On Schlicht Mappings to Domains Convex in One Direction.

On Schlicht Mappings to Domains Convex in One Direction

by WALTER HENGARTNER and GLENN SCHOBER[1])

1. Principal Results

Our first result is a characterization of schlicht mappings from the unit disk $|z|<1$ onto domains in the (u, v)-plane which are convex in the v-direction. We consider only domains D which permit the normalization that $z=\pm 1$ correspond, in some sense, to the right and left extremes of D. The characterization is in terms of the condition

$$\operatorname{Re}\{(1-z^2)f'(z)\}\geqslant 0, \qquad |z|<1. \tag{1}$$

Except in the degenerate case $\operatorname{Re}\{(1-z^2)f'(z)\}\equiv 0$, this means geometrically that each circular arc (and line segment) joining $z=-1$ to $z=1$ in the unit disk corresponds to an analytic arc in D which may be represented as a function $v=v(u)$.

Secondly, for Steiner symmetric domains, i.e., domains that are both convex in the v-direction and symmetric with respect to the u-axis, we show that level sets inherit the Steiner symmetry.

We next determine coefficient bounds for functions satisfying (1). If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, we show that

$$|a_n|\leqslant|f'(0)| \quad \text{for} \quad n=1,2,3,\ldots. \tag{2}$$

Actually more stringent estimates are obtained.

Functions satisfying (1) admit elementary distortion estimates for $|f'(z)|$ and $|f(z)-f(0)|$. The upper estimates are similar to those for the class of convex mappings; however, the lower bounds are new and different.

In addition, functions satisfying (1) give rise to an interesting "Viertelsatz" type theorem: *If f satisfies* (1), $f(0)=0$, *and* $|f'(0)|=1$, *then* $f(|z|<1)$ *contains all points of the disk* $|w|<\frac{1}{2}\log 2$. The constant $\frac{1}{2}\log 2$ is sharp, and it is interesting that it falls strictly between the constant $\frac{1}{4}$ for the general class of normalized schlicht mappings and the constant $\frac{1}{2}$ for the subclass of convex mappings. Moreover, if we assume also that $f(|z|<1)$ is Steiner symmetric, then the constant becomes $\frac{1}{2}$ just as for convex mappings.

---

[1]) This research was supported in part by NSF grants GP7505 and GP11603.

2. Characterization

A domain in the (u, v)-plane is *convex in the v-direction* if it contains together with each pair of points with the same abscissa, the entire segment joining them. We consider the class Σ of such domains D which admit a schlicht mapping f of $|z|<1$ onto D with the following normalization: There exist points z_n' converging to $z=1$ and z_n'' converging to $z=-1$ such that

$$\left.\begin{aligned} \lim_{n\to\infty} \operatorname{Re}\{f(z_n')\} &= \sup_{|z|<1} \operatorname{Re}\{f(z)\} \\ \lim_{n\to\infty} \operatorname{Re}\{f(z_n'')\} &= \inf_{|z|<1} \operatorname{Re}\{f(z)\}. \end{aligned}\right\} \tag{3}$$

This normalization means that the prime ends $f(1)$ and $f(-1)$, which lie in the extended plane, are in some sense the right and left extremes of D. The class Σ contains domains such as the right half plane and the strip $|v|<1$. In the latter case both $f(1)$ and $f(-1)$ are prime ends at ∞.

If D is convex in the v-direction, it follows from standard results in the theory of prime ends that existence of a mapping with normalization (3) is equivalent to assuming that there exists one vertical ray in $\mathbf{C}-D$ which meets ∂D from above and another which meets ∂D from below. Consequently, additional examples of domains in Σ are domains which are Steiner symmetric with respect to the real axis, except for the plane itself. On the other hand, the upper half plane and the plane slit vertically to ∞ are not in Σ (see also Section 6).

The following theorem shows that schlicht mappings from $|z|<1$ onto domains of Σ with normalization (3) are completely characterized by the condition (1).

THEOREM 1. *Suppose f is analytic and non-constant for $|z|<1$. Then we have*

$$\operatorname{Re}\{(1-z^2)f'(z)\} \geqslant 0, \quad |z|<1 \tag{1}$$

if and only if

(a) *f is schlicht on $|z|<1$*

(b) *$f(|z|<1)\in\Sigma$, and*

(c) *f is normalized by* (3).

