Semicontinuity of the Face-Function of a Covex Set

Semicontinuity of the Face-Function of a Convex Set

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Dedicated to Hugo Hadwiger on His Sixtieth Birthday

Introduction

Throughout this paper, X is assumed to be a closed subset of a topological linear space E; other conditions on X and E are stated explicitly, sometimes in standing hypotheses at the beginning of a section. For each point x of X, $F(x)$ is the closure of the union of $\{x\}$ with all segments in X that cross x; that is,

$$F(x) = \mathrm{cl}\,\{y : [y, y + \lambda(x - y)] \subset X \text{ for some } \lambda > 1\}.$$

The set-valued function F is called the *face-function* of X, for when X is a finite-dimensional convex body or the boundary of such a body, $F(x)$ is the smallest face of X that includes x. (In this case the closure in the definition of $F(x)$ is redundant.) The set of all points of X at which the face-function F is lower [resp. upper] semicontinuous is denoted by X_l [resp. X_u].

Though some of our theorems apply to infinite-dimensional sets, this introduction describes only the finite-dimensional results. One of our two main results is the following.

THEOREM A. *If X is the boundary of a d-dimensional convex body then the face-function F of X is lower semicontinuous almost everywhere in the sense of category and upper semicontinuous almost everywhere in the sense of measure. However, when $d \geqslant 3$ an example of Corson* [4] *shows that F may be lower semicontinuous almost nowhere in the sense of measure and upper semicontinuous almost nowhere in the sense of category.*

Actually, the statement that F is upper semicontinuous almost everywhere in the sense of measure is proved only for $d \leqslant 3$, so part of the above theorem remains as a conjecture.

Now let K be a compact convex set in a locally convex E and let $C(K)$ be the set of all continuous real-valued functions on K. For each $f \in C(K)$ let f_e denote the restriction to K of the pointwise supremum of all continuous affine functions on E that are majorized on K by f. The function f_e, which is plainly real-valued, convex, and lower semicontinuous, is here called the *envelope* of f. Let K_e denote the set of

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all points of K at which all envelopes are continuous. For a finite-dimensional K, the points of K_e were shown by Witsenhausen [17, p. 20] to be precisely the points of convergence of a certain approximate algorithm for a type of minimax stochastic control problem associated with K. Plainly $\operatorname{int} K \subset K_e$, and Witsenhausen proved that $K_e = K$ when K is a polytope as well as when K is strictly convex. Kruskal [11] reported Witsenhausen's conjecture that $K_e = K$ for all finite-dimensional K, disproved it by a 3-dimensional example, and conjectured that $K_e = K$ for all 2-dimensional K. Our second main result is the following.

THEOREM B. *If X is the boundary of a d-dimensional compact convex body K, and $X_e = X \cap K_e$, then $X_e \subset X_l$ and X_e is almost all of X in the sense of category. Further, $X_e = X$ when $d = 2$.*

Thus Kruskal's conjecture is proved and it is seen that Witsenhausen's algorithm has good convergence properties with respect to category, though not necessarily with respect to measure.

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Preparatory Remarks

Semicontinuity is used in the sense of Choquet [3] and Fort [5]. If T is a topological space, $\mathbf{S}(T)$ the set of all closed subsets of T, and N_δ a net on a directed set D into $\mathbf{S}(T)$, then

$\underline{\lim}_{\delta \in D} N_\delta = \{t \in T$: for each neighborhood U of t, N_δ eventually intersects $U\}$,

$\overline{\lim}_{\delta \in D} N_\delta = \{t \in T$: for each neighborhood U of t, N_δ frequently intersects $U\}$.

If Z is a topological space and S is a function on Z into $\mathbf{S}(T)$, then S is said to be *lower* [resp. *upper*] *semicontinuous* at the point z_0 of Z provided that $\underline{\lim}_{\delta \in D} S(z_\delta) \supset S(z_0)$ [resp. $\overline{\lim}_{\delta \in D} S(z_\delta) \subset S(z_0)$] for each net z_δ that converges to z_0 in Z. When the point z_0 admits a countable neighborhood base, these definitions are equivalent to the corresponding ones that involve sequences rather than nets. When T is compact they are equivalent to requiring that for each open set U which intersects $S(z_0)$ [resp. contains $S(z_0)$] the set $\{z \in Z: U$ intersects $S(x)\}$ [resp. $\{z \in Z: U$ contains $S(z)\}$] is a neighborhood of z_0.

