# On the Symplectic Cobordism Ring.

# On the Symplectic Cobordism Ring

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## I. Introduction

This paper continues the study of $\Omega_*^{Sp}$, the ring of cobordism classes of weakly symplectic manifolds along lines initiated by Liulevicius, [6] and Stong [10]. $\Omega_n^{Sp}$ is computed for $n<25$ and, modulo extension problems, for $n<30$ (along with information on the ring structure) and some information is obtained on the image of the map $\Omega_*^{Sp}\to\Omega_n^0$, to wit:

THEOREM 1.1. *$\Omega_n^{Sp}\to\Omega_n^0$ is trivial for $n<32$; for $n=32$ the image is given by* $[RP(2)]^{16}$.

The first stage in our work is the (partial) calculation of the $E_2$ term of the mod 2 Adams spectral sequence for $\Pi_*(MSp)$ using an algebraic spectral sequence essentially contained in Adams [1] and Liulevicius [6], [7].

In the second stage we employ "Riemann-Roch" relations on the cohomology characteristic number of symplectic manifolds to establish the impossibility of certain classes in the Adams spectral sequence persisting to $E_\infty$; this leads immediately to a determination of certain differentials. Also applicable to the calculation of differentials (although dispensable for our range of calculation) are the Cartan formulae for higher order cohomology operations and complete knowledge of $\Omega_*^{SU}$ and its Adams spectral sequence (due to Anderson, Brown and Peterson [4]).

Finally, we make some remarks about the extension problem in the $E_\infty$ term.

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## II. The $E_2$ Term

Let $BSp(n)$ denote the universal classifying space for $Sp(n)$ vector bundles and let $MSp(n)$ be the associated Thom space; $MSp$ will designate the associated Thom spectrum. Note that $MSp$ is a ringed spectrum and, according to Novikov [5] we have a ring isomorphism $\Omega_*^{Sp}\cong\Pi_*(MSp)$. Milnor [9] has shown that $\Pi_*(MSp)$ has no odd torsion; we therefore consider only the mod 2 Adams spectral sequence which we denote $E_*^{*,*}(MSp)$. To compute $E_2^{*,*}(MSp)=Ext_A^{*,*}(H^*(MSp), Z_2)$ [$A$ will be

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the mod 2 Steenrod algebra, and cohomology will be mod 2 unless otherwise indicated] we first summarize results contained essentially in Liulevicius [6].

If $H$ is a graded vector space let $H'$ denote the vector space isomorphic to $H$ as an ungraded vector space but such that the isomorphism $H \to H'$ doubles grading (and preserves any multiplication or comultiplication that $H$ may possess.) Let $A''$ be given structure as a graded $A$-module by having $A$ act via the map $A \xrightarrow{\alpha} A \xrightarrow{\alpha} A$ where $\alpha$ is the dual of the squaring map on the dual of $A$. Let $S$ be the graded coalgebra over $Z_2$ such that $S^* \cong Z_2[x_2, x_4, x_5, \ldots, x_i, \ldots]$ $x_i$ of grading $i$, $i \neq 2^a - 1$.

PROPOSITION 2.1. *$H^*(MSp) \cong A'' \otimes S''$ as a graded coalgebra and $A$-module ($A$ operating on $S''$ trivially).*

PROPOSITION 2.2. $E_2^{**}(MSp) = Ext_A(A'', Z_2) \otimes S''^*$

$$= Ext_B(Z_2, Z_2) \otimes Z_2[V_2, V_4, V_5, \ldots, V_i, \ldots]$$
$$(i \neq 2^a - 1,\ V_j \in Ext_A^{0,4j}(H^*(MSp), Z_2)$$

where $B$ is the normal and Hopf subalgebra of $A$ such that $A//B \cong A''$.

If we describe $A$ in terms of the Milnor basis [8] $B$ may be described as the Hopf subalgebra of $A$ generated (additively) by $\{St^I\}$, $I = (C_0, C_1, \ldots, C_j, \ldots)$ $0 < C_j \leq 4$ for all $j$.

It is convenient to employ a special notation for elements of $B$.

Let $I = (C_0, C_1, \ldots, C_j, \ldots)$ $(0 \leq C_j < 4)$. Substitute for each non-zero $C_j$ the symbol

$1_j$ if $C_j = 1$

$2_j$ if $C_j = 2$

$1_j 2_j$ if $C_j = 3$

delete zeros and commas and bracket the resulting concatenation; this will be taken as the new symbol for $St^I$, e.g. $St^{(1,3,2,0,1,2)}$ becomes $[1_0 1_1 2_1 2_2 1_4 2_5]$.

Let $C(n)$ be the Hopf aubalgebra of $B$ generated (multiplicatively) by the elements $[1_0]$, $[2_j]$ for $j < n$, $[1_j]$ for $j > n$. As a Hopf algebra $C(n) \cong {}'C(n) \otimes {}''C(n)$ where ${}'C(n)$ is generated by the elements $[1_j]$, $[2_j]$ for $j < n$ and ${}''C(n)$ by the elements $[1_j]$, $j > n$. Note that $Ext_B^{s,t}(Z_2, Z_2) \to Ext_{C(n)}^{s,t}(Z_2, Z_2)$ induced by the injection $C(n) \to B$ is an isomorphism onto for $t - s < 2^{n+2} - 2$.

