

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Band:** 46 (1971)

**Artikel:** On a Marinescu Structure on ... (X)  
**Autor:** Binz, E. / Feldmann, W.  
**DOI:** <https://doi.org/10.5169/seals-35533>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 17.11.2024

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## On a Marinescu Structure on $\mathcal{C}(X)$

E. BINZ AND W. FELDMAN<sup>1)</sup>

### 0.1. Introduction

For any completely regular topological space  $X$  the  $\mathbf{R}$ -algebra  $\mathcal{C}(X)$ , the set of all continuous real-valued functions of  $X$  endowed with the pointwise defined operations, can be represented as the union of subalgebras, each of which is canonically identified with  $\mathcal{C}(Y)$  for some locally compact space  $Y$ . On each of those subalgebras  $\mathcal{C}(Y)$  there is a natural topology, namely the topology of compact convergence.

The collection of all filters on  $\mathcal{C}(X)$  which have as a basis a convergent filter in one of those subalgebras, defines a certain type of convergence structure (Limitierung [1]) on  $\mathcal{C}(X)$ , a so called Marinescu structure. The algebra  $\mathcal{C}(X)$  endowed with this structure is referred to as  $\mathcal{C}_I(X)$ .

A well-written study of Marinescu structures can be found in [7].

The purpose of this note is to give a description of  $\mathcal{C}_I(X)$ . Here we state some of the properties of  $\mathcal{C}_I(X)$ .

The evaluation map  $\omega$  from the cartesian product  $\mathcal{C}_I(X) \times X$  into the reals is continuous. Assigning to each set  $A \subset X$  the set

$$\{f \in \mathcal{C}(X) \mid f(q) = 0 \text{ for all } q \in A\},$$

we obtain a one-to-one correspondence between the collection of all closed (proper) ideals in  $\mathcal{C}_I(X)$  and all non-empty closed subsets of  $X$ . In particular, every closed maximal ideal consists of all functions in  $\mathcal{C}(X)$  vanishing on a fixed point  $p$  in  $X$ .

Of some interest to us is the initial topology on  $\mathcal{C}(X)$  determined by all continuous seminorms of  $\mathcal{C}_I(X)$ . This topology turns out to be the topology of compact convergence.

As a consequence  $\mathcal{C}_I(X)$  and  $\mathcal{C}_{\infty}(X)$ , the  $\mathbf{R}$ -vector space  $\mathcal{C}(X)$  together with above mentioned topology, have the same dual spaces. In addition we find that the properties of  $\mathcal{C}_I(X)$  listed so far hold also for  $\mathcal{C}_c(X)$ , the algebra  $\mathcal{C}(X)$  equipped with the continuous convergence structure [1]. We therefore investigate whether  $\mathcal{C}_I(X)$  and  $\mathcal{C}_c(X)$  are identical. On a space  $X$  having a countable neighbourhood basis for each point, the identity of  $\mathcal{C}_I(X)$  and  $\mathcal{C}_c(X)$  is equivalent to the local compactness of  $X$ . This is a corollary to the more general theorem 8.

---

<sup>1)</sup> Parts of this paper are contained in the thesis of the second author.

### 1.1. Definition of the Convergence Structure

Let  $X$  be a completely regular topological space. We denote the Stone-Ćech compactification of  $X$  by  $\beta X$ . It is well-known that every continuous map from  $X$  into a compact space  $C$  can be extended to a continuous map from  $\beta X$  into  $C$ . Since  $X$  is a dense subspace of  $\beta X$ , this extension is unique.

By  $\mathcal{C}(X)$ , we mean the  $\mathbf{R}$ -algebra of all continuous real-valued functions on  $X$  (under the pointwise defined operations). Every function  $f$  in  $\mathcal{C}(X)$  can be regarded as a map from  $X$  into  $\dot{\mathbf{R}}$ , the one-point compactification of the reals. Hence we can extend  $f$  to a function from  $\beta X$  into  $\dot{\mathbf{R}}$ . Clearly if  $f$  is bounded, then the extension is still real-valued. For any  $f \in \mathcal{C}(X)$ , the extension of  $f$  to  $\beta X$ , as a function with values in  $\dot{\mathbf{R}}$ , is again denoted by  $f$ . Let  $K_f \subset \beta X$  be the pre-image under  $f$  of the point  $\infty \in \dot{\mathbf{R}}$ . Since  $f : \beta X \rightarrow \dot{\mathbf{R}}$  is continuous,  $K_f$  is a compact subset of  $\beta X$ . The function  $f$  restricted to  $X$  is of course real-valued, and thus  $K_f$  must be a subset of  $\beta X \setminus X$ , the complement of  $X$  in  $\beta X$ . For any space  $Y$  such that

$$X \subset Y \subset \beta X,$$

we identify each continuous real-valued function on  $Y$  with its restriction to  $X$ . Therefore given any compact set  $K \subset \beta X \setminus X$ , the algebra  $\mathcal{C}(\beta X \setminus K)$  is contained in  $\mathcal{C}(X)$ . In particular, the subalgebra  $\mathcal{C}(\beta X \setminus K_f)$  contains  $f$ . We now conclude that

$$\mathcal{C}(X) = \bigcup_{K \subset \beta X \setminus X} \mathcal{C}(\beta X \setminus K),$$

where  $K$  ranges through all compact subsets of  $\beta X \setminus X$ .