As the term is used here, a *body* is a set that has nonempty interior. Hausdorff k-dimensional measure is denoted μ_k.

Now suppose that X is a closed convex set or is the boundary of such a set. Then a *face* of X is defined as a convex subset C of X such that C contains $[p, q]$ whenever $p, q \in X$ and C intersects $]p, q[$. For $x \in X$, the smallest face of X that includes x is the set

$$F_a(x) = \{y: [y, y + \lambda(x - y)] \subset X \text{ for some } \lambda > 1\}.$$

If X is finite-dimensional, then $F(x)=F_a(x)$ and F deserves its name of *face-function.* However, the reader should be warned that when X is infinite-dimensional the set $F(x)$ need not be a face of X. Suppose, for example, that X is a compact Choquet simplex. Then it may happen that the set $\operatorname{ext} X$ of all extreme points of X is dense in X (Thue Poulsen [16]). On the other hand, it follows from results of Alfsen [1, p. 101] and Størmer [15, p. 257] that $\operatorname{ext} X$ is closed if and only if $F(x)$ is a face of X for all $x \in X$.

Lower Semicontinuity of the Face-Function

Recall the standing hypotheses that X is a closed subset of a topological linear space E, F is the face-function of X, and X_l is the set of all points of X at which F is lower semicontinuous.

1.1 THEOREM. *If X is locally compact, separable, and metrizable, then X_l is a dense G_δ subset of X.*

Proof. Let $X_1, X_2, \ldots$ be a sequence of compact sets whose union is X. For each pair of positive integers i and j let W_{ij} denote the set of all points x of X for which there exists a point $y \in X_j$ and there exists a sequence x_δ in X converging to x such that

(a) $[y, y+\lambda(x-y)] \subset X$ for some $\lambda \in [1+i^{-1}, 1+i]$, and

(b) the distance from the point y to the set $\underline{\lim} F(x_\delta)$ is at least i^{-1}.

It follows by routine arguments that $X \sim X_l = \bigcup_{i,j=1}^{\infty} W_{ij}$ and each set W_{ij} is closed. Hence X_l is a G_δ set.

For each positive integer i and each point x of X, let $F_i(x) = \operatorname{cl}\{y : [y, y+\lambda(x-y)] \subset X$ for some $\lambda \in [1+i^{-1}, 1+i]\}$. As each F_i is everywhere upper semicontinuous, it follows from Theorem 12 of Fort [5] and the Baire category theorem that there is a dense G_δ subset V of X such that each function F_i is lower semicontinuous at each point of V. Consider an arbitrary point v of V and sequence x_δ in X converging to v. For each i,

$$F_i(v) \subset \underline{\lim}\, F_i(x_\delta) \subset \underline{\lim}\, F(x_\delta),$$

and as the last set is closed it follows that

$$F(v) = \operatorname{cl} \bigcup_1^{\infty} F_i(v) \subset \underline{\lim}\, F(x_\delta). \quad \square$$

For the X of Theorem A, 1.1 implies X_l is a dense G_δ subset of X. If $d \leqslant 2$ the set $\operatorname{ext} X$ is closed and consequently $X_l = X$. We now describe a simple example showing that if $d \geqslant 3$ X_l may be a small subset of X in the sense of measure. Let C be the unit circle $\{(\alpha, \beta, 0) : \alpha^2+\beta^2=1\}$, let $p=(0, 0, 1)$, and for each $c \in C$ and $\lambda \in [0, 1]$ let

$$A(c, \lambda) = [c, c+\lambda(p-c)] \cup [c, c+\lambda(-p-c)].$$

For each $\eta \in [0, \frac{1}{2}]$ let C^η be a Cantor set in C such that $\mu_1(C \sim C^\eta) < \eta$, let K^η be the convex hull of the set

$$\Big(\bigcup_{c\in C^\eta} A(c, 1-\eta)\Big)\cup\Big(\bigcup_{c\in C\sim C^\eta} A(c, 1-2\eta)\Big),$$

and let X^η denote the boundary of K^η. Then

$$\bigcup_{c\in C^\eta} A(c, 1-\eta)\subset X^\eta \sim X^\eta_l,$$

whence it follows that $\lim_{\eta\to 0}\mu_2(X^\eta_l)/\mu_2(X^\eta)=0$ even though X^η converges to the set $\bigcup_{c\in C}A(c, 1)$ as $\varepsilon\to 0$.

