: G \times G \to G \# G$ is the projection (see [1]). If m is of infinite order in $[G \# G, G]$ then $m^*(x) \neq 0$ for some $x \in P^*(m_0)$ by Lemma 5.1. But

$m^*(x)\neq 0$ means exactly $x\notin P^*(m_1)$ because q induces a monomorphism in cohomology. Thus $x\in P^*(m_0)$ and $x\notin P^*(m_1)$, i.e.

$$P^*(m_0) \neq P^*(m_1)$$

if m is of infinite order. Therefore m must be of finite order. By Lemma 5.1 there are only finitely many elements of finite order in $[G\# G, G]$. Hence the number of group like structures $m_1 = m_0 + q\,m$ is finite.

e) Now we prove the last statement of Theorem I (continued) and assume $\beta_j(G) > \gamma_j(G) \neq 0$ for some j. Let X etc be as in b), and consider an arbitrary integral matrix $\Phi\neq 0$ with $\gamma_j(G)$ columns and $\beta_j(G)-\gamma_j(G)$ rows. By Lemma 5.4 there exists an integer $N>0$ and elements $\theta(k)\in \mathbf{E}(G)$, k any integer, such that $\theta(k)^*$ is represented by a sequence $\{\theta(k)_1, \dots \theta(k)_l\}$ of matrices of the following form:

$$\begin{aligned}\theta(k)_n &= \begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}, \quad n<j\\ &= \begin{pmatrix} I & (kN)\Phi\\ 0 & I\end{pmatrix}, \quad n=j\\ &= \begin{pmatrix} I & 0\\ 0 & \Phi''_n\end{pmatrix}, \quad n>j.\end{aligned}$$

Now let m be any group like structure on G. Consider the infinite sequence

$$\theta(1)*m, \dots \theta(k)*m, \dots \tag{*}$$

of group like structures on G. We assert that $\theta(k)*m=\theta(k')*m$ implies $k=k'$. For let m induce $m^*\in\Delta(B)$. Then

$$m^* = \varphi * \delta_0$$

for some $\varphi\in\mathbf{A}$, as observed in c). Thus $\theta(k)*m=\theta(k')*m$ implies

$$(\theta^{-1}(k)^*\,\varphi)*\delta_0 = (\theta^{-1}(k')^*\,\varphi)*\delta_0 .$$

By definition $\theta(k)^*\in\mathbf{A}$. Hence $\theta^{-1}(k)^*\,\varphi\in\mathbf{A}$ for all k. Since $\mathbf{A}$ is a system of representatives of $\mathrm{Aut}_A B$ modulo $\mathrm{Aut}_{HA}(B, \delta_0)$ we conclude $\theta^{-1}(k)^*\,\varphi=\theta^{-1}(k')^*\,\varphi$, i.e.

$$\theta(k)^* = \theta(k')^* .$$

This is only possible if $k=k'$. Thus $k\neq k'$ implies $\theta(k)*m\neq\theta(k')*m$, i.e. the multiplications (*) are all different from each other. The proof of the entire Theorem I is now complete.

7. Proof of Theorem II

Let X be a basis of $P^*(G)$ as in Corollary 5.3 and consider the homomorphism

$$h_2 : [G\times G, G] \to \mathrm{Hom}\,(F(X), B\otimes B)$$

with $B=S(X)$. We will construct an infinite sequence $m_1, m_2, \ldots$ of multiplications of G such that

$$h_2(m_i) \cong h_2(m_j) \quad \operatorname{rel} \operatorname{Aut}_A B \tag{1}$$

if and only if $i=j$;

$$h(m_i) \quad \text{is non-associative for all} \quad i. \tag{2}$$

From this we obtain the proof of Theorem II as follows. Let $h': \mathbf{E}(G) \to \operatorname{Aut}_A B$ be as in § 6. Then $h(m_i) \not\cong h(m_j)$ rel $\operatorname{Aut}_A B$ obviously implies $h(m_i) \not\cong h(m_j)$ rel $\operatorname{Im} h'$, and this in turn implies $m_i \not\cong m_j \operatorname{rel} \mathbf{E}(G)$. The second part of Theorem II follows from (2) and the particular way the m_i are constructed.

By our hypothesis on G there exists a dimension n such that

$$\begin{gathered} X_n \neq \emptyset \\ (\bar{B} \otimes \bar{B})_n \neq 0 . \end{gathered} \tag{*}$$

Let $x = x_{n1}$ and pick monomials $a, b \in \bar{B}$ such that $a \otimes b \neq 0 \in (\bar{B} \otimes \bar{B})_n$. By Lemma 5.1 there exists an $m: G \# G \to G$ and an integer $N > 0$ such that

$$\begin{gathered} m^*(x) = N(a \otimes b) \\ m^*(x_{ik}) = 0 \quad \text{for} \quad x_{ik} \neq x . \end{gathered}$$