By  $\mathcal{C}_{co}(\beta X \setminus K)$ , we mean the algebra  $\mathcal{C}(\beta X \setminus K)$  endowed with the topology of compact convergence. The convergence structure, being the subject of our investigation, is the finest of all convergence structures on  $\mathcal{C}(X)$  making the inclusion maps from  $\mathcal{C}_{co}(\beta X \setminus K)$  into  $\mathcal{C}(X)$  continuous for every compact subset  $K \subset \beta X \setminus X$ . We denote the algebra  $\mathcal{C}(X)$  together with this convergence structure by  $\mathcal{C}_I(X)$ , and notice that this is simply the inductive limit, in the category of convergence spaces, (see [7]) of the family

$$\{\mathcal{C}_{co}(\beta X \setminus K) : K \text{ a compact subset of } \beta X \setminus X\} \quad (*)$$

with the ordering defined by inclusion. Of course the inclusion map from  $\mathcal{C}_{co}(\beta X \setminus K)$  into  $\mathcal{C}_{co}(\beta X \setminus K')$  is continuous whenever  $K$  is contained in  $K'$ . Since all the spaces considered in (\*) are locally convex topological  $\mathbf{R}$ -algebras,  $\mathcal{C}_I(X)$  is indeed a Marinescu space as introduced by H. Jarchow in [7]. We leave it to the reader to verify that  $\mathcal{C}_I(X)$  is a convergence  $\mathbf{R}$ -algebra [1], meaning that the operations are continuous.

## 1.2. Completeness of $\mathcal{C}_I(X)$

A filter  $\Theta$  on a commutative convergence group  $G$  is called Cauchy if  $\Theta - \Theta$  converges to zero, where “ $-$ ” denotes the difference operation in  $G$ . If every Cauchy filter in  $G$  converges to some element in  $G$ , then the group is said to be complete.

**THEOREM 1.** *For any completely regular topological space  $X$ , the convergence algebra  $\mathcal{C}_I(X)$  is complete.*

*Proof.* Let  $\Theta$  be a Cauchy filter on  $\mathcal{C}_I(X)$ . We must find a function  $t \in \mathcal{C}_I(X)$  such that  $\Theta$  converges to  $t$ . Here, we remark that a filter  $\Psi$  on  $\mathcal{C}_I(X)$  converges to a function  $g$  in  $\mathcal{C}_I(X)$  if and only if there is a compact  $K \subset \beta X \setminus X$  such that  $\mathcal{C}(\beta X \setminus K)$  contains  $g$  and  $\Psi$  has a base in  $\mathcal{C}_{co}(\beta X \setminus K)$  which is a filter convergent to  $g$  in this space. Now the filter  $\Theta - \Theta$  has a base  $\Phi$  in  $\mathcal{C}_{co}(\beta X \setminus K)$  with  $\Phi$  convergent to zero for some compact  $K \subset \beta X \setminus X$ . Hence any element  $A$  of  $\Phi$  contains  $M - M$  where  $M \in \Theta$ . We will show that  $M$  itself is in  $\mathcal{C}(\beta X \setminus K')$  for some compact  $K' \subset \beta X \setminus X$ . Let  $g$  be a fixed element in  $M$ . For each  $f \in M$ , the function  $f - g$  is in  $M - M$ , and thus in  $\mathcal{C}(\beta X \setminus K)$ . This means that

$$f^{-1}(\infty) \subset g^{-1}(\infty) \cup K.$$

Therefore  $M$  is contained in  $\mathcal{C}(\beta X \setminus K')$  where  $K'$  stands for  $g^{-1}(\infty) \cup K$ . It follows that  $\Theta$  has a base in  $\mathcal{C}(\beta X \setminus K')$ , call it  $\Theta'$ . Since

$$\mathcal{C}(\beta X \setminus K) \subset \mathcal{C}(\beta X \setminus K'),$$

the filter  $\Theta' - \Theta'$  on  $\mathcal{C}_{co}(\beta X \setminus K')$  has  $\Phi$  as a base, and thus  $\Theta'$  is a Cauchy filter in  $\mathcal{C}_{co}(\beta X \setminus K')$ . The completeness of  $\mathcal{C}_{co}(\beta X \setminus K')$  implies that  $\Theta'$  itself converges to some function  $t \in \mathcal{C}(\beta X \setminus K')$ . Hence  $\Theta$  converges to  $t$  in  $\mathcal{C}_I(X)$  as desired.

## 1.3. Closed Ideals in $\mathcal{C}_I(X)$

By an ideal, we mean of course a *proper* ideal. It is evident that for every non-empty subset  $S$  of  $X$  the ideal

$$I(S) = \{f \in \mathcal{C}(X) : f(S) = \{0\}\}$$

is closed in  $\mathcal{C}_I(X)$ . We conjecture that all closed ideals in  $\mathcal{C}_I(X)$  are precisely of this form.

To prove this, let  $J \subset \mathcal{C}_I(X)$  be a closed ideal. We call the set of all points  $p \in X$  with the property that every function  $f \in J$  vanishes on  $p$  the null-set of  $J$ , and denote this set by  $N_X(J)$ . It is exactly the intersection of all zero-sets  $Z_X(f)$  where  $f$  runs through  $J$ . By  $Z_X(f)$ , we mean  $\{x \in X : f(x) = 0\}$ . Since for any function  $f \in J$ , there



is a bounded function  $g \in J$  such that  $Z_X(f) = Z_X(g)$ , we can represent  $N_X(J)$  as