Up to this point, our discussion has been aimed at the case in which X is the boundary of a finite-dimensional convex body. When X is the boundary of an infinite-dimensional convex body (or, in particular, is the unit sphere of an infinite-dimensional separable Banach space), we do not know whether X_l is necessarily nonempty or dense in X. Now suppose, on the other hand, that X is an entire convex set. X_l is dense in X when X is a body. But what happens when X is an infinite-dimensional compact convex set? In the metrizable case, 1.1 implies X_l is almost all of X in the sense of category. In the locally convex case, $\operatorname{cl}\operatorname{con}X_l=X$ by the Krein-Milman theorem (for plainly $X_l\supset \operatorname{ext}X$), but we do not know whether X_l must be dense in X. If X is compact and convex but nonmetrizable and E is not locally convex, must X_l be nonempty?

Continuity of Envelope Functions

Throughout this section, X is the boundary of a compact convex set K in a locally convex space. The set K_e is as defined in the introduction, and $X_e=X\cap K_e$.

2.1 PROPOSITION. *For any $f\in C(K)$ and any $p\in K$, $f_e(p)$ is the infimum of all numbers of the form $\sum_1^n\alpha_i f(k_i)$, where $p=\sum_1^n\alpha_i k_i$ is an expression of p as a convex combination of points k_i of K. For any closed face L of K, the envelope of f's restriction to L is equal to the restriction to L of f's envelope.*

Proof. The first assertion follows from Lemma 9.6 and Proposition 4.5 of Phelps [14]. The second assertion is an immediate consequence of the first one and the definition of a face. □

The following formalizes the idea behind Kruskal's example [11] of a 3-dimensional K for which $K_e\neq K$.

2.2 THEOREM. *If F is the face-function of K and $F(k)$ is a face for all $k\in K$, then $K_e\subset K_l$; that is, F is lower semicontinuous at each point of K_e.*

Proof. Consider an arbitrary point p of $K\sim K_l$, and let k_δ be a net converging to p in K such that $\underline{\lim}\, F(k_\delta)\not\supset F(p)$. By passing to a subnet if necessary, we can find a point q of $F(p)$ and a neighborhood V of q that is disjoint from C, the closure of the union of the sets $F(k_\delta)$. There exist points $v \in V \cap F(p)$ and $w \in K$ and a number $\lambda \in [0, 1]$ such that $p = \lambda v + (1-\lambda) w$. Now let $f(c) = 1$ for all $c \in C \cup \{w\}$, let $f(v) = 0$, and extend f to a member of $C(K)$. As f is identically 1 on each of the faces $F(k_\delta)$, it follows from the second part of 2.1 that $f_e(k_\delta) = 1$ for all δ. However, any affine function g majorized on K by f must have

$$g(p) = \lambda g(v) + (1-\lambda) g(w) \leqslant 1 - \lambda,$$

so that $f_e(p) < 1$, f_e is discontinuous at p, and $p \notin K_e$. □

2.3 THEOREM. *If K is metrizable then the set of points at which the restriction to X of every envelope function on K is continuous forms a dense G_δ subset of X.*

Proof. Since K is metrizable there is a sequence f_δ of functions uniformly dense in $C(K)$. It follows from [14, p. 19(c) and 4.5] that the sequence f_{δ_e} of envelopes is uniformly dense in the collection of all envelope functions on K. Therefore the set in question is the intersection of the sets of points of continuity of the restrictions of the f_{δ_e}, each of which is well known to be a dense G_δ subset of X. The desired conclusion is then an immediate consequence of the Baire category theorem. □

2.4 COROLLARY. *If K is a metrizable compact convex subset of a locally convex space then K_e is a dense G_δ subset of K.*

2.5 COROLLARY. *If X is the boundary of a d-dimensional compact convex body K then $X_e \subset X_l$ and X_e is a dense G_δ subset of X.*

Proof. The first statement of 2.5 follows from 2.2. In view of 2.3 it suffices for the second to show that a real-valued, lower semicontinuous, convex function f on K is continuous at every point of continuity of its restriction to X. This is left to the reader. □

We do not know, in general, whether $X_e = X_l$.

