Nonlinear Functional Equations and Eigenvalue Problems in Nonseparable Banach Spaces[1])

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1. Let X be a real reflexive Banach space and A, B nonlinear mappings of X into the conjugate space X^*, with A of monotone type and B compact. In the last years, much interest in nonlinear functional analysis has been concentrated on the problem of determining useful conditions under which the functional equation

$$Au = 0 \tag{1}$$

or the eigenvalue problem

$$Au = tBu \quad \text{for some real } t \tag{2}$$

admit solutions (which possibly satisfy additional restrictions).

For A satisfying certain asymptotic conditions (such as A coercive or A^{-1} bounded), various results on the solvability of equation (1) have been obtained (e.g. Brézis [3], Browder [4, 6, 8, 9], Browder-Hess [13], Leray-Lions [22], Minty [23]). There is an alternative type of hypothesis one may impose on the mapping A in order to get existence theorems for equation (1), namely the hypothesis that $A = A_0$ is homotopic to a mapping A_1 which commutes with a group $\mathscr{G}$ of transformations on the spaces X and X^*, with $\mathscr{G}$ having elements of finite order (in particular A_1 odd). Under the assumption that X is *separable*, several mathematicians have derived existence theorems involving homotopy arguments, making use of an approximation method of Galerkin type (e.g. Browder [8, 9, 10, 11], Browder-Petryshyn [14]). (For a completely different approach see Hess [19]). Though most of the concrete reflexive Banach spaces occurring in applications are separable, it is necessary for the investigation of certain specific problems to have a similar approach in nonseparable spaces. For that reason, Nečas [24] has recently given a method which works in nonseparable spaces, and which is extended in the writer's papers [17, 18].

One way of attacking the eigenvalue problem (2), is by variational methods (e.g. Browder [5], Hess [16], Krasnoselskii [21], Vainberg [26]). In [7, 8], Browder has developed a theory for nonlinear eigenvalue problems in *separable* spaces based on Galerkin approximations. This latter approach has the advantage that it does not involve the theory of infinite-dimensional manifolds (Lusternik's principle), and that it permits to prove the existence of an infinite number of distinct normalized eigenfunctions (Lusternik-Schnirelman theory) under milder differentiability hypotheses.

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It is our purpose in the present note to describe an easy argument of Galerkin approximation type which allows to prove both existence theorems and results on eigenvalue problems in *nonseparable* Banach spaces. In contrary to the Galerkin approximation method in separable spaces, which is based on an a priori given injective approximation scheme, our method consists in recursively constructing a suitable scheme. The main result is the Proposition proved in Section 2. In Section 3 we apply the conclusions of the Proposition to the functional equation (1), assuming that $A=A_0$ is homotopic to an odd mapping A_1. The result is closely related to that of Nečas [24], but it seems that our proof is simpler. A brief discussion follows of how our theory can be used in order to study nonlinear equations of Hammerstein type in nonreflexive Banach spaces. In Section 4 we finally show the applicability of the Proposition to the treatment of nonlinear eigenvalue problems in nonseparable spaces.

2. For X a real Banach space and X^* its conjugate space, we let (w, u) denote the duality pairing between elements $w\in X^*$ and $u\in X$. An operator A defined on a closed set $C\subset X$, with range contained in X^*, is said to be of *type (S)* if it satisfies the condition: *for any sequence* $\{u_n\}\subset C$ *converging weakly to some* $u\in X$, *for which* $\lim(Au_n, u_n-u)=0$, *its strong convergence follows.* Mappings of type (S) have been introduced by Browder [7] and have shown to form a very useful class of operators of monotone type for homotopy considerations and eigenvalue problems. The mapping A is further *bounded* if it maps bounded sets onto bounded sets. Let Λ be the set of all finite-dimensional subspaces of X, ordered by inclusion. For $F\in\Lambda$, j_F denotes the injection mapping of F into X. If the operator A maps $C\subset X$ into X^*, the *Galerkin approximant* $A_F: C\cap F\to F^*$ is defined by $A_F=j_F^*Aj_F$. In the following we use the symbols "$\to$" and "$\rightharpoonup$" to denote strong and weak convergence, respectively.