$$\bigcap_{g \in J^\circ} Z_X(g),$$

where  $J^\circ$  denotes the collection of all bounded functions in  $J$ . Furthermore, the set  $J^\circ$  is a closed ideal in  $\mathcal{C}_{\infty}(\beta X)$ , and is therefore of the form  $I(N_{\beta X}(J^\circ))$  where  $N_{\beta X}(J^\circ)$  is a non-empty subset of  $\beta X$ . Evidently the ideal  $J \subset I(N_X(J))$ . We will show that  $J$  is all of  $I(N_X(J))$ . First, we verify that  $J^\circ$  contains all the bounded functions in  $I(N_X(J))$ . Since  $J^\circ$  consists of all functions in  $\mathcal{C}(\beta X)$  vanishing on  $N_{\beta X}(J^\circ)$ , it is enough to prove that any bounded element of  $I(N_X(J))$  vanishes on  $N_{\beta X}(J^\circ)$ . Clearly we are done as soon as we know that  $\overline{N_{\beta X}(J^\circ)}$  is the closure of  $N_X(J)$  in  $\beta X$ . Assume, to the contrary, that  $N_{\beta X}(J^\circ)$  contains  $\overline{N_X(J)}$ , the closure in  $\beta X$  of  $N_X(J)$ , as a proper subset. For a point  $q \in N_{\beta X}(J^\circ)$  outside of  $\overline{N_X(J)}$ , we choose in  $\beta X$  a closed neighborhood  $U$  of  $q$  disjoint from  $\overline{N_X(J)}$ . There exists a function  $g \in \mathcal{C}(\beta X)$  such that  $g(q) = 1$  and  $g$  vanishes on the complement of  $U$ . We assert that  $g \in J \cap \mathcal{C}(\beta X \setminus K)$ , where  $K$  denotes the compact set  $U \cap N_{\beta X}(J^\circ)$  contained in  $\beta X \setminus X$ . Clearly  $J \cap \mathcal{C}(\beta X \setminus K)$  is a closed ideal in  $\mathcal{C}_{\infty}(\beta X \setminus K)$ , and therefore consists of all functions vanishing on its null-set. Since the bounded functions in  $J \cap \mathcal{C}(\beta X \setminus K)$  are precisely the elements of  $J^\circ$ , we conclude that  $N_{\beta X}(J^\circ) \cap \beta X \setminus K$  is the null-set of  $J \cap \mathcal{C}(\beta X \setminus K)$ . The function  $g$  vanishes on  $N_{\beta X}(J^\circ) \cap \beta X \setminus K$ , and therefore  $g$  is an element of  $J \cap \mathcal{C}(\beta X \setminus K)$  as claimed. Thus we know  $g \in J^\circ$ . On the other hand,  $g$  is not an element of  $I(N_{\beta X}(J^\circ))$ , which is of course  $J^\circ$ . Because of this contradiction, we conclude that  $N_{\beta X}(J^\circ) = \overline{N_X(J)}$ , and thus  $J^\circ$  consists of all bounded functions in  $I(N_X(J))$  where  $N_X(J)$  is not empty. To complete the proof, let  $f$  be an arbitrary element of  $I(N_X(J))$ . There is a unit  $u$  in  $\mathcal{C}(X)$  such that  $f \cdot u$  is bounded. Hence  $f \cdot u \in J^\circ$ , and therefore  $(f \cdot u) \cdot 1/u \in J$ . This implies that  $J = I(N_X(J))$ .

We now have established

**THEOREM 2.** *An ideal  $J$  in  $\mathcal{C}_I(X)$  is closed if and only if  $J = I(N_X(J))$ .*

**COROLLARY 1.** *A maximal ideal in  $\mathcal{C}_I(X)$  is closed if and only if it consists of all functions in  $\mathcal{C}(X)$  vanishing at a fixed point in  $X$ .*

For every point  $p \in X$  there is a continuous  $\mathbf{R}$ -algebra homomorphism

$$i_X(p): \mathcal{C}_I(X) \rightarrow \mathbf{R},$$

defined by  $i_X(p)(f) = f(p)$  for every  $f \in \mathcal{C}(X)$ . Assigning to each point  $p \in X$  the homomorphism  $i_X(p)$ , we obtain a map

$$i_X: X \rightarrow \mathcal{H}om \mathcal{C}_I(X),$$

where  $\mathcal{H}om \mathcal{C}_I(X)$  denotes the set of all continuous  $\mathbf{R}$ -algebra homomorphisms

from  $\mathcal{C}_I(X)$  onto  $\mathbf{R}$ . Since an element of  $\mathcal{H}om \mathcal{C}_I(X)$  is determined by its kernel, a closed maximal ideal in  $\mathcal{C}_I(X)$ , we deduce from corollary 1

**COROLLARY 2.** *The map  $i_X$  is surjective.*

#### 1.4. The associated locally convex topology of $\mathcal{C}_I(X)$

First, let us demonstrate that, in general,  $\mathcal{C}_I(X)$  is not topological; more precisely

**THEOREM 3.**  *$\mathcal{C}_I(X)$  is topological if and only if  $X$  is locally compact. If  $X$  is locally compact, then  $\mathcal{C}_I(X) = \mathcal{C}_{co}(X)$ .*

*Proof.* If  $X$  is locally compact, then  $\mathcal{C}(X)$  is of the form  $\mathcal{C}(\beta X \setminus K)$ , where  $K = \beta X \setminus X$  is a compact subset of  $\beta X$ . The inclusion map from  $\mathcal{C}_{co}(\beta X \setminus K')$  into  $\mathcal{C}_{co}(X)$  is continuous for any compact set  $K' \subset \beta X \setminus X$ . Thus  $\mathcal{C}_{co}(X)$  is the finest of all convergence structures making the inclusion maps continuous, i.e.,  $\mathcal{C}_I(X)$  coincides with  $\mathcal{C}_{co}(X)$  and hence is topological.

Conversely, assume that  $\mathcal{C}_I(X)$  is topological. Since the neighborhood filter of zero has a base in  $\mathcal{C}(\beta X \setminus K)$  for some compact  $K \subset \beta X \setminus X$  and every neighborhood of zero is absorbent, we have

$$\mathcal{C}(X) = \mathcal{C}(\beta X \setminus K).$$

If there were a compact  $K' \subset \beta X \setminus X$  strictly containing  $K$ , then the neighborhood filter of zero in  $\mathcal{C}_{co}(\beta X \setminus K')$  would be strictly coarser than the neighborhood filter of zero in  $\mathcal{C}_{co}(\beta X \setminus K)$ . This is apparent since two locally compact spaces  $Z$  and  $Z'$  are homeomorphic if and only if  $\mathcal{C}_{co}(Z)$  and  $\mathcal{C}_{co}(Z')$  are bicontinuously isomorphic (see [3]). Therefore  $K$  must be equal to  $\beta X \setminus X$  which means  $X$  is locally compact.

In view of the fact that  $\mathcal{C}_I(X)$  is not, in general, topological, we wish to determine the associated locally convex space  $\mathcal{C}_{\tau I}(X)$  of  $\mathcal{C}_I(X)$ . The topology of  $\mathcal{C}_{\tau I}(X)$  is generated by all the continuous seminorms on  $\mathcal{C}_I(X)$ .