Remark 1. If $\operatorname{Re}\{(1-z^2)f'(z)\}\geqslant 0$ and vanishes for some point in $|z|<1$, then by the minimum principle for harmonic functions it vanishes identically. In that case

$$f(z) = a_0 + i\beta \log\frac{1+z}{1-z}, \quad a_0\in\mathbf{C}, \quad \beta\in\mathbf{R}, \tag{4}$$

which defines a schlicht mapping of $|z|<1$ onto a vertical strip. In addition, f has the normalization (3) with $z=\pm 1$ corresponding to prime ends at ∞, and $f(|z|<1)\in\Sigma$.

Remark 2. The condition

$$\operatorname{Re}\{(1-z^2)f'(z)\} > 0, \qquad |z| < 1, \tag{5}$$

has an elementary geometric interpretation. If we parametrize the line segment and circular arcs γ_t, $-(\pi/2) < t < \pi/2$, joining $z = -1$ to $z = 1$ in the unit disk by

$$\gamma_t : z = z(s) = \frac{e^{s+it} - 1}{e^{s+it} + 1}, \qquad -\infty < s < \infty, \tag{6}$$

then one easily verifies that

$$\frac{d}{ds} \operatorname{Re}\{f(z(s))\} = 2 \operatorname{Re}\{[1 - z^2(s)]\, f'(z(s))\}. \tag{7}$$

Consequently, the condition (5) is equivalent to the property that the circular arcs γ_t are mapped onto analytic arcs which may be represented as functions $v = v(u)$. It follows that f has the normalization (3). Furthermore, since the region bounded by $f(\gamma_t) \cup f(\gamma_\tau) \cup f(-1) \cup f(1)$ is convex in the v-direction for every $-(\pi/2) < t < \tau < \pi/2$, we find that $f(|z| < 1)$ is also convex in the v-direction.

Remark 3. An analytic function f is *close-to-convex* if there exists a convex mapping φ such that $\operatorname{Re}\{f'(z)/\varphi'(z)\} > 0$ for $|z| < 1$. Functions satisfying (5) are special close-to-convex functions associated with $\varphi(z) = \frac{1}{2} \log[(1+z)/(1-z)]$. W. Kaplan [3] has shown that close-to-convex functions, hence functions satisfying (5), are schlicht. The geometric interpretation of Remark 2 could also be used to show that functions satisfying (5) are schlicht.

Proof of Theorem 1. If f satisfies (1), then (a), (b), and (c) follow from Remarks 1–3. The converse will be a consequence of Lemma 2.

LEMMA 1. *Suppose*
(a) f *is schlicht on* $|z| < 1$,
(b) $f(|z| < 1) \in \Sigma$,
(c) f *is normalized by* (3), *and*
(d) $\operatorname{Re} f$ *is bounded.*
Then $\operatorname{Re}\{(1 - z^2) f'(z)\} \geq 0$ *for* $|z| < 1$.

Proof: Since $u = \operatorname{Re} f$ is bounded, the radial limits $U(\theta) = \lim_{r \to 1} u(re^{i\theta})$ exist except for a set N of Lebesgue measure zero and u has the Poisson representation

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(\theta) \operatorname{Re}\left\{\frac{e^{i\theta} + z}{e^{i\theta} - z}\right\} d\theta. \tag{8}$$

Since $f(|z|<1)\in\Sigma$ and f satisfies (3), it follows from the prime end correspondence that $U(\theta_1)\leqslant U(\theta_2)$ for all θ_1, θ_2 where $U(\theta_1)$ and $U(\theta_2)$ exist and $-\pi\leqslant\theta_1<\theta_2\leqslant 0$ or $0\leqslant\theta_2<\theta_1\leqslant\pi$. By defining $U(\theta)$ for $\theta\in N$ to be the average of the one-sided limits taken over the complement of N, we may assume in (8) that U is defined at each point of $[-\pi, \pi]$, non-decreasing on $[-\pi, 0]$, and non-increasing on $[0, \pi]$. Now from the representation (8) Kaplan [3] has shown by differentiating and integrating by parts that

$$\operatorname{Re}\{(1-z^2)f'(z)\} = \frac{-1+|z|^2}{\pi}\int_{-\pi}^{\pi}\frac{\sin\theta\, dU(\theta)}{|e^{i\theta}-z|^2} \geqslant 0. \tag{9}$$