$C(n)$ is normal in $C(n+1)$, ${}'C(n)$ is normal in $C'(n+1)$, $C(n+1)//C(n) \cong {}'C(n+1)//{}'C(n) \cong K(n)$ where $K(n)$ is an exterior algebra on one generator of grading $2^{n+2} - 2$.

We now give a slight improvement of a spectral sequence due to Adams [3]:

Let $\Gamma$ be a connected Hopf algebra over $Z_2$, $\Lambda$ a normal (but not necessarily central) and Hopf subalgebra of $\Gamma$. Let $F(\Gamma^*)$ be the cobar construction [1] on $\Gamma^*$, the dual of $\Gamma$. By definition

$$F(\Gamma^*) = \sum_{s \geq 0} I(\Gamma^*)^{\otimes s}$$

where $\otimes s$ denotes $s$-fold tensor product and $I$ denotes augmentation ideal. $F(\Gamma^*)$ is a bigraded cochain complex with cup products (see [3] for gradings, differentials and cup products) and its cohomology is just $Ext_{\Gamma}^{*,*}(Z_2, Z_2)$. We define a filtration, the $\Lambda$-filtration on $F(\Gamma^*)$ by setting

$$a_1 \otimes a_2 \cdots \otimes a_r \in F(\Gamma^*)^{(p)} \qquad (a_i \in I(\Gamma^*))$$

if $a_i$ annihilates $I(\Lambda)$ for $p$-values of $i$. Let $\Omega = \Gamma // \Lambda$.

PROPOSITION 2.3. The $\Lambda$-filtration of $F(\Gamma^*)$ gives rise to a trigraded spectral sequence with products, $E_*^{*,*,*}$ such that

$$E_1^{*,*,*} = Ext^{*,*}(Z_2, Z_2) \otimes F(\Omega^*)$$

$$E_1^{s,t,u} = \sum_{u'+u''=u} Ext_{\Omega}^{t,u'}(Z_2, Z_2) \otimes F(\Omega^*)^{s,u''}$$

$$d_r: E_r^{s,t,u} \to E_r^{s+r,t-r+1,u}$$

$$d_r(x \cdot y) = d_r(x)\, y + x d_r(y)$$

$$E_{r+1} = H(E_r; d_r);$$

$E_{\infty}^{s,n-s,t}$ gives a composition series for $Ext_{\Gamma}^{s,t}(Z_2, Z_2)$; the product structure of $E_{\infty}$ as the limit of the $E_r$ agrees with the product structure of $E_{\infty}$ as the graded group associated to a filtration of $Ext_{\Gamma}^{*,*}(Z_2, Z_2)$.

*Proof.* We refer to the proof of Theorem 2.3.1 of [3]. We must verify that the condition $\Lambda$ central in $\Gamma$ may properly be weakened to $\Lambda$ normal in $\Gamma$. Now actually the effect upon the proof in [3] of this weakening of the hypothesis is that the maps

$$\mu_*: \bar{B}(\Lambda) \otimes (\bar{\Gamma})^p \to \bar{B}(\Gamma)^{(p)}$$
$$\mu': B(\Omega) \otimes (\bar{\Omega})^p \to C^{(p)}$$

([3], p. 40) fail to be chain maps (that is they do not commute with the differentials). The reduction

$$\mu'_*: \bar{B}(\Lambda) \otimes (\bar{\Omega})^p \to \bar{B}(\Gamma)^{(p)} / \bar{B}(\Gamma)^{(p-1)}$$

will however remain a chain map since in fact $\mu'$ deviates from being a chain map precisely by an error term in $\bar{B}(\Gamma)^{(p-1)}$.

We could also give the $E_2$ term as $Ext_{\Omega}(Z_2, Ext_{\Gamma}(Z_2, Z_2))$ but this would require a prior knowledge of the action of $\Omega$ on $Ext_{\Gamma}(Z_2, Z_2)$; on the other hand the $E_1$ term requires $F(\Omega^*)$ which is awkward in the general case but very simple for our applications in which $\Omega = K(n)$.

In fact let $\Gamma = C(n+1)$, $\Lambda = C(n)$, $\Omega = \Gamma // \Lambda = K(n)$, and let the associated spectral sequence be denominated ${}_nE$. Then $F(K(n)^*)$ is simply a polynomial algebra $Z_2[k_n]$, $k_n$

of bigrading $(1, 2^{n+2}-2)$, so we write

$$_nE_1 = Ext_{C(n)}(Z_2, Z_2) \otimes Z_2[k_n]$$
$$_nE_1^{s,t,u} = Ext_{C(n)}^{t,u-s(2^{n+2}-2)}(Z_2, Z_2) \otimes k_n^s.$$

PROPOSITION 2.4. (Liulevicius [6]) $\mathrm{Ext}_{C(1)}^{*,*}(Z_2, Z_2)$ has multiplicative generators $q_0, k_0, \tau_0, \omega_0, q_j$ $(j>1)$ of bigradings $(1, 1)$, $(1, 2)$, $(3, 7)$, $(4, 12)$, $(1, 2^{j+1}-1)$ respectively subject to the relations

$$q_0k_0 = 0 \quad k_0^3 = 0 \quad k_0\tau_0 = 0 \quad \tau_0^2 = q_0^2\omega_0;$$

$\mathrm{Ext}_{C(1)}^{*,*}(Z_2, Z_2)$ thus is a free $Z_2[\omega_0, q_2, q_3, \ldots, q_j, \ldots]$ module.