Let

$$p: \mathcal{C}_I(X) \rightarrow \mathbf{R}$$

be a continuous seminorm. We construct a seminorm  $\tilde{p}$  which majorizes  $p$  and is more convenient to work with. For a compact set  $K \subset \beta X \setminus X$ , we denote by  $p_K$  the restriction of  $p$  to  $\mathcal{C}(\beta X \setminus K)$ . Clearly

$$p_K: \mathcal{C}_{co}(\beta X \setminus K) \rightarrow \mathbf{R}$$

is continuous. Therefore we can find a compact set  $Q_K \subset \beta X \setminus K$  such that a constant multiple  $\alpha$  of the seminorm

$$s_{Q_K}: \mathcal{C}_{co}(\beta X \setminus K) \rightarrow \mathbf{R},$$

defined by  $s_{Q_K}(f) = \sup_{q \in Q_K} |f(q)|$ , majorizes  $p_K$ . This implies that for any function  $f \in \mathcal{C}(\beta X \setminus K)$ ,

$$\tilde{p}_K(f) = \sup \{p_K(g) : |g| \leq |f| \text{ and } g \in \mathcal{C}(\beta X \setminus K)\}$$

is a real number less than or equal to  $\alpha s_{Q_K}(f)$ . Since for every function  $g \in \mathcal{C}(X)$  the relation  $|g| \leq |f|$  implies that  $g \in \mathcal{C}(\beta X \setminus K)$ , we know that

$$\tilde{p}(f) = \sup \{p(g) : |g| \leq |f| \text{ and } g \in \mathcal{C}(X)\}$$

is identical to  $\tilde{p}_K(f)$ . Of course every function in  $\mathcal{C}(X)$  is an element of  $\mathcal{C}(\beta X \setminus K)$  for some compact  $K \subset \beta X \setminus X$ . It is not difficult to verify that the maps

$$\tilde{p} : \mathcal{C}_I(X) \rightarrow \mathbf{R}$$

and

$$\tilde{p}_K : \mathcal{C}_{\infty}(\beta X \setminus K) \rightarrow \mathbf{R} \text{ for any compact } K \subset \beta X \setminus X,$$

sending each  $f \in \mathcal{C}(X)$  to  $\tilde{p}(f)$  and each  $f \in \mathcal{C}(\beta X \setminus K)$  to  $\tilde{p}_K(f)$  respectively, are seminorms. Since  $\tilde{p}$  restricted to  $\mathcal{C}(\beta X \setminus K)$  is  $\tilde{p}_K$ , we conclude that  $\tilde{p}$  itself is a continuous seminorm. Furthermore,  $\tilde{p}$  has the following properties

$$\tilde{p}(f) = \tilde{p}(|f|) \text{ for all } f \in \mathcal{C}(X)$$

and

$$\tilde{p}(f) \leq \tilde{p}(g) \text{ for all } f, g \in \mathcal{C}(X) \text{ with } |f| \leq |g|.$$

**LEMMA 1.** *The kernel  $P$  of  $\tilde{p}$ , the set of all functions  $f \in \mathcal{C}(X)$  with  $\tilde{p}(f) = 0$ , is a closed ideal in  $\mathcal{C}_I(X)$  consisting of all elements in  $\mathcal{C}(X)$  vanishing on a compact subset of  $X$ .*

*Proof.*  $P$  is clearly a linear subspace of  $\mathcal{C}(X)$ . To show it is an ideal, let  $g \in P$ . For an arbitrary element  $f \in \mathcal{C}(X)$ , we consider

$$((-n \vee f) \wedge n)$$

where  $n$  denotes the function of constant value  $n \in \mathbf{N}$ . Now

$$\tilde{p}(g \cdot ((-n \vee f) \wedge n)) \leq \tilde{p}(g \cdot n) = n \cdot \tilde{p}(g)$$

and hence  $g \cdot ((-n \vee f) \wedge n) \in P$ . The Fréchet filter generated by the sequence

$$(g \cdot ((-n \vee f) \wedge n))_{n \in \mathbf{N}}$$

converges to  $g \cdot f$  in  $\mathcal{C}_I(X)$ . Since  $P$  is obviously closed,  $g \cdot f$  is an element of  $P$ . Thus  $P$  is a closed ideal in  $\mathcal{C}_I(X)$ , and therefore consists of all functions in  $\mathcal{C}(X)$  vanishing on its non-empty null-set  $Q \subset X$  (see theorem 2). It only remains to prove that  $Q$  is compact. We can express  $P$  as the union of the kernels of  $\tilde{p}_K$  for all compact  $K \subset \beta X \setminus X$ .

On the other hand, the kernel  $P_K$  of  $\tilde{p}_K$  contains the kernel  $H_K$  of  $s_{Q_K}$ . Hence we have

$$N_{\beta X \setminus K}(P_K) \subset N_{\beta X \setminus K}(H_K).$$

But  $N_{\beta X \setminus K}(H_K)$  is nothing else but  $Q_K$ . Since  $Q$  is contained in the intersection of the null-sets of  $P_K$ ,

$$Q \subset \bigcap_K Q_K,$$

where  $K$  runs through all compact subsets of  $\beta X \setminus X$ . The fact that  $\bigcap_K Q_K$  is a compact subset of  $X$  implies that  $Q$  is compact.

