Perturbation of Closed Operators and Their Adjoints.

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1. Introduction

Let T denote a densely defined linear operator of a Banach space X into a Banach space Y, and let the linear operator A be bounded from X into Y with domain $D(A)=X$. Then the adjoint of $T+A$ is T^*+A^*. In general, if both T and A are densely defined, but unbounded, it is only known that $(T+A)^* \supset T^*+A^*$, provided $D(T)\cap D(A)$ is dense in X. In this note we show that $(T+A)^*=T^*+A^*$ for perturbations A sufficiently small with respect to T measured in the gap topology on the set $C(X, Y)$ of closed linear operators with domain in X and range contained in Y.

THEOREM. *Let X, Y be Banach spaces, $\{T(t)\}_{0\leqslant t\leqslant 1}$ a (in the gap topology) continuous family of densely defined operators in $C(X, Y)$, and $\{S(t)\}_{0\leqslant t\leqslant 1}$ a continuous family of operators in $C(Y^*, X^*)$. Assume $S(t)\subset T(t)^*$ for all t. If $S(t)=T(t)^*$ for some t, then the equality holds for all t.*

If T and A are two linear operators from X into Y, with $D(A)\supset D(T)$, then A is said to be T-bounded if there exist constants $a, b\geqq 0$ such that $\|Au\|\leqq a\|u\|+b\|Tu\|$ for all $u\in D(T)$. The greatest lower bound of all possible values $b\geqq 0$ is called T-bound of A. For $T\in C(X, Y)$ and A a T-bounded operator with relative bound smaller than one, the family $\{T(t)=T+tA\}_{0\leqslant t\leqslant 1}$ of operators in $C(X, Y)$ is continuous in the gap topology ([4], theorem IV. 2.14). Thus, we get as a consequence the following result, which is of interest in the applications:

COROLLARY 1. *Let X, Y denote Banach spaces, and let T be a densely defined operator in $C(X, Y)$. Suppose A is a T-bounded operator such that A^* is T^*-bounded, with both relative bounds smaller than one. Then $T+A$ is closed, and $(T+A)^*=T^*+A^*$.*

This Corollary leads to a new proof of the interior regularity of weak solutions of linear elliptic partial differential equations.

We finally generalize Corollary 1 in a way to observe that the property of linear manifolds to be cores[1]) of operators T, T^* is invariant under small perturbations of these operators. This result extends a similar assertion by F. E. Browder ([1], Theorem 22) inasmuch as we do not assume any existence of regular points of the operator T.

It was pointed out by Professor Browder that our results are strongly connected to those in [2] and [3].

[1]) A linear manifold D_0 contained in the domain of a closed linear operator T is called a *core* of T if the closure of the restriction of T to D_0 is again T.

2. Proof of the Theorem

We consider first closed linear manifolds $M, N, \ldots$ of a Banach space Z. Let S_M denote the unit sphere of M. For any two closed linear manifolds M, N of Z, we set [2])

$$\delta(M, N) = \sup_{u \in S_M} \operatorname{dist}(u, N), \tag{1}$$

$$\hat{\delta}(M, N) = \max(\delta(M, N), \delta(N, M)). \tag{2}$$

If $M=0$, (1) has no meaning; in this case we define $\delta(0, N)=0$ for any N. $\hat{\delta}(M, N)$ is called the *gap* between M and N. Though $\hat{\delta}$ is in general not a proper distance function, it is equivalent to such a function. In [4], Lemma IV.2.2, it is proved that if M, N are two closed linear manifolds of Z, and if $u \in Z$, then

$$(1 + \delta(M, N)) \operatorname{dist}(u, M) \geqq \operatorname{dist}(u, N) - \|u\| \cdot \delta(M, N). \tag{3}$$

LEMMA 1. *Let Z be a Banach space, M, N, N' closed linear manifolds in Z, where $N' \subset N$. If*

$$\delta(N, M) < \tfrac{1}{3}, \quad \delta(M, N') < \tfrac{1}{3}, \tag{4}$$

then $N' = N$.

Proof. Suppose $N' \neq N$. For any $\varepsilon > 0$, there exists an element $u \in N$ such that $\|u\| = 1$ and $\operatorname{dist}(u, N') > 1 - \varepsilon$ (see [4], Lemma III.1.12). Replacing N by N' in (3) we get

$$(1 + \delta(M, N')) \operatorname{dist}(u, M) \geqq \operatorname{dist}(u, N') - \|u\| \, \delta(M, N') > 1 - \varepsilon - \delta(M, N').$$

Since $\delta(N, M) \geqq \operatorname{dist}(u, M)$ and $\varepsilon > 0$ is arbitrary, we obtain

$$(1 + \delta(M, N')) \, \delta(N, M) \geqq 1 - \delta(M, N'),$$

which is a contradiction because of (4), q.e.d.