PROPOSITION. *Let X a real reflexive Banach space, C a closed subset of X, I a closed interval in R^1, and $A(u, t)$ a mapping of $C\times I$ into X^* with the following properties:*

(i) *For fixed t, $A(u, t): C\to X^*$, is bounded, continuous, and of type (S);*

(ii) *$A(u, t)$ is uniformly continuous in t with respect to u in bounded subsets of C.*

Let $\{E_n\}_{n=1}^{\infty}$ be a given increasing sequence in Λ with $C\cap E_1\neq\emptyset$. Suppose to each $F\in\Lambda$ with $F\supset E_1$ there exist elements $u_F\in C\cap F$ and $t_F\in I$ such that $j_F^*A(u_F, t_F)=0$, and assume said elements are uniformly bounded for $F\supset E_1$.

Then $A(u_0, t_0)=0$ for some $u_0\in C$ and $t_0\in I$. Moreover, there exists an increasing sequence $\{F_n\}$ in Λ with $F_n\supset E_n$ for each n, such that for some subsequence $\{n(k)\}$ of $\{n\}$, $u_{F_{n(k)}}\to u_0$ and $t_{F_{n(k)}}\to t_0$.

Proof. We construct the asserted sequence $\{F_n\}$ in Λ as follows:

(a) We set $F_1=E_1$.

(b) Suppose we have already constructed $F_1 \subset \cdots \subset F_n$, and let $u_n = u_{F_n} \in C \cap F_n$ and $t_n = t_{F_n} \in I$ denote the described elements corresponding to F_n such that $j^*_{F_n} A(u_n, t_n) = 0$. There exists $v_n \in X$, $\|v_n\| = 1$, such that $|(A(u_n, t_n), v_n)| \geqq \frac{1}{2} \|A(u_n, t_n)\|$. We then choose $F_{n+1} \supset F_n + E_{n+1} + [v_n]$.

By hypothesis, the sequences $\{u_n\}$ and $\{t_n\}$ are bounded. We may pass to infinite subsequences and assure that $u_n \rightharpoonup u_0 \in X$ and $t_n \to t_0 \in I$. It follows from condition (ii) that

$$\|A(u_n, t_n) - A(u_n, t_0)\| \to 0 \quad (n \to \infty). \tag{3}$$

We assert that

$$(A(u_n, t_0), w) \to 0 \quad (n \to \infty) \tag{4}$$

for all $w \in X_0 = \text{closure}\{\bigcup_{j=1}^{\infty} F_j\}$. Indeed, if w lies in some F_j and $n \geqq j$, we have

$$|(A(u_n, t_0), w)| \leqq |(A(u_n, t_n), w)| + |(A(u_n, t_0) - A(u_n, t_n), w)|,$$

where the first term on the right side vanishes, while the second term tends to 0 as $n \to \infty$ according to (3). Because of the boundedness of the sequence $\{A(u_n, t_0)\}$, (4) extends to all $w \in X_0$. We now get

$$|(A(u_n, t_0), u_n - u_0)| \leqq |(A(u_n, t_n), u_n)| + |(A(u_n, t_0) - A(u_n, t_n), u_n)| + |(A(u_n, t_0), u_0)|.$$