Next, we will show that  $\tilde{p}$  is majorized by a constant multiple of the supremum seminorm  $s$  over  $Q$ . Let  $f \in \mathcal{C}(X)$ , and consider

$$g = ((-s(\mathbf{f}) \vee f) \wedge s(\mathbf{f})).$$

By the previous lemma, we have

$$\tilde{p}(f - g) = 0.$$

Furthermore,

$$|\tilde{p}(f) - \tilde{p}(g)| \leq \tilde{p}(f - g),$$

and hence  $\tilde{p}(f) = \tilde{p}(g)$ . From the inequality  $|g| \leq s(\mathbf{f})$ , we conclude that

$$\tilde{p}(f) \leq \tilde{p}(s(\mathbf{f})) = s(f) \tilde{p}(\mathbf{1}).$$

Therefore we have proved

**THEOREM 4.** *The associated locally convex space of  $\mathcal{C}_I(X)$  is  $\mathcal{C}_{\infty}(X)$ .*

The associated locally convex space of  $\mathcal{C}_I(X)$  coincides with the locally convex inductive limit of the family

$$\{\mathcal{C}_{\infty}(\beta X \setminus K) : K \text{ is a compact subset of } \beta X \setminus X\}.$$

Thus we may state

**COROLLARY 1.** *The locally convex inductive limit of the family*

$$\{\mathcal{C}_{\infty}(\beta X \setminus K) : K \text{ is a compact subset of } \beta X \setminus X\}$$

*is  $\mathcal{C}_{\infty}(X)$ .*

For any convergence vector space  $E$  over  $\mathbf{R}$ , its dual  $\mathcal{L}(E)$  is identical with the dual of the associated locally convex space of  $E$ . Therefore

**COROLLARY 2.**  $\mathcal{L}(\mathcal{C}_I(X)) = \mathcal{L}(\mathcal{C}_{\infty}(X)).$

### 1.5. Functorial Properties of $\mathcal{C}_I(X)$

Let  $X$  and  $Y$  denote completely regular topological spaces. Every continuous map

$$t: X \rightarrow Y$$

induces a homomorphism

$$t^*: \mathcal{C}_I(Y) \rightarrow \mathcal{C}_I(X),$$

defined by  $t^*(f) = f \circ t$  for every  $f \in \mathcal{C}(Y)$ . To see that  $t^*$  is continuous, we consider the restrictions

$$t_K^*: \mathcal{C}_{co}(\beta Y \setminus K) \rightarrow \mathcal{C}_I(X)$$

where  $t_K^*$  denotes  $t^* \upharpoonright \mathcal{C}(\beta Y \setminus K)$ , and verify that  $t_K^*$  is continuous for every compact set  $K \subset \beta Y \setminus Y$ . To this end, we extend  $t$  to a map

$$\tilde{t}: \beta X \rightarrow \beta Y.$$

For each compact  $K \subset \beta Y \setminus Y$ , we know  $\tilde{t}^{-1}(K)$  is a compact subset of  $\beta X \setminus X$ . Furthermore, for a compact  $K \subset \beta Y \setminus Y$  the map  $t_K^*$  is induced by

$$\tilde{t} \upharpoonright (\beta X \setminus \tilde{t}^{-1}(K)): \beta X \setminus \tilde{t}^{-1}(K) \rightarrow \beta Y \setminus K,$$

which we denote by  $t_K$ . That is,  $t_K^*(f) = f \circ t_K$  for all  $f \in \mathcal{C}(\beta Y \setminus K)$ . Clearly

$$t_K^*: \mathcal{C}_{co}(\beta Y \setminus K) \rightarrow \mathcal{C}_{co}(\beta X \setminus \tilde{t}^{-1}(K))$$

is continuous for every compact  $K \subset \beta Y \setminus Y$ , and therefore  $t^*$  itself is continuous.

On the other hand, let

$$u: \mathcal{C}_I(Y) \rightarrow \mathcal{C}_I(X)$$

be a continuous  $\mathbf{R}$ -algebra homomorphism sending unity to unity. We will now show that  $u$  is of the form  $t^*$  where  $t$  maps  $X$  into  $Y$  continuously. The homomorphism  $u$  induces a continuous map

$$u^*: \mathcal{H}om_s \mathcal{C}_I(X) \rightarrow \mathcal{H}om_s \mathcal{C}_I(Y)$$

defined by  $u^*(h) = h \circ u$  for every  $h \in \mathcal{H}om \mathcal{C}_I(X)$ . The index  $s$  denotes the topology of pointwise convergence. Corollary 2 of theorem 2 implies that the map  $i_Z: Z \rightarrow \mathcal{H}om_s \mathcal{C}_I(Z)$  is a homeomorphism for any completely regular topological space  $Z$ . Thus we have a continuous map  $t$  from  $X$  into  $Y$  defined by  $t = i_Y^{-1} \circ u^* \circ i_X$ . Now it is easy to verify that  $t^*$  is equal to  $u$ .

To summarize these facts, we state

**THEOREM 5.** *A homomorphism*

$$u: \mathcal{C}_I(Y) \rightarrow \mathcal{C}_I(X)$$

taking unity to unity is continuous if and only if there exists a continuous map  $t: X \rightarrow Y$  such that  $u = t^*$ .

For maps  $t: X \rightarrow Y$  and  $s: Y \rightarrow Z$  between completely regular topological spaces, we have the obvious identities

$$(s \circ t)^* = t^* \circ s^*$$

and

$$\text{id}_X^* = \text{id}_{\mathcal{C}(X)}.$$

## 1.6. Realcompact Spaces

Let  $X$  be a completely regular topological space. As before, the zero-set  $Z_{\beta X}(f)$  of a function  $f \in \mathcal{C}(\beta X)$  means the set of all points  $p \in \beta X$  where  $f$  vanishes.

Here, we consider the collection

$$(**) \quad \{\mathcal{C}_{co}(\beta X \setminus Z_{\beta X}): Z_{\beta X} \subset \beta X \setminus X \text{ is a zero-set}\}$$

This is a subfamily of the family of all topological algebras  $\mathcal{C}_{co}(\beta X \setminus K)$  for  $K$  a compact subset of  $\beta X \setminus X$ . As in section 1.1, it is clear that the union of all  $\mathcal{C}(\beta X \setminus Z_{\beta X})$  for  $Z_{\beta X}$  a zero-set outside of  $X$  is again  $\mathcal{C}(X)$ . Under the natural ordering (as in section 1.1), the collection  $(**)$  is an inductive system, and we denote the inductive limit of this system by  $\mathcal{C}_{I'}(X)$ .