We include a second, more geometric proof. From integrating (8) by parts directly we obtain a Riemann-Stieltjes representation

$$u(z) = \text{constant} - \int_{-\pi}^{\pi}\omega(z,\theta)\, dU(\theta) \tag{10}$$

where $\omega(z, \theta)$ denotes the harmonic measure in the unit disk at the point l.c. with

$$\omega(e^{it},\theta) = \begin{cases} 1 & \text{if} \quad -\pi<t<\theta \\ 0 & \text{if} \quad \theta<t<\pi. \end{cases} \tag{11}$$

If z_1 and z_2 are two points with $\operatorname{Re} z_1<\operatorname{Re} z_2$ and lying on the same circular arc or line segment joining $z=-1$ to $z=1$ in the unit disk, then we have

$$\left.\begin{aligned} &\omega(z_1,\theta)>\omega(z_2,\theta) \quad \text{if} \quad -\pi<\theta<0 \\ &\omega(z_1,\theta)<\omega(z_2,\theta) \quad \text{if} \quad 0<\theta<\pi \\ &\omega(z_1,\theta)=\omega(z_2,\theta) \quad \text{if} \quad \theta=0, \pm\pi. \end{aligned}\right\} \tag{12}$$

The representation (10), monotonicity of U, and relations (12) imply that $u(z_1)\leqslant u(z_2)$. It follows from Remark 2, in particular (7), that $\operatorname{Re}\{(1-z^2)f'(z)\}\geqslant 0$.

We now remove hypothesis (d) from Lemma 1 by a process of exhaustion.

LEMMA 2. *Suppose*
(a) *f is schlicht on* $|z|<1$,
(b) $f(|z|<1)\in\Sigma$, *and*
(c) *f is normalized by* (3).
Then

$$\operatorname{Re}\{(1-z^2)f'(z)\}\geqslant 0, \quad |z|<1. \tag{13}$$

Proof. Assume at first that $D=f(|z|<1)$ is unbounded in both the positive and negative u-directions. Let $w_n'\in D$ be a sequence of points with $\operatorname{Re} w_n'$ tending to $+\infty$

and w_n' tending to the prime end $f(1)$ as $n \to \infty$. Similarly, let $w_n'' \in D$ be a sequence of points with $\operatorname{Re} w_n''$ tending to $-\infty$ and w_n'' tending to the prime end $f(-1)$ as $n \to \infty$. Then $D_n = \{w \in D\colon \operatorname{Re} w_n'' < \operatorname{Re} w < \operatorname{Re} w_n'\}$ is in Σ.

Let F_n and F be schlicht mappings of the unit disk onto D_n and D, respectively, with normalization $F_n(0) = F(0) = f(0)$, $F_n'(0) > 0$, $F'(0) > 0$. Let $\zeta_n' = F_n^{-1}(w_n')$, $\zeta_n'' = F_n^{-1}(w_n'')$, $\zeta' = F^{-1}(f(1))$, and $\zeta'' = F^{-1}(f(-1))$. Since D_n converges monotonically to D, F_n converges locally uniformly to F by the Caratheodory kernel theorem (cf. [1, p. 46]). Hence $\varphi_n = F^{-1} \circ F_n$ converges locally uniformly to the identity. In fact, $\Delta_n = \varphi_n(|z| < 1)$ is a Jordan domain, and the specific convergence of D_n to D implies that $\partial \Delta_n$ converges to $|z| = 1$ in the sense of Fréchet. By the convergence theorem of Radó (cf. [1, p. 50]) φ_n converges to the identity uniformly on $|z| \leqslant 1$. In particular, $\lim_{n\to\infty} \varphi_n(\zeta') = \zeta'$ and $\lim_{n\to\infty} \varphi_n(\zeta'') = \zeta''$. Since the sequences $\{w_n'\}$ and $\{w_n''\}$ converge to the prime ends $f(1)$ and $f(-1)$, respectively, we also have $\lim_{n\to\infty} \varphi_n(\zeta_n') = \zeta'$ and $\lim_{n\to\infty} \varphi_n(\zeta_n'') = \zeta''$. Consequently, $\lim_{n\to\infty} \zeta_n' = \zeta'$ and $\lim_{n\to\infty} \zeta_n'' = \zeta''$.