On the right side of this estimate, the first summand vanishes, the middle term tends to 0 because of (3), and the last approaches 0 according to (4), since the weak limit u_0 of the sequence $\{u_n\} \subset X_0$ lies in X_0. Property (S) of the mapping $A(u, t_0)$ implies that $u_n \to u_0$. Hence $u_0 \in C$, $A(u_n, t_0) \to A(u_0, t_0)$, and

$$A(u_n, t_n) \to A(u_0, t_0) \tag{5}$$

because of the continuity of the mapping $A(u, t_0)$ in u and the estimate (3). We infer that, according to (4),

$$(A(u_0, t_0), w) = 0 \quad \text{for all} \quad w \in X_0. \tag{6}$$

We finally prove that $A(u_0, t_0) = 0$. Suppose to the contrary that $A(u_0, t_0) \neq 0$. Then, by (5), $\|A(u_n, t_n)\| \geqq d > 0$ for some constant d and all $n \geqq n_0$, which implies that

$$|(A(u_n, t_n), v_n)| \geqq d/2 > 0$$

for $n \geqq n_0$. But (5) and the fact that some subsequence of $\{v_n\}$ (denoted again by $\{v_n\}$) converges weakly to an element $v_0 \in X_0$ have as a consequence that

$$(A(u_n, t_n), v_n) \to (A(u_0, t_0), v_0),$$

the expression on the right being 0 according to (6). This contradiction shows that $A(u_0, t_0)=0$, q.e.d.

3. We apply the Proposition in order to obtain results on the existence of solutions of the functional equation (1).

THEOREM 1 [2]). *Let X a real reflexive Banach space, G an open bounded subset of X containing 0 and symmetric about the origin, and $A_t u = A(u, t)$ a mapping of* $\mathrm{cl}(G) \times [0, 1]$ *into X^* as follows:*

(i) *For fixed t, A_t is a bounded continuous mapping of type (S);*

(ii) *$A(u, t)$ is uniformly continuous in t with respect to $u \in \mathrm{cl}(G)$;*

(iii) *A_1 is odd on* $\mathrm{bdry}(G)$, *i.e. $A(-u, 1) = -A(u, 1)$ for $u \in \mathrm{bdry}(G)$.*

Assume that $A(u, t) \neq 0$ for all $u \in \mathrm{bdry}(G)$ and all $t \in [0, 1]$. Then the equation $A_0 u = 0$ has a solution u_0 in G.

Theorem 1 follows by the classical Borsuk theorem [2, 15, 21], the invariance of the Brouwer degree under homotopies, and arguments which have become standard in the theory of mappings of monotone type (e.g. [3, 4, 6, 8, 9, 13, 17, 18, 22, 23]) from

LEMMA 1. *Let $E \in \Lambda$ be given. Then under the assumptions of Theorem 1 there exists $F \in \Lambda$, $F \supset E$, such that $j_F^* A(u, t) \neq 0$ for all $u \in \mathrm{bdry}(G) \cap F$ and all $t \in [0, 1]$.*

Proof of Lemma 1. Suppose to each $F \in \Lambda$ with $F \supset E$ we can find elements $u_F \in \mathrm{bdry}(G) \cap F$ and $t_F \in [0, 1]$ such that $j_F^* A(u_F, t_F) = 0$. Applying the Proposition with $C = \mathrm{bdry}(G)$ and $I = [0, 1]$, we are led to a contradiction to the assumptions of Theorem 1, q.e.d.

DEFINITION. *A mapping A from X to X^* is said to be pseudo-monotone if for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ and* $\limsup (Au_n, u_n - u) \leqq 0$, *it follows that for all $v \in X$,* $\liminf (Au_n, u_n - v) \geqq (Au, u - v)$.

Pseudo-monotone mappings have been introduced by Brézis [3] and have grown increasingly important in the discussion of nonlinear elliptic boundary value problems [3, 11, 13, 22]. Everywhere defined continuous monotone operators from X to X^* (i.e. mappings A satisfying $(Au - Av, u - v) \geqq 0$ for all u, v in X) are pseudo-monotone.

For pseudo-monotone operators we have the following extension of Theorem 1.

THEOREM 2. *Let G a convex open bounded subset of the real reflexive Banach space X, with $0 \in G$ and G symmetric about 0. Suppose the mapping $A_t u = A(u, t): X \times [0, 1] \to X^*$ satisfies the conditions:*

(i) *For fixed t, A_t is bounded, continuous, and pseudo-monotone;*

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[2]) For G a subset of a Banach space, $\mathrm{cl}(G)$ denotes its closure and $\mathrm{bdry}(G)$ its boundary.