2.6 PROPOSITION. $X_e = X$ *if* $d \leqslant 2$.

Proof. If $F(x)$ is a segment it follows from the proof of 2.5 that f_e is continuous at x. If x is an extreme point then $f_e(x) = f(x)$ as a result of 2.1. It follows easily that f_e is upper semicontinuous at x. □

Upper Semicontinuity of the Face-Function

The standing hypotheses for this section are the same as for the preceding section. In addition, for any set Y with face-function G, the *inside* $I(Y)$ is defined as the set of all points $y \in Y$ such that $G(y) = Y$. (This notion is due to Michael [12].)

3.1 THEOREM. *If X is the boundary of a convex body and Y is a maximal convex subset of X then $I(Y) \subset X_u$.*

Proof. Let F and G be the face-functions of X and Y respectively, and consider an arbitrary point $y \in I(Y)$. Then $G(y) = Y$ by definition, and from the maximality of Y it follows that $F(y) = Y$. We want to show that for an arbitrary net x_δ converging to y in X, and an arbitrary point $p \in \overline{\lim} F(x_\delta)$, it is true that $p \in F(y)$. As $[p, y] \subset X$, there is a supporting hyperplane H of X such that $[p, y] \subset H$. Any segment in X that has y as an inner point must lie in H, whence $Y \subset H$ and it follows from the maximality of Y that

$$Y = X \cap H \ni p. \ \square$$

A subset U of a set X is said to be *ubiquitous* in X provided that every point of X belongs to U or is an endpoint of a segment in U.

3.2 THEOREM. *If E is complete, metrizable, and separable, and X is the boundary of a convex body in E, then X_u is the union of all sets of the form $I(Y)$, where Y is a maximal convex subset of X. The set X_u is ubiquitous in X.*

Proof. With U denoting the union in question, it follows from 3.1 that $U \subset X_u$. Any point x of X lies in a maximal convex subset Y of X, and as Y is closed a construction of Michael [12, 5.1] and Klee [7, 2.6] produces a point $y \in I(Y)$. It is easily verified that $]x, y] \subset I(Y)$ and thus U is ubiquitous. If $x \notin U$, then $F(x)$ does not contain Y even though $F(v) = Y$ for all $v \in]x, y]$, and consequently $x \notin X_u$. Thus $X_u = U$. $\square$

If X is as described in Theorem A of the introduction, it follows from 3.2 that X_u is dense in X. We conjecture, moreover, that $\mu_{d-1}(X \sim X_u) = 0$, but are able to prove this only for $d \leqslant 3$. The proof is based on the following result concerning upper semicontinuous collections of convex sets.

3.3 THEOREM. *Let $\mathbf{C}$ be an upper semicontinuous collection of compact convex subsets of Euclidean n-space E^n such that $\bigcup \mathbf{C}$ is a Borel set, and let $W = \bigcup_{C \in \mathbf{C}} (C \sim I(C))$. Then $\mu_n(W) = 0$ when $n \leqslant 2$.*

The theorem is trivial when $n = 1$, for then W is countable. Suppose that $n = 2$ and let $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{L}$ denote respectively the collections of all 0-, 1-, and 2-dimensional members of $\mathbf{C}$. Then $\bigcup \mathbf{J}$ is a G_δ set relative to $\bigcup \mathbf{C}$ and contributes nothing to W, while $\mathbf{L}$ is an F_σ set and contributes nothing to $\mu_2(W)$. Thus we may (and will) assume without loss of generality $\mathbf{C} = \mathbf{K}$.

Choose a small positive η——$\eta < 1/100$ will surely suffice. For each pair p, q of distinct points of E^2 whose coordinates are all rational, let $\mathbf{K}(p, q)$ denote the collection of all members of $\mathbf{K}$ that have one endpoint within $\eta \|p - q\|$ of p and the other within $\eta \|p - q\|$ of q; the set of former endpoints is denoted by $W(p, q)$. Then $\mathbf{K} = \bigcup_{p,q} \mathbf{K}(p, q)$ and $W = \bigcup_{p,q} W(p, q)$, and with the aid of upper semicontinuity it follows that for each p, q, $\bigcup \mathbf{K}(p, q)$ is a G_δ set relative to $\bigcup \mathbf{K}$. To complete the proof it suffices to show $\mu_2 W(p, q)=0$.