Now let μ_n be the Möbius transformation with $\mu_n(1) = \zeta_n'$, $\mu_n(-1) = \zeta_n''$, and $\mu_n(i) = F^{-1}(f(i))$. Then μ_n converges uniformly in $|z| \leqslant 1$ to a Möbius transformation μ which takes 1, -1, i onto ζ', ζ'', $F^{-1}(f(i))$, respectively. Note that $F \circ \mu = f$ by uniqueness. Define $f_n = F_n \circ \mu_n$. Since f_n maps $|z| < 1$ onto D_n, $f(1) = w_n'$, and $f(-1) = w_n''$, Lemma 1 implies that $\operatorname{Re}\{(1 - z^2) f_n'(z)\} \geqslant 0$. Finally, (13) follows from the locally uniform convergence of f_n to f.

If D is unbounded only in the positive u-direction, define $D_n = \{w \in D\colon \operatorname{Re} w < \operatorname{Re} w_n'\}$ where w_n' is as before and define w_n'' to be the prime end $f(-1)$. Then the result follows by the above argument. A similar procedure works when D is unbounded only in the negative u-direction, and the lemma reduces to Lemma 1 if D is bounded in both directions.

3. Steiner Symmetric Domains

One can give examples of domains in Σ whose level domains are not in Σ. However, as an application of Theorem 1 we shall show that level domains of Steiner symmetric domains are Steiner symmetric.

DEFINITION. A set S is *Steiner symmetric with respect to the real axis if* $w \in S$ *implies* $tw + (1-t)\bar{w} \in S$ *for all* $t \in [0, 1]$.

LEMMA 3. *Suppose*

(a) f *is schlicht on* $|z| < 1$,

(b) $f(|z| < 1)$ *is Steiner symmetric with respect to the real axis,*

(c) $f(0)$ *is real,* $f'(0) > 0$, *and*

(d) $\operatorname{Re} f$ *is bounded.*

Then $\operatorname{Re}\{(1-z^2)f'(rz)\}>0$ *for* $|z|<1$ *and* $0<r\leqslant 1$.

Proof. Since $\operatorname{Re} f$ is bounded, we have the Poisson representation (see (8))

$$\operatorname{Re}\{f(z)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\theta)\operatorname{Re}\left\{\frac{e^{i\theta}+z}{e^{i\theta}-z}\right\} d\theta \tag{14}$$

where U is non-decreasing on $[-\pi, 0]$ and non-increasing on $[0, \pi]$. Since $f(|z|<1)$ is symmetric, $f(\bar{z})=\overline{f(z)}$, $U(-\theta)=U(\theta)$, and $\int_{-\pi}^{\pi} dU(\theta)=0$. It follows that

$$\left.\begin{aligned} f'(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\theta)\frac{\partial}{\partial z}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) d\theta = \frac{i}{2\pi z}\int_{-\pi}^{\pi} U(\theta)\frac{\partial}{\partial \theta}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) d\theta \\ &= \frac{-i}{2\pi z}\int_{\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, dU(\theta) \end{aligned}\right\} \tag{15}$$

and

$$f'(rz) = \frac{-i}{2\pi r z}\int_{-\pi}^{\pi}\left[\frac{e^{i\theta}+rz}{e^{i\theta}-rz} - \frac{1+z}{1-z}\right] dU(\theta) = \frac{-i}{\pi r}\int_{-\pi}^{\pi}\frac{(r-e^{i\theta})\, dU(\theta)}{(e^{i\theta}-rz)(1-z)}. \tag{16}$$

Therefore

$$\left.\begin{aligned} \operatorname{Re}\{(1-z^2)f'(rz)\} &= \left[\frac{(|z|^2-1)}{\pi}\int_{-\pi}^{\pi}\frac{(\sin\theta)\, dU(\theta)}{|e^{i\theta}-rz|^2}\right] \\ &+ \left[\frac{(r^2-1)\operatorname{Im}\{z\}}{\pi r}\int_{-\pi}^{\pi}\frac{dU(\theta)}{|e^{i\theta}-rz|^2}\right]. \end{aligned}\right\} \tag{17}$$

Using the symmetry we see that both terms in brackets are non-negative. Since $f'(0)>0$, $\operatorname{Re}\{(1-z^2)f'(rz)\}$ is strictly positive by the minimum principle for harmonic functions.