(ii) $A(u, t)$ *is continuous in* t*, uniformly with respect to* $u \in \mathrm{cl}(G)$;

(iii) A_1 *is odd on* bdry(G).

If there exists $\varepsilon > 0$ *such that* $\|A(u, t)\| \geqq \varepsilon$ *for all* $u \in \mathrm{bdry}(G)$ *and* $t \in [0, 1]$*, then the equation* $A_0 u = 0$ *is solvable in* G.

Proof. By a recent result of Troyanski [25] we can assume without loss of generality that both X and X^* are locally uniformly convex spaces. Let J denote the (single-valued) normalized duality mapping from X to X^* given by

$$Ju = \{q \in X^* : (q, u) = \|q\| \|u\|, \|q\| = \|u\|\}.$$

For each $\lambda > 0$ and $t \in [0, 1]$, the mapping $B_t^{(\lambda)} = A_t + \lambda J$ is then continuous and of type (S). By the boundedness of G, there exists $\varepsilon_0 > 0$ such that $B_t^{(\lambda)} u \neq 0$ for all $u \in \mathrm{bdry}(G)$, $t \in [0, 1]$, and $0 \leqq \lambda < \varepsilon_0$. Hence for fixed $\lambda \in (0, \varepsilon_0)$, the mapping $B_t^{(\lambda)} u$ satisfies the assumptions of Theorem 1, and there exists an element $u_\lambda \in G$ with $(A_0 + \lambda J) u_\lambda = 0$. Taking a sequence $\{\lambda_n\} \to 0^+$ and assuming that $u_n = u_{\lambda_n} \rightharpoonup u_0 \in \mathrm{cl}(G)$, we obtain $A_0 u_n = -\lambda_n J u_n \to 0$ and $\lim (A_0 u_n, u_n - u_0) = 0$. By the pseudo-monotonicity of A_0,

$$0 = \lim (A_0 u_n, u_n - v) \geqq (A_0 u_0, u_0 - v)$$

for all $v \in X$. This implies that $A_0 u_0 = 0$ and $u_0 \in G$, q.e.d.

We show now how our theory can be applied to the investigation of nonlinear equations of Hammerstein type

$$u + TFu = f$$

in a nonreflexive Banach space X. Here F denotes a (nonlinear) mapping of X to X^*, T a linear operator of X^* to X, and $f \in X$ a given element. Without assuming that T is compact (which case leads back to the now-classical theory of compact operators in Banach spaces), it seems to be the first time that Hammerstein equations are considered by methods of operators of monotone type in a nonreflexive space X. Former investigations were restricted to equations in a reflexive space X, or in the conjugate space X^* of some Banach space X (e.g. [1, 3, 12, 18, 20]).

DEFINITION. *A bounded linear monotone operator* T *of* X^* *into* X *is said to be angle-bounded if there exists a constant* $\gamma \geqq 0$ *such that for all* v, w *in* X^*,

$$|(v, Tw) - (w, Tv)| \leqq \gamma (v, Tv)^{1/2} (w, Tw)^{1/2}.$$

LEMMA 2. *Let* X *an arbitrary real Banach space,* F *a pseudo-monotone mapping of* X *to* X^**, and* T *an angle-bounded linear operator of* X^* *to* X*. Then the equation* $u + TFu = f$ *in* X *can be reduced to an equivalent equation* $Av = 0$ *in a Hilbert space* H*, with* A *a pseudo-monotone mapping of* H *into itself. If* X^* *is nonseparable, then* H *has the same property in general.*

Proof. By the natural imbedding, we identify X with a subspace of X^{**} and consider T as an (angle-bounded) mapping of X^* to X^{**}. By a result of Browder-Gupta [12] (cf. also Amann [1], Hess [20]), there exist a Hilbert space H (whose norm and inner product we denote by $\|.\|_H$ and $(.,.)_H$, respectively), a continuous linear mapping S of X^* to H with range dense in H, and a monotone linear bijective mapping C of H onto H, such that $T=S^*CS$ and $(C^{-1}v, v)_H \geqq d\|v\|_H^2$ for all $v \in H$, with $d>0$. Since T has range contained in X and $CS(X^*)$ is dense in H, it follows that the range of S^* is contained in $X \subset X^{**}$.