This proposition may easily be reproven using $_0E$; $\mathrm{Ext}_{C(0)}^{*,*}(Z_2, Z_2)$ is of course just $Z_2[q_0, q_1, \ldots, q_j, \ldots]$ since $C(0)$ is an exterior algebra.

We now consider $^nE$, $n>0$. Since $C(n) = {'C}(n) \otimes {''C}(n)$ and $''C(n)$ is an exterior algebra we may write

$$Ext_{C(n)}(Z_2, Z_2) = Ext_{'C(n)} \otimes Z_2[q_{n+1}] \otimes Z_2[q_{n+2}, \ldots, q_j, \ldots]$$
$$Ext_{C(n+1)}(Z_2, Z_2) = Ext_{'C(n+1)} \otimes Z_2[q_{n+2}, \ldots, q_j, \ldots]$$
$$_nE = {}_n'E \otimes Z_2[q_{n+2}, \ldots, q_j, \ldots]$$

where $_n'E$ is the spectral sequence associated to $'C(n+1)//'C(n) \cong K(n)$.

THEOREM 2.5. *All $d_r$ in $_nE$ are trivial for $r>3$. i.e. $_nE_4 \cong {}_nE_\infty$.*

*Proof.* We define a second grading, the "$n$-grading" on $C(n+1) = {'C}(n+1) \otimes {''C}(n+1)$ by assigning $n$-grading 0 to $[1_0]$, to $[2_j]$, $j<n$ and to $[1_j]$, $j>n+1$ and $n$-grading 1 to $[2_n]$. It is readily verified that this gives a well defined grading on $C(n+1)$ and induces a fourth grading on $_nE$ so that

$$Ext_{C(n)}(Z_2, Z_2) \in {}_nE_1^{*,*,*,0}$$
$$q_{n+1} \in {}_nE_1^{1,0,2^{n+2}-1,1}$$
$$k_n \in {}_nE_1^{0,1,2^{n+2}-2,1}$$
$$q_j \in {}_nE_1^{1,0,2^{j+1}-1,0} \quad (j>n+1)$$

and so that $d_r$ preserves the fourth grading. Direct computation shows that $q_{n+1}^4$ persists to $_nE_\infty$ and that $_nE_r$ can be regarded as a free $Z_2[q_{n+1}^4]$ module for $r=1$; we show indirectly that this is true for all $r$. Assume then that $_nE_r$ is a free $Z_2[q_{n+1}^4]$ module. To show that $_nE_{r+1}$ is also a free $Z_2[q_{n+1}^4]$ module we must show that if $u \in {}_nE_r$, $d_ru = [q_{n+1}^4] \cdot w$ then there is a $u' \in {}_nE_r$ such that $d_ru' = w$. Since $d_ru = [q_{n+1}^4] \cdot w$ $u$ has fourth grading greater by 4 than that of $w$ and a second grading 2 less than that of $w$. In $_nE_1$ the fourth grading minus the second grading gives the power of $q_{n+1}$, hence $u$ is the image of an element of $_nE_1$ which is there divisible by $q_{n+1}^4$. By induction this divisibility persists to ${}_nE_r$ so

$$u = [q_{n+1}^4]\, u', \quad d_r u = [q_{n+1}^4]\, d_r u' = [q_{n+1}^4 w]$$

and $u' = w$.

Now suppose $d_r u \neq 0$, $r > 3$. Then as before $u = [q_{n+1}^4]\, u'$, $d_r u = [q_{n+1}^4]\, d_r u'$, $d_r u \neq 0$ giving an absurd infinite regress and so proving the theorem.

Now let $\omega_n$ denote the class in $Ext_{C(n+1)}(Z_2, Z_2)$ corresponding to the image of $q_{n+1}^4$ in ${}_nE_\infty$. We will (in line with our general notational policy) also refer to the corresponding (unique) element of $Ext_B(Z_2, Z_2)$ as $\omega_n$.

COROLLARY 2.6. *$Ext_B(Z_2, Z_2)$ is a free $Z_2[\omega_0, \omega_1, \ldots, \omega_j, \ldots]$ module. Note that we do have relations in $Ext_B(Z_2, Z_2)$ of the form $u \cdot v = w\omega_n$ where $u$, $v$ are not divisible by $\omega_n$.*

Let us consider the element $k_j$ which arises in ${}_jE_1$ and persists by reason of grading considerations to ${}_jE_\infty$ to give an element of $Ext_{C(j+1)}^{1, 2^{j+2}-2}(Z_2, Z_2)$ which cannot (again for grading considerations) be killed in ${}_{j'}E(j' > j)$ and so determines an element (also called) $k_j \in Ext_B(Z_2, Z_2)$. Direct computation shows that these $k_j$ together with $q_0 \in Ext_B^{1,1}(Z_2, Z_2)$ complete the description of $Ext_B^{1,*}(Z_2, Z_2)$. (The elements $[2_j]$, $[1_0]$ constitute a minimal set of polynomial generators of $B$; $k_j$, $q_0$ are their duals.)