For any integer r we define a multiplication m'_r by

$$m'_r = m_0 + q(m + m + \cdots + m)\,(r \text{ times})$$

(recall that $[G \times G, G]$ is written additively). Write δ_r for $h_2(m'_r)$. Let us assume that $\delta_r \cong \delta_s$ rel $\operatorname{Aut}_A B$, i.e. $(\theta \otimes \theta)\, \delta_r = \delta_s \theta$ for some $\theta \in \operatorname{Aut}_A B$:

$$\begin{array}{ccc} B & \xrightarrow{\delta_r} & B \otimes B \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \theta \otimes \theta} \\ B & \xrightarrow{\delta_s} & B \otimes B . \end{array}$$

Let $P^*(B, \delta_r)$ be the primitives of the Hopf algebra (B, δ_r). Note that

$$\begin{gathered} \delta_r = \delta_0 \quad \text{on} \quad X_k \\ P^k(B, \delta_r) = X_k \end{gathered}$$

for $k < n$ and all r. As observed in § 2 the automorphism θ is determined only up to an element of $\operatorname{Aut}_{HA}(B, \delta_r)$. Therefore we can assume

$$\theta(x_{kj}) = x_{kj} \quad \text{for} \quad k < n .$$

In dimension n we have $\theta(x) = \lambda x + w$ for some integer λ. Using the latter two relations one obtains from $(\theta \otimes \theta)\, \delta_r(x) = \delta_s \theta(x)$ by a direct computation $(Nr - \lambda N s)\, a \otimes b = 0$, i.e.

$$r = \lambda s$$

for some integer λ. Therefore $(r, s)=1$ implies $\delta_r \not\simeq \delta_s$ rel $\mathrm{Aut}_A B$. Let $p_1, \ldots$ be the sequence of primes and define

$$m_i = m_0 + q(p_i m).$$

This sequence of multiplications satisfies (1). The non-associativity of $h(m_i)$ is an immediate consequence of the fact that the monomials a, b used in the definition of m are of different length because their dimension is necessarily odd.

To prove the statement on the non-isomorphic algebra structures we pick a dimension $n' \neq n$ such that (*) holds for n'. Obviously the Hopf algebras corresponding to n are not isomorphic over **Q** to those corresponding to n' because their diagonals deviate from associativity in different dimensions. Therefore the corresponding Pontryagin algebras are nonisomorphic over **Q**, either.

8. Proof of Proposition Ia

We write $K = S^1 \times K(\mathbf{Z}, 2)$. The discussion of maps $K \times K \to K$ is particularly simple because $[K \times K, K]$ is isomorphic to $H^1(K \times K) + H^2(K \times K)$. Note that the Tor-term in the Künneth formula for $H^i(K)$ and $H^i(K \times K)$, $i = 1, 2$, vanishes.

Let x' generate $H^1(S^1)$, and denote by $x = x' \otimes 1$ the generator of $H^1(K)$. Then $H^1(K \times K) = \mathbf{Z} + \mathbf{Z}$ is generated by $x \otimes 1$ and $1 \otimes x$. Similarly let y' generate $H^2(K(\mathbf{Z}, 2))$ so that $y = 1 \otimes y'$ generates $H^2(K)$. Then $H^2(K \times K) = \mathbf{Z} + \mathbf{Z} + \mathbf{Z}$ is generated by $1 \otimes y$, $x \otimes x$ and $y \otimes 1$. Any map $m: K \times K \to K$ is uniquely determined by its action on $x \in H^1(K)$ and $y \in H^2(K)$. Let r be an integer, and define $m_r: K \times K \to K$ by

$$m_r^*(x) = x \otimes 1 + 1 \otimes x$$
$$m_r^*(y) = y \otimes 1 + 1 \otimes y + r(x \otimes x).$$

For all r such a map m_r exists and is furthermore a multiplication of K. A direct computation shows that m_r is a group-like structure on K for all r. Let us assume that $\theta \in \mathbf{E}(K)$ establishes an isomorphism between m_r and m_s:

$$\begin{array}{ccc} K \times K & \xrightarrow{m_r} & K \\ \downarrow{\scriptstyle \theta \times \theta} & & \downarrow{\scriptstyle \theta} \\ K \times K & \xrightarrow{m_s} & K. \end{array}$$

Since $\theta^*: H^*(K) \to H^*(K)$ is an isomorphism we necessarily have $\theta^*(x) = \varepsilon_1 x$, $\theta^*(y) = \varepsilon_2 y$ with $\varepsilon_i = \pm 1$. The relation $m_r^* \theta^*(y) = (\theta \times \theta)^* m_s^*(y)$ immediately yields $|s| = |r|$. Thus $m_r \simeq m_s$ rel $\mathbf{E}(K)$ implies $|s| = |r|$. Therefore the infinitely many group-like structures

$$m_0, m_1, \ldots, m_r, \ldots$$

on K are mutually non-isomorphic. This completes the proof of Proposition Ia.

REFERENCES

[1] M. Arkowitz and C. R. Curjel, Topology *2* (1963), 205.
[2] ———, Topology *6* (1967), 137.
[3] Sem. H. Cartan, ENS, 1959/60.
[4] B. Eckmann and P. J. Hilton, Math. Annalen *145* (1962), 227.
[5] I. M. James, Quart. J. Math. (Oxford) (2) *11* (1960), 161.
[6] W. S. Massey, Ann. of Math. (2) *62* (1955), 327.
[7] J. W. Milnor and J. C. Moore, Ann. of Math. (2) *81* (1965), 211.

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