By the above result, the equation

$$u + TFu = f \tag{7}$$

is equivalent to the equation

$$u - f + S^*CSFu = 0.$$

Since S^* is injective, there exists a uniquely determined v in H with $u-f=S^*v$, and the initial equation (7) and

$$v + CSF(S^*v + f) = 0 \tag{8}$$

are equivalent. By the bijectiveness of C, (8) holds if and only if

$$C^{-1}v + SF(S^*v + f) = 0.$$

It is readily seen that the operator A:

$$Av = C^{-1}v + SF(S^*v + f) \quad (v \in H)$$

is a pseudo-monotone mapping of H into itself. Finally, if X^* is nonseparable, the same is true in general for H as the completion of a factorspace X^* modulo some subspace (cf. the construction of H in [12]), q.e.d.

An application of Theorem 2 gives the following existence theorem of Fredholm alternative type for asymptotically homogeneous and odd Hammerstein equations.

THEOREM 3. *Let X a separable real Banach space, B a bounded continuous pseudo-monotone mapping of X to X^* which is odd and homogeneous (i.e. $B(\lambda u)=\lambda Bu$ for $\lambda \in R^1$), and $N: X \to X^*$ a bounded continuous operator with $\lim_{\|u\| \to \infty} \|u\|^{-1}\|Nu\| = 0$, and such that $B+N$ is pseudo-monotone. Let further T a linear angle-bounded operator of X^* to X. Then the range of $I+T(B+N)$ is all of X, provided $u+TBu=0$ implies that $u=0$.*

Proof[3]. In order to show that the mapping $I+T(B+N)$ is surjective, it suffices

[3]) Here we denote by "$\rightharpoonup$" weak convergence in X or H, by "$\overset{*}{\rightharpoonup}$" weak* convergence in X^*.

by Lemma 2 to prove the solvability of the equation

$$C^{-1}v + S(B+N)(S^*v+f) = 0 \tag{9}$$

in H for arbitrarily given $f\in X$. We observe that if $u+TBu=0$ only for $u=0$, then the equation $C^{-1}v+SBS^*v=0$ implies that $v=0$.

In the following let $f\in X$ be fixed. For $t\in[0, 1]$ and $v\in H$ we let

$$A_tv = C^{-1}v + (1-\tfrac{1}{2}t)\,S(B+N)(S^*v+f) - \tfrac{1}{2}tS(B+N)(S^*(-v)+f).$$

It is readily seen that the homotopy A_tv has the following properties:

(i) For fixed t, A_t is pseudo-monotone, bounded and continuous;

(ii) A_tv is continuous in t, uniformly for v in bounded sets;

(iii) $A_0v=C^{-1}v+S(B+N)(S^*v+f)$, while A_1 is odd.

The desired result on the solvability of the equation (9) follows from Theorem 2, if we prove that, assuming $C^{-1}v+SBS^*v=0$ only for $v=0$, there exists $R>0$ such that $\|A_tv\|_H\geqq 1$ for all $t\in[0, 1]$ and all $v\in H$ with $\|v\|_H\geqq R$.