The element $q_{j+1}$ must therefore perish; it can do so only in ${}_jE$ and only via $d_1 q_{j+1} = q_0 k_j$. However the class $q_{j+1} k_i (j > i)$ which arises in ${}_iE$ is not killed in ${}_jE$ as $d_1 (q_{j+1} k_i) = q_0 k_i k_j$ but $q_0 k_i k_j = 0$ in ${}_jE$ as already in ${}_iE$ $q_0 k_i$ has been killed by $q_{i+1}$.

There is a well-defined element

$$\gamma_{i,j} \in Ext_B^{2, 2^{i+2}+2^{j+2}-3}(Z_2, Z_2)$$

corresponding to $q_{j+1} k_i$.

Similarly, the element $q_0 q_{j+1}^2$ is not killed in ${}_jE$ as while $d_2 (q_{j+1}^2) = k_0 k_j^2$, $d_2 (q_0 q_{j+1}^2) = q_0 k_0 k_j^2 = d_1 (q_{j+1} k_0 k_j) = 0 \in {}_jE_2$, so $q_0 q_{j+1}^2$ determines a class

$$\tau_j \in Ext_B^{3, 2^{j+3}-1}(Z_2, Z_2).$$

(If $j = 0$ replace $d_2$ by $d_3$ in the preceding). For $j > 0$ the definition of $\tau_j$ is subject to an indeterminancy of $k_j \gamma_{0,j}$. More generally let $j(i)$, $1 \leqslant i \leqslant n$ be a strictly increasing sequence of nonnegative integers. Corresponding to

$$q_0 q_{j(1)+1}^2 \cdots q_{j(i)+1}^2 \cdots q_{j(n)+1}^2$$

we have a class

$$\tau_{j(1), \ldots, j(i), \ldots, j(n)}$$

Now suppose $0 \leqslant i < j < k$. In ${}_kE$

$\gamma_{j,k}$ is represented by $q_{k+1} k_j$

$\gamma_{i,k}$ is represented by $q_{k+i}k_i$

$\gamma_{i,j}$ is represented by $\gamma_{i,j}$ (coming from $q_{j+1}k_i$ in ${}_jE$).

There are also elements $\gamma_{i,j}k_k$ and $q_k k_i k_j$ in ${}_kE$ (persisting to ${}_kE_\infty$). We see at once that

$$k_k\gamma_{i,j} \neq 0, \quad k_i\gamma_{j,k} \neq 0, \quad k_j\gamma_{i,k} \neq 0$$

and

$$k_i\gamma_{j,k} = k_j\gamma_{i,k} + ck_k\gamma_{i,j}$$

where $c \in Z_2$ represents an extension problem. But there is an automorphism $\Pi$ on $B$ as an ungraded Hopf algebra which interchanges $[1_{j+1}]$ and $[1_{k+1}]$ and $[2_j]$ and $[2_k]$. The corresponding (non-grading preserving) automorphism $\Pi^*$ on $Ext_B(Z_2, Z_2)$ has

$$\Pi^* k_i = k_i \qquad \Pi^*\gamma_{i,j} = \gamma_{i,k}$$
$$\Pi^* k_j = k_k \qquad \Pi^*\gamma_{i,k} = \gamma_{i,j}$$
$$\Pi^* k_k = k_j \qquad \Pi^*\gamma_{j,k} = \gamma_{j,k}$$

so that

$$k_i\gamma_{j,k} = k_k\gamma_{i,j} + ck_j\gamma_{i,k}$$

Now $c=0$ would imply

$$k_i\gamma_{j,k} = k_j\gamma_{i,k} = k_k\gamma_{i,j}$$

but we have seen that $k_i\gamma_{j,k}$, $k_j\gamma_{i,k}$, $k_k\gamma_{i,j}$ must generate a two-dimensional $Z_2$-vector space in $Ext_B(Z_2, Z_2)$ so $c=1$ and we have the relation

$$k_i\gamma_{j,k} + k_j\gamma_{i,k} + k_k\gamma_{i,j} = 0$$

The symmetries of $B$ are thus very useful for the study of $Ext_B(Z_2, Z_2)$. They may be used for example to show that for $0<i<j$,

$$(\gamma_{i,j})^4 = k_i^4\omega_j + k_j^4\omega_i$$

(It is also true, but more difficult to establish that $(\gamma_{0,j})^4 = k_j^4\omega_0$.)