It is easy to see that  $\mathcal{C}_{I'}(X)$  is actually the finest convergence structure on  $\mathcal{C}(X)$  obtainable as an inductive limit of a subfamily of the family of all  $\mathcal{C}_{co}(\beta X \setminus K)$  for  $K$  a compact subset of  $\beta X \setminus X$ . Of course the identity,

$$(***) \quad \text{id}: \mathcal{C}_{I'}(X) \rightarrow \mathcal{C}_I(X),$$

is continuous. Our main concern in this section is to determine under what conditions this identity is a homeomorphism.

If every compact subset of  $\beta X \setminus X$  is contained in a zero-set in  $\beta X \setminus X$ , then clearly the identity  $(***)$  is a homeomorphism. Conversely, assume that

$$\text{id}: \mathcal{C}_I(X) \rightarrow \mathcal{C}_{I'}(X)$$

is continuous. Therefore we have a continuous injection

$$\text{id}^*: \mathcal{H}om_s \mathcal{C}_{I'}(X) \rightarrow \mathcal{H}om_s \mathcal{C}_I(X),$$

where  $\mathcal{H}om_s \mathcal{C}_{I'}(X)$  denotes the set of all continuous  $\mathbf{R}$ -algebra homomorphism

from  $\mathcal{C}_{I'}(X)$  onto  $\mathbf{R}$  together with the topology of pointwise convergence. For both  $X$  and its Hewitt realcompactification  $vX$  the convergence algebras  $\mathcal{C}_{I'}(X)$  and  $\mathcal{C}_{I'}(vX)$  are identical, since any zero-set contained in  $\beta X \setminus X$  is already contained in  $\beta X \setminus vX$  (see [6], p. 118). Thus

$$\mathcal{H}om_s \mathcal{C}_{I'}(X) = \mathcal{H}om_s \mathcal{C}_{I'}(vX).$$

In view of (I), we conclude that the map

$$i_{vX}: vX \rightarrow \mathcal{H}om_s \mathcal{C}_{I'}(X)$$

is continuous. This tells us that  $\text{id}^* \circ i_{vX}$  maps  $vX$  injectively into  $\mathcal{H}om_s \mathcal{C}_{I'}(X)$ , which is homeomorphic to  $X$ . Hence  $X$  must be realcompact.

To continue our investigation, without loss of generality we can regard  $X$  as a realcompact space. Since by assumption

$$\text{id}: \mathcal{C}_I(X) \rightarrow \mathcal{C}_{I'}(X)$$

is continuous, we know that the inclusion map from  $\mathcal{C}_{\infty}(\beta X \setminus K)$  into  $\mathcal{C}_{I'}(X)$  is continuous for any compact  $K \subset \beta X \setminus X$ . Thus the neighborhood filter of zero in  $\mathcal{C}_{\infty}(\beta X \setminus K)$  has a basis in  $\mathcal{C}_{\infty}(\beta X \setminus Z_{\beta X})$  for some zero-set contained in  $\beta X \setminus X$ . Because every neighborhood of zero in  $\mathcal{C}_{\infty}(\beta X \setminus K)$  is absorbent,  $\mathcal{C}(\beta X \setminus Z_{\beta X}) \supset \mathcal{C}(\beta X \setminus K)$  meaning that  $Z_{\beta X} \supset K$ . To summarize, we have established the following

**THEOREM 6.** *Let  $X$  be a realcompact space.  $\mathcal{C}_I(X)$  is identical to  $\mathcal{C}_{I'}(X)$  if and only if every compact set in  $\beta X \setminus X$  is contained in some zero-set in  $\beta X \setminus X$ .*

We note that in the case of a realcompact locally compact space  $X$ , the convergence algebra  $\mathcal{C}_I(X)$  coincides with  $\mathcal{C}_{I'}(X)$  if and only if  $\beta X \setminus X$  is a zero-set, i.e.,  $X$  is  $\sigma$ -compact.

More generally, assume that  $\mathcal{C}_{I'}(X)$  is topological for a realcompact space  $X$ . By arguing as in section 1.4, we conclude that  $X$  is of the form  $\beta X \setminus Z_{\beta X}$  for some zero-set  $Z_{\beta X}$ . This means that  $X$  is  $\sigma$ -compact and locally compact.

Therefore, we can state

**THEOREM 7.** *Let  $X$  be a realcompact space. The convergence algebra  $\mathcal{C}_{I'}(X)$  is topological if and only if  $X$  is locally compact and  $\sigma$ -compact.*

As an example of a realcompact space  $X$  for which  $\mathcal{C}_I(X)$  and  $\mathcal{C}_{I'}(X)$  do not coincide, consider the reals together with the discrete topology.

### 1.7. Universal Representation of $\mathcal{C}_I(X)$

For a completely regular topological space  $X$ , the homomorphism

$$d: \mathcal{C}_I(X) \rightarrow \mathcal{C}_c(\mathcal{H}om_c \mathcal{C}_I(X)),$$

defined by  $d(f)(h) = h(f)$  for all  $f \in \mathcal{C}(X)$  and all  $h \in \mathcal{H}om \mathcal{C}_I(X)$ , is called the universal representation [2] of  $\mathcal{C}_I(X)$ . The subscript  $c$  indicates the continuous convergence structure (*Limitierung der stetigen Konvergenz* [1]) on the sets  $\mathcal{H}om \mathcal{C}_I(X)$  and  $\mathcal{C}(\mathcal{H}om_c \mathcal{C}_I(X))$ .

We first investigate the continuous convergence structure on  $\mathcal{H}om \mathcal{C}_I(X)$ .