Suppose that to each n we can find elements $v_n\in H$ with $\|v_n\|_H\geqq n$, $t_n\in[0, 1]$, and $e_n\in H$ with $\|e_n\|_H<1$, such that $A_{t_n}v_n=e_n$. We may assume that $t_n\to t\in[0, 1]$. Setting $w_n=\|v_n\|_H^{-1}v_n$, we then obtain

$$\begin{aligned}&C^{-1}w_n + (1-\tfrac{1}{2}t)\,SB(S^*w_n+\|v_n\|_H^{-1}f) + \tfrac{1}{2}tSB(S^*w_n-\|v_n\|_H^{-1}f)\\ &\quad= \tfrac{1}{2}(t_n-t)\,\{SB(S^*w_n+\|v_n\|_H^{-1}f) - SB(S^*w_n-\|v_n\|_H^{-1}f)\}\\ &\qquad- (1-\tfrac{1}{2}t_n)\,\|v_n\|_H^{-1}\,SN(S^*v_n+f) + \tfrac{1}{2}t_n\|v_n\|_H^{-1}\,SN(S^*(-v_n)+f)\\ &\qquad+ \|v_n\|_H^{-1}\,e_n\to 0 \quad (n\to\infty).\end{aligned}$$

Because of the separability of X, the weak* topology on closed balls in X^* is metrizable, and balls in X^* are thus weak* sequentially compact. By passing to infinite subsequences, we may assure that $w_n\rightharpoonup w$ in H, $B(S^*w_n+\|v_n\|_H^{-1}f)\overset{*}{\rightharpoonup}a$ and $B(S^*w_n-\|v_n\|_H^{-1}f)\overset{*}{\rightharpoonup}b$ in X^*. It follows that $S^*w_n\pm\|v_n\|_H^{-1}f\rightharpoonup S^*w$ in X, $C^{-1}w_n\rightharpoonup C^{-1}w$ in H, and $C^{-1}w+(1-\tfrac{1}{2}t)\,Sa+\tfrac{1}{2}tSb=0$. We further infer that

$$\begin{aligned}&(C^{-1}w_n, w_n-w)_H + (1-\tfrac{1}{2}t)\,(B(S^*w_n+\|v_n\|_H^{-1}f), (S^*w_n+\|v_n\|_H^{-1}f)-S^*w)\\ &\quad+ \tfrac{1}{2}t(B(S^*w_n-\|v_n\|_H^{-1}f), (S^*w_n-\|v_n\|_H^{-1}f)-S^*w)\to 0.\end{aligned}$$

We assume that $0<t\leqq 1$ (the case $t=0$ is treated similarly) and choose further infinite subsequences such that the three limits $\lim(C^{-1}w_n, w_n-w)_H$, $\lim(B(S^*w_n+\|v_n\|_H^{-1}f), (S^*w_n+\|v_n\|_H^{-1}f)-S^*w)$, and $\lim(B(S^*w_n-\|v_n\|_H^{-1}f), (S^*w_n-\|v_n\|_H^{-1}f)-S^*w)$ exist. By the pseudo-monotonicity property of the mappings C^{-1} and B, all of the three limits are 0. Hence, again by pseudo-monotonicity, $a=b=BS^*w$, and consequently $C^{-1}w+SBS^*w=0$.

Since $(C^{-1}v_n, v_n)_H \geqq d\|v_n\|_H^2$, we conclude that $(C^{-1}w_n, w_n)_H \geqq d > 0$. Moreover $(C^{-1}w_n, w_n)_H \to (C^{-1}w, w)_H$. Thus $w \neq 0$, q.e.d.

Remark. Theorem 3 remains true for X nonseparable, but reflexive.

4. Our principal methodological result on nonlinear eigenvalue problems which extends the corresponding Theorem 1 of Browder [7] to mappings in nonseparable spaces is

THEOREM 4. *Let X a real reflexive Banach space, C a closed subset of X, and A, B continuous mappigs of C into X^*, with A bounded and of type (S) and B compact. Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence in Λ with $C \cap E_1 \neq \emptyset$. Suppose to each $F \in \Lambda$ with $F \supset E_1$ there exist elements $u_F \in C \cap F$ and $t_F \in R^1$ such that $j_F^* A u_F = t_F j_F^* B u_F$, and assume u_F and t_F remain uniformly bounded for $F \supset E_1$.*

Then there exists a sequence $\{F_n\}$ in Λ with $F_n \supset E_n$ for each n, such that for some subsequence $\{n(k)\}$ of $\{n\}$, $u_{F_{n(k)}} \to u_0 \in C$, $t_{F_{n(k)}} \to t_0 \in R^1$, and $Au_0 = t_0 Bu_0$.