For each point $x \in W(p, q)$, and each ε with $0<\varepsilon<\eta$, let $S(x)$ be the member of $\mathbf{K}(p, q)$ that has x as one of its endpoints and let x_ε be the point of $S(x)$ whose distance from x is ε. Let $W_\varepsilon(p, q)=\{x_\varepsilon : x \in W(p, q)\}$. It follows with the aid of upper semicontinuity that each set $W_\varepsilon(p, q)$ is an $F_{\sigma\delta}$ set relative to $\bigcup \mathbf{K}(p, q)$ and hence is μ_2-measurable. To complete the proof it suffices to show that

$$(*) \qquad \|x_\varepsilon - y_\varepsilon\| \geqslant \|x - y\|/2$$

for all $x, y \in W(p, q)$ and $0<\varepsilon<\eta$, for it then follows that $\mu_2 W_\varepsilon(p, q) \geqslant \mu_2 W(p, q)/4$, and, as the various sets $W_\varepsilon(p, q)$ are pairwise disjoint, a contradiction would ensue if $\mu_2 W(p, q)>0$.

Let $L(x)$ denote the line containing $S(x)$, $L'(x)$ the part of $L(x) \sim S(x)$ adjoining x, and $L''(x)$ the rest of $L(x)$. If $L(x)$ and $L(y)$ are parallel the inequality (*) is trivial, so we assume that $L(x)$ and $L(y)$ intersect and, for notational simplicity, that their point of intersection is the origin 0. There are four possibilities: (a) $0 \in L'(x) \cap L'(y)$; (b) $0 \in L''(x) \cap L''(y)$; (c) $0 \in L'(x) \cap L''(y)$; (d) $0 \in L''(x) \cap L'(y)$. Denoting $\|x\|$ and $\|y\|$ by α and β respectively, in case (a), $x_\varepsilon = x + \varepsilon x \alpha^{-1}$ and $y_\varepsilon = y + \varepsilon y \beta^{-1}$, while in case (b) $x_\varepsilon = x - \varepsilon x \alpha^{-1}$ and $y_\varepsilon = y - \varepsilon y \beta^{-1}$. The two cases will be treated together. We have

$$\begin{aligned}
\|x_\varepsilon - y_\varepsilon\| &= \|x \pm \varepsilon x \alpha^{-1} - (y \pm \varepsilon y \beta^{-1})\| \\
&= \|x(1 \pm \varepsilon \alpha^{-1}) - y(1 \pm \varepsilon \alpha^{-1}) \pm \varepsilon y(\alpha^{-1} - \beta^{-1})\| \\
&= \|(1 \pm \varepsilon \alpha^{-1})(x - y) \pm \varepsilon y(\beta - \alpha)\alpha^{-1}\beta^{-1}\| \\
&\geqslant |1 \pm \varepsilon \alpha^{-1}| \cdot \|x - y\| - \varepsilon \alpha^{-1}\|x - y\|.
\end{aligned}$$

It follows that $\|x_\varepsilon - y_\varepsilon\| \geqslant \|x - y\|$ in case (a) and

$$\|x_\varepsilon - y_\varepsilon\| \geqslant (1 - 2\varepsilon\alpha^{-1})\|x - y\| \geqslant \|x - y\|/2$$

in case (b) because $\varepsilon \leqslant \alpha/4$ in this case. Finally, in case (c), let θ denote the acute angle between L and M and suppose first that 0 comes between y and y_ε. Then by the law of cosines

$$\begin{aligned}
\|x_\varepsilon - y_\varepsilon\|^2 &= (\varepsilon - \beta)^2 + (\varepsilon + \alpha)^2 - 2(\varepsilon - \beta)(\varepsilon + \alpha)\cos\theta \\
&= 2\varepsilon(\varepsilon - \beta + \alpha)(1 - \cos\theta) + \alpha^2 + \beta^2 + 2\alpha\beta\cos\theta \\
&> \alpha^2 + \beta^2 - 2\alpha\beta\cos(\pi - \theta) = \|x - y\|^2.
\end{aligned}$$