THEOREM 2. *Suppose D is a domain which is Steiner symmetric with respect to the real axis. Let f be a schlicht mapping of* $|z|<1$ *onto D such that* $f(0)$ *is real and* $f'(0)>0$. *Then the level sets* $f(|z|<r)$, $0<r<1$, *are also Steiner symmetric with respect to the real axis.*

Proof. Denote by f_n $(n=1, 2, 3, \ldots)$ the schlicht mapping of $|z|<1$ onto the domain $\{w\in D\colon -n<\operatorname{Re} w-f(0)<n\}$ such that $f_n(0)=f(0)$ and $f_n'(0)>0$. It follows from Lemma 3 that $\operatorname{Re}\{(1-z^2)f_n(rz)\}>0$. By the Caratheodory kernel theorem (cf. [1, p.

46]) the functions f_n converge locally uniformly to f. Therefore $\mathrm{Re}\{(1-z^2)f'(rz)\}>0$. By applying Theorem 1 to $g(z)=f(rz)$ we find that $g(|z|<1)=f(|z|<r)$ is convex in the v-direction. The symmetry with respect to the real axis is obvious because $f(\bar{z})=\overline{f(z)}$.

Remarks. M. S. Robertson [5] studied a class of mappings onto domains convex in the v-direction. He did not require our normalization (3), but rather assumed either regularity at the boundary or that level sets inherited the convexity property. In view of Theorem 2 mappings onto Steiner symmetric domains fall into his class. In general, however, the normalization we require gives us results of a different nature without any regularity assumptions.

R. S. Gupta [2] gave an erroneous proof of Theorem 2 (see Mathematical Reviews *37* (1969), #6452). It would also appear that the lower bound he asserts for $|f'(z)|$ for a Steiner symmetric mapping f is incorrect since it does not approach zero as $|z|\to 1$ as it must for any domain with a reentrant corner.

4. Coefficient Estimates

The following lemma is well known (e.g., [1, p. 167]).

LEMMA 4. *If* $g(z)=1+\sum_{n=1}^{\infty} c_n z^n$ *is analytic for* $|z|<1$ *and has positive real part, then* $|c_n|\leqslant 2$ *for all* n.

We now show that functions satisfying (1) have stringent coefficient bounds.

THEOREM 3. *If* $f(z)=a_0+(\alpha+i\beta)z+\sum_{n=2}^{\infty} a_n z^n$ *is analytic for* $|z|<1$ *and satisfies* $\mathrm{Re}\{(1-z^2)f'(z)\}\geqslant 0$, *then*

$$|a_n| \leqslant \alpha \qquad \text{for} \quad n=2,4,6,\ldots \tag{18}$$

and

$$|a_n| \leqslant \left(1-\frac{1}{n}\right)\alpha+\frac{1}{n}|\alpha+i\beta| \quad \text{for} \quad n=1,3,5,\ldots. \tag{19}$$

Consequently,

$$|a_n| \leqslant |f'(0)| \qquad \text{for} \quad n=1,2,3,4,\ldots. \tag{20}$$

Remark. Equality in (18), (19), and (20) is achieved by $f(z)=(1-z)^{-1}$. Moreover, (18) is sharp among bounds which depend on *both* $\alpha=\mathrm{Re}\{f'(0)\}$ *and* $\beta=\mathrm{Im}\{f'(0)\}$ for the functions

$$f(z)=\frac{\alpha}{1-z}+\frac{i\beta}{2}\log\frac{1+z}{1-z} \qquad (\alpha>0). \tag{21}$$

In addition, (19) is sharp in the same sense for $n=1$ (obviously) and for $n=3$ for the functions

$$\left.\begin{aligned} f(z) = {}& \tfrac{1}{2}\alpha(1-e^{2i\theta})^{-1}\left[(1-e^{i\theta})^2\log(1+z)-(1+e^{i\theta})^2\log(1-z)\right. \\ & \left.+4e^{i\theta}\log(1-e^{i\theta}z)\right]+\tfrac{1}{2}i\beta\log\frac{1+z}{1-z}, \qquad \theta=\tfrac{1}{2}\arg(\alpha+i\beta), \end{aligned}\right\} \tag{22}$$

for all $\alpha>0$ and β.