Proof. Follows immediately from the Proposition, with $I = R^1$ and $A(u, t) = Au - tBu$.

As an application to the "selfadjoint" case where A and B are the derivatives of two functions, we get the following extension of Theorem 3 in [7] and Theorem 14 in [8]:

THEOREM 5. *Let f, h continuously differentiable real-valued functions defined on the (not necessarily separable) real reflexive Banach space X, with f' bounded and of type (S) and h' compact. Suppose that for a given constant c the level set $M_c(f) = \{u \in X : f(u) = c\}$ is nonempty and bounded, and that for $u \in M_c(f)$, $(f'u, u) \neq 0$. Suppose further that there exists a point $v_0 \in M_c(f)$ and a constant $d > 0$ such that for all $u \in M_c(f)$ for which $h(u) \geqq h(v_0)$, $(h'u, u) \geqq d$.*

Then h assumes its maximum on $M_c(f)$ at a point u_0 which is a solution of the equation $f'u_0 = t_0 h'u_0$ for some real number t_0.

Proof. By the continuity of f, the level set $M_c(f)$ is closed in X. Let F an arbitrary element of Λ with $M_c(f) \cap F \neq \emptyset$, and let f_F, h_F denote the restrictions of f and h to F. The functions f_F and h_F are continuously differentiable on F, with $(f_F)' = j_F^* f' j_F$, $(h_F)' = j_F^* h' j_F$. We set $M_{c,F}(f) = M_c(f) \cap F$. Since $((f_F)'u, u) = (f'u, u) \neq 0$ for all $u \in M_{c,F}(f)$, $M_{c,F}(f)$ is a compact manifold of codimension 1 in F. Thus there exists $u_F \in M_{c,F}(f)$ such that $h(u_F) = \sup_{u \in M_{c,F}(f)} h(u)$. By the Lagrange multiplier method,

$$(h_F)'u_F = \lambda_F (f_F)'u_F \tag{10}$$

for some real λ_F.

Let $\{w_n\}$ be a sequence in $M_c(f)$ with $h(w_n) \to m = \sup_{u \in M_c(f)} h(u)$. We choose an increasing sequence $\{E_n\}_{n=1}^{\infty}$ in Λ such that $E_1 \supset \{v_0, w_1\}$, while $w_n \in E_n$ for $n \geqq 2$.

In order to prove the applicability of Theorem 4 with $C=M_c(f)$, we show that for $F\in\Lambda$, $F\supset E_1$, the corresponding numbers $(\lambda_F)^{-1}$ of (10) are uniformly bounded. Indeed, it follows from (10) that $|(h'u_F, u_F)|=|\lambda_F|\,|(f'u_F, u_F)|$, where $|(h'u_F, u_F)|\geqq d>0$ and $|(f'u_F, u_F)|\leqq k_0$ for each $F\in\Lambda$ with $F\supset E_1$. Thus $|\lambda_F|\geqq k_1>0$, and we can write $(f_F)'u_F=t_F(h_F)'u_F$, with $t_F=(\lambda_F)^{-1}$ uniformly bounded.

By Theorem 4 there exists a sequence $\{F_n\}$ in Λ with $F_n\supset E_n$ for each n, such that $u_{F_{n(k)}}\to u_0\in M_c(f)$, $t_{F_{n(k)}}\to t_0\in R^1$, and $f'u_0=t_0h'u_0$. Since $w_n\in E_n\subset F_n$, $h(w_n)\leqq \leqq h(u_{F_n})$. In this last relation the left side converges to m, while $h(u_{F_{n(k)}})\to h(u_0)$ by continuity of h. Hence $h(u_0)=\sup_{u\in M_c(f)}h(u)$, q.e.d.

In a similar way one generalizes Theorem 15 of [8].

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