The space  $\mathcal{H}om_c \mathcal{C}_c(X)$  is homeomorphic to  $X$  [3], and thus the continuous convergence structure on  $\mathcal{H}om \mathcal{C}_c(X)$  is the topology of pointwise convergence. Since the evaluation map

$$\omega: \mathcal{C}_I(X) \times X \rightarrow \mathbf{R}$$

(defined by  $\omega(f, p) = f(p)$  for all  $f \in \mathcal{C}(X)$  and all  $p \in X$ ) is continuous, the identity

$$\text{id}: \mathcal{C}_I(X) \rightarrow \mathcal{C}_c(X)$$

is continuous. Furthermore, the sets  $\mathcal{H}om \mathcal{C}_I(X)$  and  $\mathcal{H}om \mathcal{C}_c(X)$  are identical (corollary 2 of theorem 2) which means that

$$\text{id}: \mathcal{H}om_c \mathcal{C}_c(X) \rightarrow \mathcal{H}om_c \mathcal{C}_I(X)$$

is continuous. On the other hand the identity map from  $\mathcal{H}om_c \mathcal{C}_I(X)$  into  $\mathcal{H}om_s \mathcal{C}_I(X)$  is clearly continuous (the subscript  $s$  indicates the topology of pointwise convergence). It follows that

$$\mathcal{H}om_c \mathcal{C}_I(X) = \mathcal{H}om_s \mathcal{C}_I(X),$$

which is homeomorphic to  $X$  via the map  $i_X$  defined earlier. Therefore

$$i_X^*: \mathcal{C}_c(\mathcal{H}om_c \mathcal{C}_I(X)) \rightarrow \mathcal{C}_c(X)$$

is a bicontinuous isomorphism, and of course  $i_X^* \circ d$  is the identity map on  $\mathcal{C}(X)$ .

Our main problem is thus to determine whether  $\mathcal{C}_I(X)$  and  $\mathcal{C}_c(X)$  coincide. So far, we can say the following

**THEOREM 8.** *Let  $X$  be a completely regular topological space. If there is a point  $q$  in  $X$  having a countable base of neighborhoods and no compact neighborhood, then  $\mathcal{C}_c(X)$  cannot be an inductive limit of topological vector spaces over  $\mathbf{R}$ .*

*Proof.* Any inductive limit of topological vector spaces over  $\mathbf{R}$  has the property that for each filter  $\Phi$  converging to zero, there exists a coarser filter  $\Phi'$  convergent to zero with

$$\lambda \cdot \Phi' = \Phi'$$

for every real number  $\lambda$  unequal to zero.



Our aim is to show that under the assumption of the theorem,  $\mathcal{C}_e(X)$  fails to satisfy this condition.

Let  $\{Q_m\}_{m \in \mathbb{N}}$  be a countable collection of open sets in  $X$  that form a base for the neighborhood filter at  $q$ . We define inductively a certain system of nested neighborhoods of  $q$ . Let  $N_1 = X$  and let  $\{O_{1,\alpha}\}$  be an open covering of  $X$  with no finite subcovering. Set

$$U_1 = O_1^q \cap Q_1,$$

where  $O_1^q$  is a member of  $\{O_{1,\alpha}\}$  containing  $q$ . Assume that the closed respectively open neighborhoods  $N_i$  and  $U_i$  are defined. Choose  $N_{i+1}$  to be a closed neighborhood of  $q$  contained in  $U_i$ , and let  $\{O_{i+1,\alpha}\}$  be a covering of  $N_{i+1}$  by open sets in  $X$  having no finite subcovering. We pick  $U_{i+1}$  to be an open neighborhood of  $q$  contained in

$$O_{i+1}^q \cap Q_{i+1} \cap N_{i+1},$$

where  $O_{i+1}^q$  is a member of  $\{O_{i+1,\alpha}\}$  with  $q \in O_{i+1}^q$ . With this system of respectively closed and open neighborhoods of  $q$ ,

$$N_1 \subset U_1 \subset N_2 \subset U_2 \dots,$$

we construct a filter  $\Theta$  that does not satisfy the condition mentioned above. Let

$$T_n = \left\{ f \in \mathcal{C}(X) : f(N_n) \subset \left[ \frac{-1}{n}, \frac{1}{n} \right] \right\}$$

and let

$$T_x = \{ f \in \mathcal{C}(X) : f(W_x) = \{0\} \}$$

for  $x \neq q$ , where we choose  $W_x$  as follows: Since  $x \neq q$ , the point  $x$  lies in  $N_r$  but not in  $N_{r+1}$  for some natural number  $r$ . Let  $W_x$  be a closed neighborhood of  $x$  contained in

$$\bigcap_{j=1}^r O_j^x \cap N_{r+1}$$

where  $O_j^x$  is a member of the covering system  $\{O_{j,\alpha}\}$  containing  $x$ . It is clear that the sets  $\{T_n : n \in \mathbb{N}\}$  and  $\{T_x : x \in X \text{ and } x \neq q\}$  generate a filter  $\Theta$  convergent to zero in  $\mathcal{C}_e(X)$ . Assume that there exists a coarser filter  $\Theta'$  in  $\mathcal{C}_e(X)$  convergent to zero with

$$\lambda \cdot \theta' = \theta'$$

for every real number  $\lambda \neq 0$ . To the interval  $[-1, 1]$ , there is a set  $F' \in \Theta'$  and a neighborhood  $N_k$  of  $q$  such that

$$F'(N_k) = \{ f(p) : f \in F' \text{ and } p \in N_k \}$$

is a subset of  $[-1, 1]$ . For  $\lambda$  equal to  $1/2k$ , we have

$$\frac{1}{2k} F'(N_k) \subset \left[ -\frac{1}{2k}, \frac{1}{2k} \right],$$

and  $(1/2k) F' \in \Theta'$ . Thus  $(1/2k) F'$  contains a finite intersection of elements of the form  $T_n$  and  $T_x$ , say

$$\bigcap_{n \in \tilde{N}} T_n \cap \bigcap_{x \in \tilde{X}} T_x,$$

where  $\tilde{N}$  is a finite subset of  $\mathbf{N}$  and  $\tilde{X}$  is a finite subset of  $X \setminus \{q\}$ . Now we claim that