Proof of Theorem 3. If $\mathrm{Re}\{(1-z^2)f'(z)\}$ vanishes at some point, then from Remark 1 we have

$$f(z)=a_0+i\beta\log\frac{1+z}{1-z}=a_0+\sum_{k=0}^{\infty}\frac{i\beta}{2k+1}z^{2k+1}. \tag{23}$$

Clearly (18) and (19) are satisfied.

If $\mathrm{Re}\{(1-z^2)f'(z)\}$ does not vanish, then the function

$$g(z)=\frac{(1-z^2)f'(z)-i\beta}{\alpha}=1+\frac{1}{\alpha}\sum_{n=1}^{\infty}\left[(n+1)\,a_{n+1}-(n-1)\,a_{n-1}\right]z^n \tag{24}$$

satisfies the hypothesis of Lemma 4. Therefore for $n=1, 2, 3, \ldots$

$$\frac{1}{\alpha}\left|(n+1)\,a_{n+1}-(n-1)\,a_{n-1}\right|\leqslant 2 \tag{25}$$

and by the triangle inequality

$$(n+1)\,|a_{n+1}|\leqslant(n-1)\,|a_{n-1}|+2\alpha. \tag{26}$$

This implies

$$\left.\begin{aligned} n\,|a_n| &\leqslant 0\,|a_0|+n\alpha && \text{for} \quad n=2,4,6,\ldots, \quad \text{and} \\ n\,|a_n| &\leqslant 1\,|a_1|+(n-1)\,\alpha && \text{for} \quad n=1,3,5,\ldots, \end{aligned}\right\} \tag{27}$$

from which (18) and (19) follow.

5. Distortion Theorems

We turn to the distortion theory of functions satisfying (1).

LEMMA 5. *Suppose g is analytic for $|z|<1$ and has nonnegative real part. Then g*

has the representation

$$g(z)=\frac{g(0)+\overline{g(0)}\,G(z)}{1-G(z)} \tag{28}$$

where $|G(z)|\leqslant|z|$.

Proof. If $g(0)=0$, then $g\equiv 0$ and we may choose $G\equiv 0$. Otherwise, the function

$$G(z)=\frac{g(z)-g(0)}{g(z)+\overline{g(0)}} \tag{29}$$

is analytic for $|z|<1$, vanishes at the origin, and satisfies $|G(z)|\leqslant 1$. Therefore by Schwarz's lemma $|G(z)|\leqslant|z|$, and the representation follows.

THEOREM 4. *If* f *is analytic for* $|z|<1$ *and satisfies* $\operatorname{Re}\{(1-z^2)f'(z)\}\geqslant 0$, *then for* $|z|\leqslant r<1$

$$\frac{(1-r)\,|f'(0)|}{(1+r)(1+r^2)}\leqslant|f'(z)|\leqslant\frac{|f'(0)|}{(1-r)^2} \tag{30}$$

and for $|z|=r<1$

$$\tfrac{1}{2}|f'(0)|\log\frac{(1+r)^2}{1+r^2}\leqslant|f(z)-f(0)|\leqslant\frac{r\,|f'(0)|}{1-r}. \tag{31}$$

Remark. The upper bounds in (30) and (31) are sharp for

$$f(z)=\frac{z}{1-z}, \tag{32}$$

and the lower bounds in (30) and (31) are sharp for

$$f(z)=\frac{i}{2}\log\frac{(1-iz)^2}{(1-z^2)}. \tag{33}$$

It is interesting to note that both upper bounds are the same as for the class of convex functions, while the lower bounds are of a new form.