For closed linear manifolds M, N of Z, with $M^{\perp}$, $N^{\perp}$ denoting the orthogonal complement in Z^* of M and N respectively, we have ([4], Theorem IV.2.9)

$$\hat{\delta}(M, N) = \hat{\delta}(M^{\perp}, N^{\perp}). \tag{5}$$

LEMMA 2. *Let $\{M(t)\}_{0 \leq t \leq 1}$ be a (in the gap topology) continuous family of closed linear manifolds of a Banach space Z, and let $\{N(t)\}_{0 \leq t \leq 1}$ be a similar family in the dual space Z^*. Assume $N(t) \subset M(t)^{\perp}$ for all t. If $N(t) = M(t)^{\perp}$ holds for some t, then it holds for all t.*

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[2]) see [4], chapt. IV, § 2.

Proof. The families $\{M(t)\}$ and $\{N(t)\}$ are uniformly continuous on $[0, 1]$. Also $\{M(t)^{\perp}\}$ has the same property by (5).

Suppose $N(t_0)=M(t_0)^{\perp}$ for some $t_0\in[0, 1]$. By the stated uniform continuity, there is an $\varepsilon>0$ (independent of t_0) such that $\hat{\delta}(N(t), M(t_0)^{\perp})=\hat{\delta}(N(t), N(t_0))<\frac{1}{3}$ and $\hat{\delta}(M(t)^{\perp}, M(t_0)^{\perp})<\frac{1}{3}$ if $|t-t_0|<\varepsilon$. Then $N(t)=M(t)^{\perp}$ by Lemma 1. Thus the desired property for t propagates to all $t\in[0, 1]$ in a finite number of steps, q.e.d.

Let us now consider the set $C(X, Y)$ of all closed linear operators from X into Y. If $S, T\in C(X, Y)$, their graphs $G(S)$, $G(T)$ are closed linear manifolds of the Banach space $Z=X\times Y$, where the norm is chosen to be $\|\{u, v\}\|=(\|u\|^2+\|v\|^2)^{1/2}$ for $u\in X$, $v\in Y$, implying that $Z^*=X^*\times Y^*$. We set

$$\delta(S, T)=\delta(G(S), G(T)),\quad \hat{\delta}(S, T)=\hat{\delta}(G(S), G(T)); \tag{6}$$

$\hat{\delta}(S, T)$ is then called the *gap* between S and T.

Proof of the Theorem. We apply Lemma 2 with $Z=X\times Y$, $Z^*=X^*\times Y^*$, $M(t)=G(T(t))$ and $N(t)=G'(-S(t))$ (the inverse graph of $-S(t)$), noting that $M(t)^{\perp}=G(T(t))^{\perp}=G'(-T(t)^*)\supset G'(-S(t))=N(t)$ for all t, and that $\hat{\delta}(G(S_1), G(S_2))=\hat{\delta}(G'(S_1), G'(S_2))$, q.e.d.

3. Interior Regularity of Solutions of Linear Elliptic Equations

In the Sobolev spaces $H^s(R^n)$ (s an integer) being the completion of the set $C_0^\infty(R^n)$ with respect to the norm $\|u\|_s^2=\int(1+|k|^2)^s|\hat{u}(k)|^2\,dk$ (where $\hat{u}$ denotes the Fourier transformed of the function u), we consider linear elliptic differential expressions[3] $\mathcal{S}=\sum_{|p|\leqslant m}a_p(x)\,D^p$ of order m with coefficients $a_p\in C^\infty(R^n)$ which are bounded together with all their derivatives. Under these assumptions, $\mathcal{S}$ maps the space $H^{s+m}(R^n)$ continuously into $H^s(R^n)$. Let $\mathcal{S}^*$ denote the formal adjoint expression, $\mathcal{S}^*=\sum_{|p|\leqslant m}(-1)^{|p|}\,D^p(\bar{a}_p(x).)$. Then[4] $(\mathcal{S}^*u, v)_0=(u, \mathcal{S}v)_0$ for all u, $v\in C_0^\infty(R^n)$ and, by a limiting process, for all $u\in H^s(R^n)$, $v\in H^t(R^n)$ with $s+t\geqq m$.

The following theorem on interior regularity of weak solutions of linear elliptic equations is well known:

THEOREM. *Let $\mathcal{S}$ be a linear elliptic differential expression of order m, defined in an open domain $\Omega\subset R^n$, with coefficients in $C^\infty(\Omega)$. Suppose $u\in L^2_{\mathrm{loc}}$ is a weak solution of the equation $\mathcal{S}u=f$, $f\in H^t_{\mathrm{loc}}$ $(t\geqq 0)$, i.e.*

$$(u, \mathcal{S}^*\varphi)_0=(f, \varphi)_0$$

for all $\varphi\in C_0^\infty(\Omega)$. Then $u\in H^{t+m}_{\mathrm{loc}}$.