If, instead, y_ε comes between y and 0, then

$$\|x_\varepsilon - y_\varepsilon\|^2 = (\varepsilon + \alpha)^2 + (\beta - \varepsilon)^2 + 2(\varepsilon + \alpha)(\beta - \varepsilon)\cos\theta$$

and

$$\|x - y\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\cos\theta.$$

Hence $\|x_\varepsilon - y_\varepsilon\| \geqslant 2^{-\frac{1}{2}}\|x-y\|$ if and only if $\varepsilon^2+\alpha^2+(\beta-\varepsilon)^2+2\alpha(\beta-\varepsilon)\cos\theta+4\varepsilon\alpha+$ $+4\varepsilon(\beta-\varepsilon)\cos\theta \geqslant 2\varepsilon(\beta-\varepsilon)+2\varepsilon\alpha\cos\theta$. As $\beta-\varepsilon \geqslant 0$, $\varepsilon^2+(\beta-\varepsilon)^2 \geqslant 2\varepsilon(\beta-\varepsilon)$, and $4\varepsilon\alpha \geqslant 2\varepsilon\alpha\cos\theta$, the desired conclusion follows. The same argument handles case (d). □

The assumption of upper semicontinuity was used only to get measurability, but simple examples show this assumption cannot be completely discarded. In addition, some geometric condition related to the straightness of the segments is necessary. For consider a Jordan region J in the plane whose boundary curve has positive 2-dimensional measure, let **C** be a continuous collection of parallel segments filling all but two antipodal points of a circular disk D, and let **K** be the collection of arcs obtained from the members of **C** under a homeomorphism of D onto J. Then **K** is a continuous collection of pairwise disjoint arcs whose endpoints form a set of positive 2-dimensional measure.

For any collection S of line segments in E^d let $P(\mathrm{S})$ denote the set of all endpoints of the members of S. A subset M of E^d is called an *endset* provided that $M=P(\mathrm{S})$ for some collection S of pairwise disjoint segments. It follows from 3.3 that when $d \leq 2$ the d-dimensional Lebesgue measure of any compact (and hence any measurable) endset in E^d is zero. However, for $d=4$ (and hence for all $d \geq 4$) Bruckner and Ceder [2] have produced in E^d a compact endset of positive d-measure. Their construction is based on Nikodym's example [13] of a Cantor set X of positive measure in E^2 such that for each point x of X there is a line in E^2 intersecting X only at x. It is unknown whether a compact endset in E^3 must be of measure zero [10].

3.4 THEOREM. *If X is the boundary of a d-dimensional convex body then $\mu_{d-1}(X \sim X_u)=0$ when $d \leqslant 3$.*

Proof. No restriction on d is required until the end of the argument, when 3.3 is used. Assuming without loss of generality that the convex body in question is compact and is situated in a d-dimensional *Euclidean* space E^d, we show first that the body may also be assumed smooth.

Suppose X' is the boundary of a compact convex body Y' in E^d, and let X'_u denote the set of all points at which the face-function of X' is upper semicontinuous. Let B denote the unit ball of E^d and let $Y=Y'+B$, a smooth convex body. Let X denote the boundary of Y, and for each point $x \in X$ let x' denote the unique point of X' nearest to x. It is well-known that $\|x'_1-x'_2\| \leqslant \|x_1-x_2\|$ for all $x_1, x_2 \in X$, and hence the mapping $'$ carries sets of zero $(d-1)$-dimensional Hausdorff measure onto such sets. Thus the reduction to the case of smooth convex bodies is justified if we can show that the set $X \sim X_u$ is carried onto the set $X' \sim X'_u$ by the mapping $'$. For this it suffices, in view of 3.2, to show that for any maximal convex subset M' of X' there is a maximal convex $M \subset X$ such that the restriction to M of the mapping $'$ simply translates M onto M'. Indeed, choose $x' \in I(M')$, let J be a closed halfspace supporting X' at x', and let b be a point at which B is supported by a translate of J. Then the set $M=M'+b$ is a maximal convex subset of X, with $m'=m-b$ for all $m\in M$.