It is clear that $Ext_B(Z_2, Z_2)$ contains a free polynomial ring generated by the $\omega_i$ and the $k_j$ $(j>0)$. As an application we observe that the spectral sequences show that $k_2\tau_{0,1}$ is the unique non-zero element in $Ext_B^{6,35}(Z_2, Z_2)$. On the other hand $[(\gamma_{0,1})^2(\gamma_{0,2})]^4 = k_1^8 k_2^4 \omega_0^3 \neq 0$ so we have, using symmetry, $i, j>0$, $i \neq j$

$$k_i\tau_{0,j} = (\gamma_{0,j})^2(\gamma_{0,1}).$$

Let us fix $\tau_j$, $j>0$, by requiring that $k_0\tau_j = 0$. Then we have (by computations such as those above):

THEOREM 2.7. *The following are some of the relations holding in* $Ext_B(Z_2, Z_2)$. *(In each case* $0 \leqslant i < j < k < l$.*)*

$$q_0k_i = 0 \quad k_0k_i^2 = 0 \quad q_0\gamma_{i,j} = k_0k_ik_j \quad k_0k_ik_jk_k = 0 \quad (\gamma_{i,j})^4 = k_i^4\omega_j + k_j^4\omega_i$$

$$k_i\gamma_{j,k} + k_j\gamma_{i,k} + k_k\gamma_{i,j} = 0 \quad \gamma_{i,j}\gamma_{k,l} + \gamma_{i,k}\gamma_{j,l} + \gamma_{i,l}\gamma_{j,k} = 0 \quad k_0\tau_i = 0$$

$$k_j\tau_j \equiv 0 \bmod k_j^2\gamma_{0,j} \quad k_j\tau_k \equiv 0 \bmod k_k^2\gamma_{0,j} \quad k_j\tau_0 \equiv 0 \bmod k_0^2\gamma_{0,j} \quad \tau_0^2 = q_0^2\omega_0$$

$$\tau_j^2 \equiv q_0^2\omega_j \bmod k_j^2(\gamma_{0,j})^2, \quad k_0\tau_{0,j} = 0, \quad k_j\tau_{0,j} = (\gamma_{0,j})^3,$$

$$k_k\tau_{0,j} = (\gamma_{0,k})(\gamma_{0,j})^2, \quad k_0(\gamma_{i,j})^2 = 0, \quad \tau_I\tau_J = (q_0\tau_K\Pi_{i\in I\cap J}\omega_i) + \text{other terms}$$

where $K=(I\cup J)-(I\cap J)$, $\tau_\phi = q_0$. Although we have not described $Ext_B(Z_2, Z_2)$ completely (there are even irreducible generators which have not been mentioned) the above information is sufficient to yield Table I which describes $Ext_B^{s,t}(Z_2, Z_2)$ for $t-s \leqslant 31$; actions of $q_0$ and $k_0$ are described by vertical and diagonal lines respectively.