Proof of Theorem 4. By Lemma 5 we may write

$$(1-z^2)f'(z)=\frac{f'(0)+\overline{f'(0)}\,G(z)}{1-G(z)} \tag{34}$$

where $|G(z)|\leqslant|z|$. If $f'(0)=0$, then (30) follows immediately. If $f'(0)\neq 0$, then direct estimates of

$$|f'(z)|=|f'(0)|\left|1+\frac{\overline{f'(0)}}{f'(0)}G(z)\right||1-G(z)|^{-1}\,|1-z^2|^{-1} \tag{35}$$

for $|z| \leqslant r$ yield the upper bound $|f'(0)|(1+r)(1-r)^{-1}(1-r^2)^{-1} = |f'(0)|(1-r)^{-2}$ and the lower bound $|f'(0)|(1-r)(1+r)^{-1}(1+r^2)^{-1}$. Now the upper bound in (31) follows directly from (30) since for $|z| \leqslant r$

$$|f(z) - f(0)| = \left|\int_0^z f'(z)\,dz\right| \leqslant \int_0^r \frac{|f'(0)|}{(1-\varrho)^2}\,d\varrho = \frac{r\,|f'(0)|}{1-r}. \tag{36}$$

Finally, to verify the lower bound in (31), let ζ be a point where $|f(z) - f(0)|$ assumes its minimum on $|z| = r$. Then the straight line segment σ from $f(0)$ to $f(\zeta)$ lies in $f(|z|<1)$. Therefore, using the lower bound in (30), we have for $|z| = r$

$$|f(z) - f(0)| \geqslant |f(\zeta) - f(0)| = \int_\sigma |dw| = \int_{f^{-1}(\sigma)} |f'(z)|\,|dz|$$

$$\geqslant \int_0^r \frac{(1-\varrho)\,|f'(0)|}{(1+\varrho)(1+\varrho^2)}\,d\varrho = \tfrac{1}{2}|f'(0)| \log\frac{(1+r)^2}{1+r^2}.$$

The following theorem is an immediate consequence of the lower bound in (31).

THEOREM 5. *If f is analytic for $|z|<1$ and satisfies* $\mathrm{Re}\{(1-z^2)f'(z)\} \geqslant 0$, *then $f(|z|<1)$ contains all points of the disk $|w - f(0)| < \frac{1}{2}|f'(0)| \log 2$.*

Remarks. The constant $\frac{1}{2}\log 2$ is best possible. It is sharp for the function (33), which leaves the origin fixed, has $|f'(0)| = 1$, and maps the unit disk onto the vertical strip $|\mathrm{Re}\, w| < \pi/2$ with a vertical slit from $\frac{1}{2} i \log 2$ to ∞.

According to the Koebe-Bieberbach "Viertelsatz" for schlicht functions normalized by $f(0) = 0$, $|f'(0)| = 1$, the domain $f(|z|<1)$ contains a disk about the origin of radius $\frac{1}{4}$. For the subclass of convex mappings the constant can be improved to $\frac{1}{2}$. The corresponding constant $\frac{1}{2}\log 2$ of Theorem 5 is particularly interesting because it falls strictly between $\frac{1}{4}$ and $\frac{1}{2}$.

If $f(|z|<1)$ is Steiner symmetric, we shall show that the constant $\frac{1}{2}\log 2$ can be improved to $\frac{1}{2}$.

THEOREM 6. *Suppose*

(a) *f is schlicht on $|z|<1$,*

(b) *$f(|z|<1)$ is Steiner symmetric with respect to the real axis,*

(c) *$f(0)$ is real and $f'(0) > 0$.*

Then $f(|z|<1)$ contains all points of the disk $|w - f(0)| < \frac{1}{2} f'(0)$.

Remark. The theorem is sharp for the functions

$$F_{p,q}(z)=\frac{2z(|p+iq|-pz)}{1-z^2}, \quad p+iq\neq 0, \tag{38}$$

which map $|z|<1$ onto (i) the plane with vertical slits from $p\pm iq$ to ∞ if $q\neq 0$, or (ii) the half-plane defined by p^{-1} Re $w<1$ if $q=0$. In all cases the distance from $F'_{p,q}(0)=0$ to the boundary of $F_{p,q}(|z|<1)$ is $|p+iq|$, which one verifies is $\frac{1}{2}F'_{p,q}(0)$.