Now suppose X is the boundary of a smooth compact convex body Y in E^d, and let $\mathbf{M}$ denote the collection of all maximal convex subsets of X. For each $M\in\mathbf{M}$ there is a supporting hyperplane H of X such that $M=X\cap H$, whence it follows from smoothness that the members of M are pairwise disjoint and by a routine argument that $\mathbf{M}$ is upper semicontinuous. Let S denote the boundary of a d-dimensional simplex that contains Y and let ϱ be a radial mapping of S onto X. Let $T_0,\ldots, T_d$ be the $(d-1)$-dimensional faces of S, and for $0\leqslant i\leqslant d$ let

$$\mathbf{M}_i=\{(\varrho^{-1}M)\cap T_i: M\in\mathbf{M}\},\quad W_i=\bigcup_{C\in\mathbf{M}_i}(C\sim I(C)).$$

As M_i is an upper semicontinuous collection of compact convex sets, it follows from 3.3 with $n=d-1$ that $\mu_{d-1}(W_i)=0$. But $X\sim X_u\subset\bigcup_0^d\varrho W_i$ and the mapping ϱ is known to be lipschitzian ([6]), so the desired conclusion follows when $d\leqslant 3$. □

3.5 PROPOSITION. *If X is the boundary of a d-dimensional convex body then X_u is an $F_{\sigma\delta}$ set.*

Proof. Let A denote the set of all points x of X such that x lies in two or more maximal convex subsets of X. For each i, let A_i denote the set of all $x\in X$ such that there are two hyperplanes H' and H'' supporting X at x and forming an angle of at least $1/i$ and there are points $x'\in X\cap H'$ and $x''\in X\cap H''$ such that

$$\|x'-x\|\geqslant 1/i\leqslant\|x''-x\|.$$

Then $A=\bigcup_1^\infty A_i$ and a routine compactness argument shows each set A_i is closed. Thus A is an F_σ set. Let $S=X\sim A$, and for each $p\in S$ let M_p denote the unique maximal convex subset of X containing p. Then S is a G_δ set.

For $1\leqslant j\leqslant d-1$, the *j-interior* $\operatorname{int}_j X$ is defined as the set of all points x of X for which there exists a j-dimensional flat J such that x is interior to $X\cap J$ relative to J. Plainly $\operatorname{int}_{d-1}X$ is open relative to X, and a routine compactness argument shows each set $\operatorname{int}_j X$ is an F_σ set.

For each pair of positive integers i and j with $j<d$ let

$S_{ij}=\{p\in S: M_p$ contains a j-dimensional ball of radius $\geqslant 1/i\}$.

Then S_{ij} is closed relative to S and it follows from 3.2 that

$$X_u=S\sim\bigcup_{j=1}^{d-1}\bigcup_{i=1}^{\infty}[(X\sim\operatorname{int}_j X)\cap S_{ij}],$$

the difference of a G_δ and a $G_{\delta\sigma}$. □

If X is as described in the conjecture stated in the introduction, it follows from 3.5 that X_u is an $F_{\sigma\delta}$ set, and from 3.4 that X_u is almost all of X in the sense of measure (at least when $d\leqslant 3$). Of course $X\sim X_u$ is countable when $d=2$. We now describe some examples with $d=3$ showing X_u may fail to be either an F_σ set or a G_δ set, and it may be of the first category in X.

Suppose X is the boundary of a d-dimensional convex body and all the maximal convex subsets of X are 0- or 1-dimensional. By 3.2, $X \sim X_u$ is the set Z of all endpoints of 1-dimensional maximal convex subsets of X. Now suppose, in addition, that both Z and $(\text{ext} X) \sim Z$ are dense in X. As $(\text{ext} X) \sim Z$ is a G_δ set, it follows from the Baire category theorem that Z is not a G_δ set and hence X_u is not an F_σ set. With $d=3$, an X in which the sets Z and $(\text{ext} X) \sim Z$ are both dense can be constructed by a procedure used by Klee [8, pp. 99–103] for constructing an X in which the set $\exp X$ of all exposed points of x is not a G_δ set. (A point x of X is *exposed* provided that there is a supporting hyperplane H of X such that $X \cap H = \{x\}$.)