TABLE I

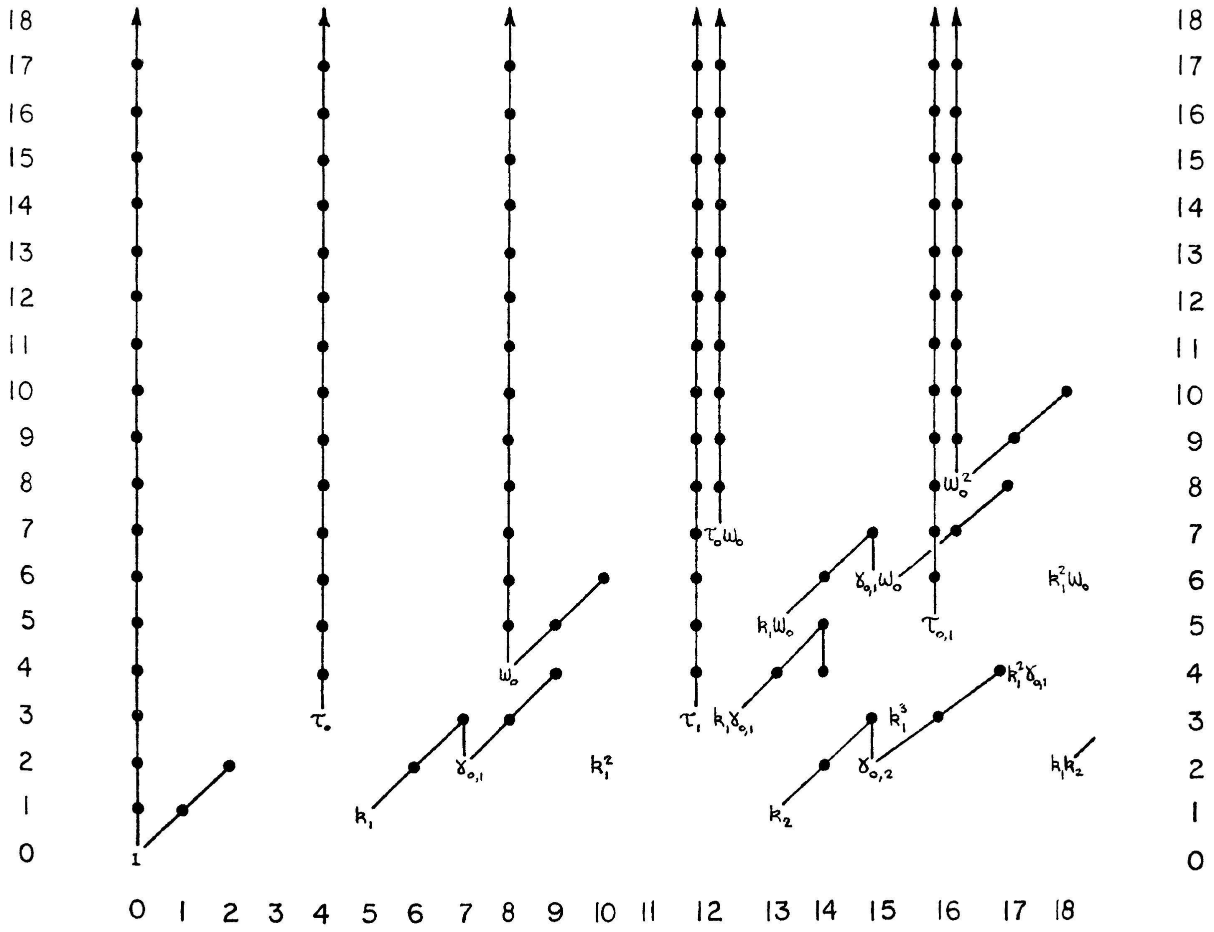
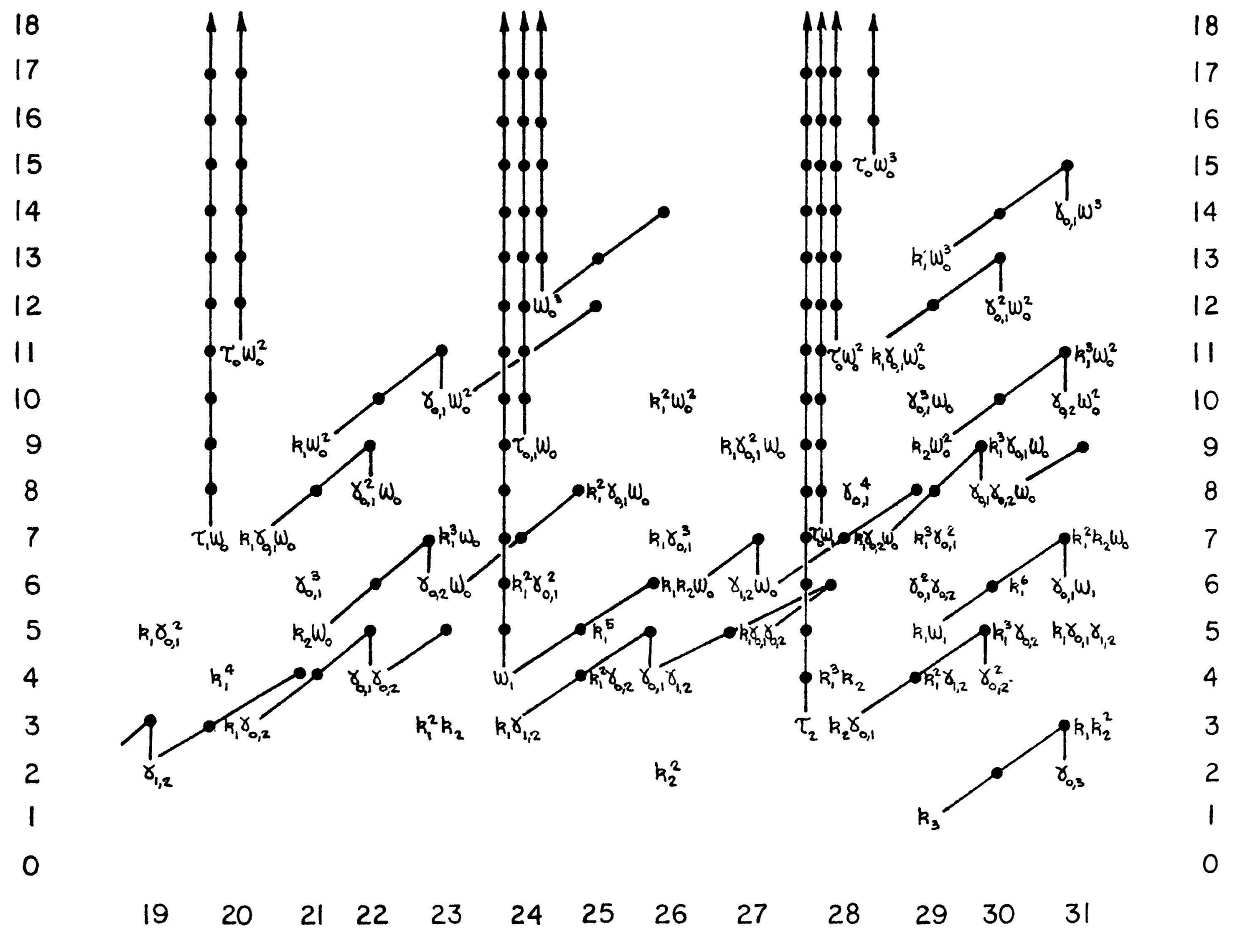


*Table I (Continued)*




## III. Characteristic Numbers and the Differentials

Let $H^{**}(BSp; Q)$ denote the direct product of the $H^n(BSp; Q)$.