[3] Script letters denote formal differential expressions, latin letters the induced differential operators acting in some Sobolev space $H^s(R^n)$, with prescribed domain of definition.

[4] $(.,.)_0$ denotes the L^2 scalar product.

The theorem can be reduced to a similar *global* assertion on the solutions of elliptic equations involving expressions with "almost constant" coefficients, defined on the whole of R^n. It is rather easy to show interior regularity of solutions of elliptic equations $\mathcal{T}u=f$ with $\mathcal{T}$ having constant coefficients. Based on these properties of elliptic expressions with constant coefficients which we assume to be known, we prove as an application of Corollary 1 the following

PROPOSITION. *Let* $\mathcal{S}=\sum_{|p|\leqq m} a_p(x)\, D^p$ *be a linear elliptic differential expression with coefficients* $a_p\in C^\infty(R^n)$ *which are bounded together with all their derivatives. Suppose there exists* $x_0\in R^n$ *such that with the notation* $a_p^0=a_p(x_0)$,

$$\Big|\sum_{|p|=m} a_p^0\xi^p\Big| \geqq \alpha|\xi|^m, \quad \Big|\sum_{|p|=m} (a_p(x)-a_p^0)\,\xi^p\Big| \leqq \beta|\xi|^m \tag{7}$$

for all $\xi\in R^n$, *where* $\beta<\alpha$.
If $u\in H^s(R^n)$ *and* $\mathcal{S}u=f\in H^s(R^n)$, *then* $u\in H^{s+m}(R^n)$.

Proof. Let $\mathcal{T}=\sum_{|p|\leqq m} a_p^0 D^p$ denote the elliptic differential expression with constant coefficients, and set $\mathcal{A}=\sum_{|p|\leqq m}(a_p(x)-a_p^0)\,D^p$. Let T be the operator induced by $\mathcal{T}$ in the space $H^s(R^n)$, with $D(T)=H^{s+m}(R^n)$. The first assumption in (7) implies that to each $\varepsilon>0$ there exists a constant $b(\varepsilon)$ such that

$$\|Tu\|_s \geqq (\alpha-\varepsilon)\,\|u\|_{s+m} - b(\varepsilon)\,\|u\|_s \tag{8}$$

for all $u\in D(T)$, as is seen by applying the Gårding inequality to the elliptic differential expression $\mathcal{T}^*\mathcal{T}-(\alpha-\varepsilon)^2(1+\Delta)^m$. We infer that the operator T is closed. The spaces $H^s(R^n)$ and $H^{-s}(R^n)$ being mutually adjoint by the scalar product $(.,.)_0$, T^* is a linear operator in $H^{-s}(R^n)$, induced by $\mathcal{T}^*$. Because of the regularity properties of elliptic mappings with constant coefficients, T^* has domain $H^{-s+m}(R^n)$. Since $\mathcal{T}$ and $\mathcal{T}^*$ have conjugate complex principal parts, for each $\varepsilon>0$ an estimate of the form

$$\|T^*v\|_{-s} \geqq (\alpha-\varepsilon)\,\|v\|_{-s+m} - b^*(\varepsilon)\,\|v\|_{-s} \tag{8a}$$

holds for all $v\in D(T^*)$.

Let A and A^* be the operators induced by $\mathcal{A}$ and $\mathcal{A}^*$, respectively, acting in the same spaces as T and T^*, and with domains $D(A)=D(T)$, $D(A^*)=D(T^*)$. As a consequence of the second estimate in (7) and of Gårdings inequality, to each $\varepsilon>0$ there exist constants $c(\varepsilon, s)$ and $c^*(\varepsilon, s)$ such that

$$\|Au\|_s \leqq (\beta+\varepsilon)\,\|u\|_{s+m} + c(\varepsilon,s)\,\|u\|_s, \quad u\in D(T), \tag{9}$$

$$\|A^*v\|_{-s} \leqq (\beta+\varepsilon)\,\|v\|_{-s+m} + c^*(\varepsilon,s)\,\|v\|_{-s}, \quad v\in D(T^*). \tag{9a}$$

It follows from the inequalities (8), (9) as well as (8a), (9a) that A and A^* are relatively bounded with respect to T and T^*, with bounds smaller than one.

For all $v \in D(T^*)$,

$$(v, f)_0 = (v, (\mathscr{T} + \mathscr{A}) u)_0 = ((\mathscr{T}^* + \mathscr{A}^*) v, u)_0 = ((T^* + A^*) v, u)_0 .$$

Therefore, applying Corollary 1 (with $X = Y = H^{-s}(R^n)$) and making use of the reflexivity of the Banach spaces $H^s(R^n)$,

$$u \in D((T^* + A^*)^*) = D(T^{**} + A^{**}) = D(T + A^{**}) = D(T + A)$$
$$= D(T) = H^{s+m}(R^n), \quad \text{q.e.d.}$$

4. *Invariance of cores under perturbations.* The following slight generalization of Corollary 1 extends a result by Browder [1].