3.6 EXAMPLE. *For $d \geqslant 3$ there is a d-dimensional compact convex body with boundary X such that $\mu_{d-1}(X_l)=0$ and X_u is of the first category in X.*

Proof. Let Y' denote the union of all segments joining the point $(0, 0, 1)$ to a point of the semicircle $\{(\alpha, \beta, 0): \alpha \geqslant 0,\ \alpha^2+\beta^2=1\}$, and let Z denote the union of the semidisk $\{(\alpha, \beta, 0): \alpha \geqslant 0,\ \alpha^2+\beta^2 \leqslant 1\}$ with the triangle $\text{con}\{(0, 1, 0), (0, -1, 0), (0, 0, 1)\}$. Let $K' = \text{con}\, Y' = \text{con}\, Z$, a 3-dimensional convex body (half of a truncated circular cone) whose boundary is $Y' \cup Z$. By a slight perturbation of Y', moving Y' to a nearby position Y while leaving invariant the points of $Y' \cap Z$, Corson [4] constructs a convex body $K = \text{con}\, Y$ such that the boundary of K is $Y \cup Z$ and the following conditions are satisfied: $\text{ext}\, Y$ is dense in Y, $\exp Y$ is an F_σ set, the set $W = \text{ext}\, Y \sim$ $\sim \exp Y$ is dense in $\text{ext}\, Y$, and every exposed point of $Y \sim Z$ is an endpoint of a segment in $Y \sim Z$. It follows that $Y_u \subset Y \sim W$, $Y_l = \text{ext}\, Y$, and $Y_l \cap Y_u \subset Y \cap Z$. As W is a dense G_δ set in Y, it follows from the Baire category theorem that Y_u is not a G_δ set in Y and is of the first category in Y. Further, it follows from 3.4 that $\mu_2(Y_l)=0$.

Four copies of the above example Y (or, rather, of its intersection with a suitable halfspace) can be fitted together to form a slight perturbation of a double cone resulting in a 3-dimensional compact convex body with boundary X such that X_u is not a G_δ set in X and is of the first category in X, and $\mu_2(X_l)=0$. Successive double cones over this set X result in similar examples in higher dimensions. □

Our discussion has thus far been aimed at the case in which X is the boundary of a finite-dimensional convex body. When X is the boundary of a convex body in a metrizable linear space E that is complete and separable, it follows from 3.2 that X_u is dense in X. To see that the completeness assumption cannot be abandoned, let X be the unit sphere of the subspace of $l(\aleph_0)$ consisting of all points having only finitely many nonzero coordinates. To see that the separability assumption cannot be abandoned, let X be the unit sphere of the space $l(\aleph_1)$. In each case the set X_u is empty. Now suppose, on the other hand, that X is an entire convex set. Then $X_u \supset I(X)$ and hence X_u is dense in X if $I(X) \neq \emptyset$ (in particular, if X is centrally symmetric or is complete, metrizable, and separable). If X is a body then $X_u = \operatorname{int} X$. If X is the unit ball of $l^2(\aleph_0)$ in the weak topology, then X is compact and metrizable and X_u is an F_σ set of the first category in X. If X is the intersection of the unit ball of $l^2(\aleph_0)$ with the positive cone of $l^2(\aleph_0)$, in the weak topology, then X is compact and X_u is empty.

Continuity of the Face-Function

With X denoting a closed subset of a topological linear space E, let $X_c = X_l \cap X_u$, the set of all points at which the face-function of X is continuous. Then $X_c = \operatorname{int} X$ when X is a convex body. Suppose, on the other hand, that E is an infinite-dimensional separable Banach space, and let Z_E denote the space of all compact convex subsets of E, metrized by the Hausdorff metric. Then Z_E is a complete metric space and those $X \in Z_E$ for which X_c is nonempty form a first category subset of Z_E. To see this, note that $X_c = \emptyset$ whenever $\operatorname{ext} X$ is dense in X, and then apply two results of Klee [9, 2.1 and 2.2].

Note that $\mu_2(X_c) = 0$ in the example of 3.6; indeed, X_c is the union of a finite number of rectifiable arcs. To obtain an unbounded 3-dimensional convex body with boundary X such that X_c is empty, fit together two copies of the Y of 3.6 so as to obtain a perturbation of a circular cone that has the same base as the cone; then send the base to infinity by a projective transformation.

We do not know whether X_c may be empty when X is the boundary of a bounded convex body in a complete metrizable separable space E of dimension $d \geqslant 3$. However, it seems probable that X_c may be empty when $d > 3$ but not when $d = 3$.

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