$$N_k \not\subset \bigcup_{x \in \tilde{X}} W_x \cup N_{k+1}.$$

Our construction guarantees that for a fixed  $W_x$ , either  $W_x$  is a subset of the complement of  $N_k$  or  $W_x$  is contained in an element of the open covering  $\{O_{k,\alpha}\}$ . Furthermore,  $N_{k+1}$  is contained in  $O_k^q$ . Since the open covering  $\{O_{k,\alpha}\}$  has no finite sub-covering, the claim is true. Therefore, we can find a function  $g \in \mathcal{C}(X)$  vanishing on  $\bigcup_{x \in \tilde{X}} W_x \cup N_{k+1}$  with  $g$  taking on the value  $1/k$  for some point in  $N_k$  and  $\|g\| \leq 1/k$ . This function is certainly not in  $(1/2k) F'$  but it is in  $\bigcup_{n \in \tilde{N}} T_n \cap \bigcup_{x \in \tilde{X}} T_x$ , and this contradiction establishes the theorem.

## 2.1. Consequences for $\mathcal{C}_c(X)$

In this section, we demonstrate consequences of the theory developed in 1.1 to 1.7 in investigating closed ideals in  $\mathcal{C}_c(Y)$  for a convergence space  $Y$ , and in determining both the associated locally convex topological space of  $\mathcal{C}_c(X)$  and the dual space of  $\mathcal{C}_c(X)$ , where  $X$  is a completely regular topological space. The results we obtain can be found in [4] and [5] respectively; however, the proofs given here are simpler than those provided in [4] and [5].

First, we look at closed ideals in  $\mathcal{C}_c(Y)$ .

Let  $Y$  be an arbitrary convergence space. To this space we associate a completely regular topological space as follows: Any two points  $p, q \in Y$  are said to be equivalent if  $f(p) = f(q)$  for all real-valued continuous functions  $f$ . As usual, the set of all these functions is denoted by  $\mathcal{C}(Y)$ . The quotient set defined by the above equivalence relation is called  $Y'$ . Any function  $f \in \mathcal{C}(Y)$  defines a function

$$f': Y' \rightarrow \mathbf{R}$$

by sending each  $\bar{p} \in Y'$  to  $f(p)$ . The initial topology induced by the family

$$\{f' : f \in \mathcal{C}(Y)\}$$

is, of course, completely regular. The set  $Y'$  together with this topology is again denoted by  $Y'$ .

The obvious projection

$$\pi: Y \rightarrow Y'$$

induces an isomorphism (with respect to the usual  $\mathbf{R}$ -algebra structure)

$$\pi^*: \mathcal{C}(Y') \rightarrow \mathcal{C}(Y)$$

defined by  $\pi^*(g) = g \circ \pi$  for all  $g \in \mathcal{C}(Y')$ . This isomorphism is continuous if both algebras carry the continuous convergence structure. Hence for any closed ideal  $J$  in  $\mathcal{C}_c(Y)$  (the algebra  $\mathcal{C}(Y)$  together with the continuous convergence structure), the ideal  $\pi^{*-1}(J) \subset \mathcal{C}_c(Y')$  is closed. Since the identity map,

$$\text{id}: \mathcal{C}_I(Y') \rightarrow \mathcal{C}_c(Y')$$

is continuous, we conclude that  $\pi^{*-1}(J)$  is closed in  $\mathcal{C}_I(Y')$ . Therefore, we know by theorem 2 that it is of the form  $I(N)$  where  $N \subset Y'$  is a closed non-empty subset. It is clear that  $I(\pi^{-1}(N)) = J$ . Since an ideal of the form  $I(M)$  for any non-empty subset of  $Y$  is closed in  $\mathcal{C}_c(Y)$ , we have the following result

**THEOREM 9.** *For any convergence space  $Y$ , an ideal  $J$  in  $\mathcal{C}_c(Y)$  is closed if and only if it is of the form  $I(N_Y(J))$ .*

Another application of the theory developed in chapter 1 is the following theorem

**THEOREM 10.** *Let  $X$  be a completely regular topological space. The associated locally convex space of  $\mathcal{C}_c(X)$  is  $\mathcal{C}_{co}(X)$ .*

*Proof.* Clearly the identity from  $\mathcal{C}_c(X)$  into the locally convex topological vector space  $\mathcal{C}_{co}(X)$  is continuous. Since

$$\text{id}: \mathcal{C}_I(X) \rightarrow \mathcal{C}_c(X)$$

is also continuous, in view of theorem 4 the proof is complete.

By reasoning as in the proof of the last theorem, we obtain

**THEOREM 11.** *For any completely regular space  $X$  the spaces  $\mathcal{L}(\mathcal{C}_I(X))$ ,  $\mathcal{L}(\mathcal{C}_c(X))$ , and  $\mathcal{L}(\mathcal{C}_{co}(X))$  are identical.*

## REFERENCES

- [1] BINZ, E. and KELLER, H. H., *Funktionenräume in der Kategorie der Limesräume*, Ann. Acad. Sci. Fenn. A, I, 383 (1966), 1–21.
- [2] BINZ, E., *Bemerkungen zu limitierten Funktionenalgebren*, Math. Ann. 175 (1968), 169–184.
- [3] —, *Zu den Beziehungen zwischen  $c$ -einbettbaren Limesräumen und ihren limitierten Funktionenalgebren*, Math. Ann. 181 (1969), 45–52.

- [4] *On Closed Ideals in Convergence Function Algebras*, Math. Ann. 182 (1969), 145–153.
- [5] BUTZMANN, H.-P., *Dualitäten in  $\mathcal{C}_c(X)$* , Thesis, Universität Mannheim.
- [6] GILLMAN, L. and JERISON, M., *Rings of Continuous Functions* (Van Nostrand, Princeton, 1960).
- [7] JARCHOW, H., *Marinescu-Räume*, Comment. Math. Helv. 44 (1969), 138–163.

*Lehrstuhl für Mathematik I*  
*Universität Mannheim*  
*D-68 Mannheim, Schloss, W.-Germany*

Received January 19, 1971