COROLLARY 2. *Let X, Y be Banach spaces, $T \in C(X, Y)$ densely defined, and suppose D_0 is a core of T, D_1 a core of T^*. Let further $A \in C(X, Y)$ with $D(A) \supset D_0$, $D(A^*) \supset D_1$, such that*

$$\|Au\| \leqq a \|u\| + b \|Tu\|, u \in D_0,$$

$$\|A^* v\| \leqq a \|v\| + b \|T^* v\|, v \in D_1,$$

with $0 \leqslant b < 1$. Then $S = T + A$ is closed with $D(S) = D(T)$, $S^* = T^* + A^*$ and has domain $D(S^*) = D(T^*)$, and D_0 and D_1 are cores of S and S^*, respectively.

Proof. By T_0 we denote the restriction of T to D_0. Similar notations are used for restrictions of other operators. The T_0-boundedness of A_0 implies the T-boundedness of the closure $\tilde{A}_0$ of A_0 with conservation of the relative bound. Hence $S = T + A = T + \tilde{A}_0$ is closed, and $D(S) = D(T)$. By the same argument, A^* is T^*-bounded with bound smaller one, and consequently $S^* = (T + A)^* = T^* + A^*$ with $D(S^*) = D(T^*)$ by Corollary 1. It remains to show that D_0 and D_1 are cores of S and S^*, respectively.

Evidently $\tilde{S}_0 \subset S$. Conversely suppose $u \in D(S) = D(T)$. Then it exists a sequence $\{u_n\} \subset D_0$ with $u_n \to u$, $T_0 u_n \to Tu$. Therefore $A_0 u_n \to Au$, and we conclude that $u_n \to u$ and $S_0 u_n \to Su$. Consequently $u \in D(\tilde{S}_0)$ and $\tilde{S}_0 u = Su$, i.e. $\tilde{S}_0 \supset S$. It is analoguously seen that $(S_1^*)^\sim \subset S^*$ and $(S_1^*)^\sim \supset T^* + A^* = S^*$, q.e.d.

Example. Let T be an elliptic differential operator of order m with constant coefficients, acting in some Sobolev space $H^s(R^n)$, with domain $H^{s+m}(R^n)$. Its adjoint T^* in $H^{-s}(R^n)$ has domain $H^{-s+m}(R^n)$. It is obvious that $C_0^\infty(R^n)$ is a core of the operators T and T^*.

Consider now an elliptic differential expression $\mathscr{S}$ of order m satisfying the hypotheses of the Proposition. Let S be the operator induced by $\mathscr{S}$ in $H^s(R^n)$, with $D(S) = H^{s+m}(R^n)$. Then S^* acts in the space $H^{-s}(R^n)$ and has domain $H^{-s+m}(R^n)$. Further $C_0^\infty(R^n)$ is core of S and S^*.

REFERENCES

[1] BROWDER, F. E.: *On the spectral theory of elliptic differential operators* I. Math. Ann. *142*, 22–130 (1961).
[2] ——, *A continuity property for adjoints of closed operators in Banach spaces, and its application to elliptic boundary value problems.* Duke Math. Journal *28*, 157–182 (1961).
[3] ——, *Non-local elliptic boundary value problems.* Amer. Journ. Math. *86*, 735–750 (1964).
[4] KATO, T.: *Perturbation theory for linear operators.* Springer-Verlag Berlin/Heidelberg/New York (1966).

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Added in Proof. Making use of Corollary 1 and an argument on holomorphic operator families of type (A) ([4], chapt. VII, § 2) closely related to a method applied by R. Wüst (*Stabilität der Selbstadjungiertheit gegenüber Störungen*; Dissertation, Rhein.-Westfäl. Techn. Hochschule Aachen, 1970), one can prove the following.

PROPOSITION. *Let $T \in C(X, Y)$ be densely defined, and let A be a linear operator from X into Y having the property that $D(A) \supset D(T)$* and *$D(A^*) \supset D(T^*)$. Suppose that for each $t \in [0, 1]$, the operators $T + tA$* and *$T^* + tA^*$ are closed. Then $(T+A)^* = = T^* + A^*$.*

This result allows especially to discuss the limit case of Corollary 1 where the relative bounds of A and A^* with respect to T and T^* equal one.

LEMMA 3. *Let $T \in C(X, Y)$, and let A be T-bounded with T-bound one. Then the following assertions are equivalent:*

(i) *$T + A$ is closed;*

(ii) *There exist constants $a, b > 0$ such that for all $u \in D(T)$,*
$\|Au\| \leqslant a\|u\| + b\|(T+A)u\|$.