It is well known that there is a pairing

$$H^{**}(BSp; Q) \otimes \Omega_*^{Sp} \to Q$$

which enables us to define symplectic (rational cohomology) characteristic numbers on the symplectic cobordism classes. Each non-zero element of $H^{4n}(BSp; Q)$ defines a non-trivial evaluation map $\Omega_{4n}^{Sp} \to Q$.

For each partition $\omega$ on the integer $n$ Stong [10] has defined a class $s_\omega(e_p)\, T \in H^{**}(BSp; Q)$ with the property that if $M$ is a $4k$-dimensional symplectic manifold $S_\omega(e_p)\, T[M]$ is an integer and is an even integer if $n+k \equiv 1 \pmod 2$.

By detailed computations we have established:

PROPOSITION 3.1. *Every cohomology characteristic number on* 8 *and* 20 *dimensional symplectic manifolds is divisible by* 4; *on* 16 *and* 24 *dimensional symplectic manifolds by* 2.

In the case of 8-dimensional manifolds our calculations are easily exhibited. If $M$ is an 8-dimensional $Sp$-manifold $S_1(e_p)\,T[M]=(1/6)S_{12}(p)\,[M]$ and $S_\phi(e_p)\,T[M]==(1/240)S_2(p)+(1/144)S_{12}(p)\,[M]$ where $S_2(p)$ and $S_{12}(p)$ generate $H^8(BSp;Z)$. The condition that $(1/6)S_{12}(p)\,[M]$ be even implies that $S_{12}(p)\,[M]$ is divisible by 4; this together with the requirement that $(1/240)S_2(p)\,[M]+(1/144)\,S_{12}(p)\,[M]$ be integral gives that $S_2(p)\,[M]$ and so also any integral cohomology characteristic number is divisible by 4.

The following has been proven by D. W. Anderson (unpublished).

PROPOSITION 3.2. *Let $M$ be a connected spectrum, $S$ the sphere spectrum, $u\in E_\infty^{s,t}(M)$ such that $h_0^n\cdot u\neq 0$ for all non-negative $n$ ($h_0$ the generator of $E_\infty^{1,1}(S)$). Let $f: S\to M$ be represented by $u$. Then the image of $H^*(M,Z)$ in $H^*(S,Z)$ under $f^*$ contains elements not divisible by $2^{s+1}$. (We allow $f$ to be a map of spectra which changes indexing.)*

The proof of Proposition 3.2 depends upon the fact that it is possible to map $M$ into $K(Z)$ (the Eilenberg-MacLane spectrum) so that $u$ is in the image of the induced map $E_\infty(K(Z))\to E_\infty(M)$. The general result for $M$ then follows from the case of $K(Z)$; the proposition is however obvious for $M=K(Z)$.

PROPOSITION 3.3.

(i) $d_2v_2=\gamma_{0,1}$, (ii) $d_2v_4=\gamma_{0,2}$, (iii) $d_3v_2^2=k_1^3$,

(iv) $d_2v_5=\gamma_{1,2}$, (v) $d_3v_6=k_1^2k_2$, (vi) $d_2v_8=\gamma_{0,3}$,

*Proof.* Putting together Propositions 3.1 and 3.2 we have at once that $v_2$, $q_0v_2$, $v_4$, $v_2^2$, $v_5$, $q_0v_5$ and $v_6$ cannot persist to $E_\infty(MSp)$. This implies parts (i)–(v).

From the results of Anderson, Brown and Peterson [4] and analysis of the map of Adams spectral sequences induced by the standard map of $MSp$ to $MSU$ we may obtain (vi).

Putting Proposition 3.3 together with Proposition 2.1, Theorem 2.7 and Table I we are able to compute the Adams spectral sequence for $MSp$ completely through the 31-stem save for two unsettled questions:

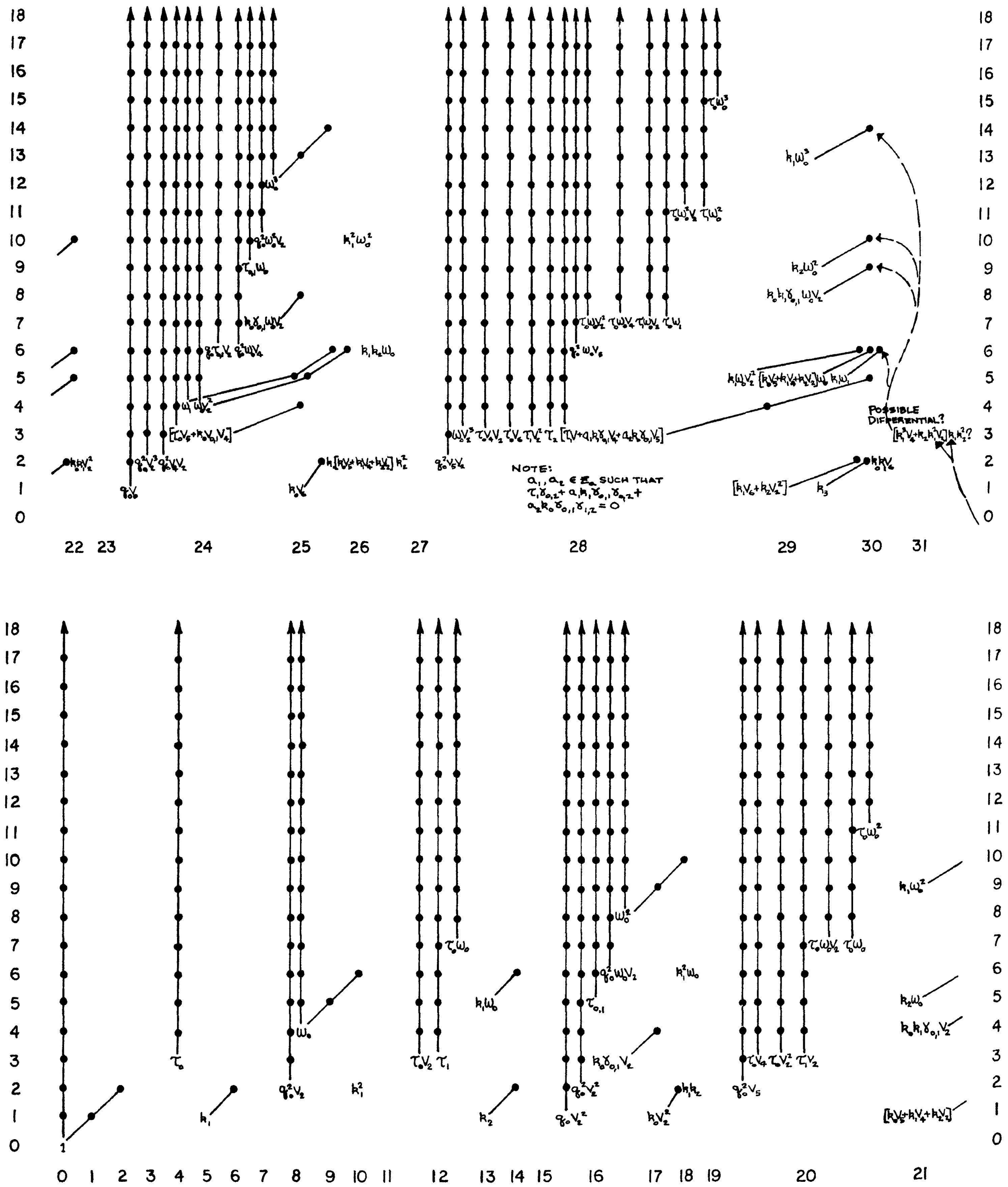
i) The value of $d_3(v_4^2)$. (We conjecture that it is $k_1k_2^2$.)

ii) Is $k_1^3v_4+k_2k_1^2v_2$ an infinite cycle?

Our results concerning $E_\infty(MSp)$ are depicted in Table II. Since $d_2(v_2^4)=d_3(v_2^4)=0$ it is now obvious that $v_2^4$ is an infinite cycle and Theorem 1.1 follows.

We now turn to the extension problems involved in passing from $E_\infty(MSp)$ to $\Omega_*^{Sp}$. $\Omega_{10}^{Sp}=Z_2\oplus Z_2$ where one $Z_2$ is the square of the $\Omega_5^{Sp}$ generator and the other is the product of the generators of $\Omega_1^{Sp}$ and $\Omega_9^{Sp}$. Multiplication by the generator of $\Omega_1^{Sp}$ induces an isomorphism between $\Omega_{13}^{Sp}$ and $\Omega_{14}^{Sp}$; this implies $\Omega_{13}^{Sp}\simeq\Omega_{14}^{Sp}\simeq Z_2\oplus Z_2$. In the 16-stem there is a non-trivial extension since the element $k_0\gamma_{0,1}v_2$ goes into a non-zero element in $E_\infty(MSU)$ but $\Omega_{16}^{SU}$ is torsion free. (In fact careful analysis of

# TABLE II



characteristic numbers shows that twice the class represented by $k_0\gamma_{0,1}v_2$ is represented by $\tau_{0,1}$.) Utilizing multiplication by $k_1$ as well as $k_0$ in $E_\infty(MSp)$ it is possible to show that all elements of the 17, 18, 21 and 22 stems of $\Omega_*^{Sp}$ are of order 2.

There are clearly no extension problems in the 20-stem while in the 24-stem we have a situation corresponding to that in the 16-stem (with $k_0\gamma_{0,1}\omega_0 v_2$ and $\tau_{0,1}\omega_0$ replacing $k_0\gamma_{0,1}v_2$ and $\tau_{0,1}$ respectively). Beyond the 24-stem we have not succeeded in resolving the extension problems. Also, some very elementary questions on the ring structure are left open. For example we do not know if the product of the class represented by $k_0^2\gamma_{0,1}v_2$ by the class represented by $k_0$ is trivial or not. (If it is trivial then $\Omega_{25}^{Sp}$ is the sum of six copies of $Z_2$.)

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