Proof of Theorem 6. Without loss of generality we may assume that $f(0)=0$. Let ϱ be the distance from $f(0)=0$ to the boundary of $D=f(|z|<1)$. Then there exists at least one point $p+iq\in\partial D$ with $|p+iq|=\varrho$. Since D is Steiner symmetric and $p+iq\notin D$, we have $D\subset F_{p,q}(|z|<1)$ where the function $F_{p,q}$ is defined in (38). Now applying Schwarz's lemma to $F^{-1}_{p,q}\circ f$, we find $f'(0)\leqslant F'_{p,q}(0)$. Therefore

$$\varrho=|p+iq|=\tfrac{1}{2}F'_{p,q}(0)\geqslant\tfrac{1}{2}f'(0). \tag{39}$$

6. Concluding Remarks

We supplement our discussion by considering the classes of domains

$$\Sigma_+=\{D:(u,v)\in D\Rightarrow(u,v+t)\in D\ \forall t\geqslant 0\} \tag{40}$$

and

$$\Sigma_-=\{D:(u,v)\in D\Rightarrow(u,v+t)\in D\ \forall t\leqslant 0\}. \tag{41}$$

These classes contain all domains which are convex in the v-direction but not in Σ (compare Section 2). Some domains of Σ are also included, e.g., vertical strips. By convention we remove the entire plane from Σ_+ and Σ_-. Then for $D\in\Sigma_+\cup\Sigma_-$ there exist schlicht mappings f of $|z|<1$ onto D normalized so that there exist points z_n, z'_n converging to $z=1$ such that

$$\left.\begin{aligned}\lim_{n\to\infty}\operatorname{Re}\{f(z_n)\}&=\inf_{|z|<1}\operatorname{Re}\{f(z)\\ \lim_{n\to\infty}\operatorname{Re}\{f(z'_n)\}&=\sup_{|z|<1}\operatorname{Re}\{f(z)\}.\end{aligned}\right\} \tag{42}$$

By analogy to Theorem 1 we note the following: *Suppose f is analytic and non-constant for* $|z|<1$. *Then we have*

$$\operatorname{Im}\{(1-z)^2f'(z)\}\geqslant 0,\quad |z|<1, \tag{43}$$

if and only if

(a) *f is schlicht on* $|z|<1$,

(b) $f(|z|<1)\in\Sigma_+$, *and*

(c) *f is normalized by* (42).

A similar characterization holds if (43) *is replaced by*

$$\operatorname{Im}\{(1-z)^2 f'(z)\} \leqslant 0, \quad |z|<1, \tag{43'}$$

and (b) *by*

$(b')\, f(|z|<1) \in \Sigma_-$.

Verification of (a)–(c) from (43) or (43′) follows the same pattern as Remarks 1–3. The conditions (43) and (43′) mean geometrically that circles tangent to $z=1$ in $|z|<1$ correspond to analytic arcs which may be represented as functions $v=v(u)$. In addition, functions satisfying (43) and (43′) are close-to-convex, hence schlicht, with convex comparison functions $\varphi(z)=\pm i(1-z)^{-1}$. The verification of (43) or (43′) from (a)–(c) follows the form of Lemma 1 (see also Kaplan [3, p. 181]) and an exhaustion argument as in Lemma 2.

Since the Koebe functions $f(z)=\pm iz(1-z)^{-2}$ satisfy (43) and (43′), respectively, coefficient bounds for functions satisfying (43) or (43′) can be no better than $|a_n|\leqslant n|a_1|$, which holds for the entire class of close-to-convex functions [4]. For the same reason the elementary distortion theory corresponding to Theorems 4 and 5 for functions satisfying (43) or (43′) is the same as for the general class of schlicht functions.

REFERENCES

[1] G. M. Golusin, *Geometrische Funktionentheorie*, German edition (VEB Deutscher Verlag der Wissenschaften, Berlin 1957).
[2] R. S. Gupta, *On Steiner-symmetrized functions*, Ganita *18* (1967), 17–20.
[3] W. Kaplan, *Close-to-convex schlicht functions*, Mich. Math. J. *1* (1952), 169–185.
[4] M. O. Reade, *On close-to-convex univalent functions*, Mich. Math. J. *3* (1955), 59–62.
[5] M. S. Robertson, *Analytic functions star-like in one direction*, Amer. J. Math. *58* (1936), 465–472.

Received November 26, 1969