

Zeitschrift: Commentarii Mathematici Helvetici
Band: 47 (1972)

Artikel: Infinite Symplectic Groups over Rings
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DOI: <https://doi.org/10.5169/seals-36364>

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Infinite Symplectic Groups over Rings

GEORGE MAXWELL

§0. Introduction

Let A be a commutative ring, M a free A -module $A^{(I)}$, for some infinite set I , and $R = \text{End}_A(M)$. In a previous paper [4], we proved that normal subgroups of the group $U(R)$ of units of R must lie in congruence layers determined by the ideals of A . We now suppose that M possesses a nondegenerate alternate bilinear form (\cdot, \cdot) and prove a similar result for the infinite symplectic group

$$\text{Sp}(R) = \{u \in U(R) \mid (u(x), u(y)) = (x, y) \text{ for all } x, y \in M\},$$

at least when $\frac{1}{2} \in A$ and the form (\cdot, \cdot) is “locally hyperbolic”. The strategy of the proof is again the one mapped out by Bass in [2] and [3]. When A is a field, our results coincide with those of Spiegel [6]. One should also note that Bak [1], Vaserstein [7] and Vaserstein and Mihalev [8] have recently studied the orthogonal analogue of Bass’ results in the “stable” finite case.

§1. Locally Hyperbolic forms

A submodule N of M is called hyperbolic if $M = N \oplus N$ and $N = N_1 \oplus N_2$, where N_1 and N_2 are totally isotropic and have bases $\{e_j\}_{j \in J}$ and $\{e^j\}_{j \in J}$ such that $(e_j, e^j) = 1$ for all $j \in J$. The basis $\{e_j, e^j\}_{j \in J}$ is then called a hyperbolic basis of N . The form (\cdot, \cdot) is called locally hyperbolic if every finitely generated submodule of M is contained in a hyperbolic submodule. When A is a field, this condition is automatically satisfied. In general, it may be satisfied by assuming a priori the existence of a hyperbolic basis for all of M .

(1.1.) *Remark.* If (\cdot, \cdot) is locally hyperbolic, then for all unimodular $x \in M$ there exists a unimodular $y \in M$ such that $(y, x) = 1$. For suppose N is a hyperbolic submodule of M containing x and $\{e_j, e^j\}_{j \in J}$ is a hyperbolic basis of N . If $x = \sum(e_j a_j + e^j a^j)$, there must exist a relation $\sum(b_j a_j + b^j a^j) = 1$ since x is unimodular. It suffices to take $y = \sum(e_j b^j - e^j b_j)$.

If \mathfrak{q} is an ideal of A , the form (\cdot, \cdot) induces, as usual, an alternate bilinear form $(\cdot, \cdot)_{\mathfrak{q}}$ on the free A/\mathfrak{q} -module $M \otimes_A A/\mathfrak{q} \cong (A/\mathfrak{q})^{(I)}$ which, in general, need not be nondegenerate. However, if (\cdot, \cdot) is locally hyperbolic, then $(\cdot, \cdot)_{\mathfrak{q}}$ is clearly locally hyperbolic and is furthermore nondegenerate. For suppose $x \otimes 1 \in M \otimes_A A/\mathfrak{q}$ is such that $(x \otimes 1, y \otimes 1)_{\mathfrak{q}} = 0$ for all $y \in M$. Let N be a hyperbolic submodule of M containing

x and suppose $\{e_j, e^j\}_{j \in J}$ is a hyperbolic basis of N . If $x = \sum(e_j a_j + e^j a^j)$, we have $(x, e^j) = a_j$ and $(x, e_j) = -a^j$ so that all a_j and a^j must be in \mathfrak{q} i.e. $x \otimes 1 = 0$.

We have $\text{End}_{A/\mathfrak{q}}(M \otimes_A A/\mathfrak{q}) \cong \text{End}_{A/\mathfrak{q}}((A/\mathfrak{q})^{(I)}) \cong R/(\mathfrak{q})$, where $(\mathfrak{q}) = \{u \in U(R) \mid u(M) \subset M \cdot \mathfrak{q}\}$ is an ideal of R . The projection $R \rightarrow R/(\mathfrak{q})$ induces a homomorphism $U(R) \rightarrow U(R/(\mathfrak{q}))$ and, if we regard $M \otimes_A A/\mathfrak{q}$ as being equipped with the form $(\cdot, \cdot)_{\mathfrak{q}}$, a homomorphism $\text{Sp}(R) \rightarrow \text{Sp}(R/(\mathfrak{q}))$. The kernel of this homomorphism is denoted by $\text{Sp}(\mathfrak{q})$ and the inverse image of the center of $\text{Sp}(R/\mathfrak{q})$ by $\text{Sp}'(\mathfrak{q})$.

§2. Preliminary Results

From now on, the form (\cdot, \cdot) is assumed to be locally hyperbolic. For every unimodular $x \in M$ and every $a \in A$, the mapping $\tau(a, x)(m) = m + x \cdot a(m, x)$ belongs to $\text{Sp}(R)$ and is called a transvection. The subgroup generated by all transvections is denoted by $\text{ESp}(R)$. If \mathfrak{q} is an ideal of A and $a \in \mathfrak{q}$, $\tau(a, x)$ is called a \mathfrak{q} -transvection: the subgroup generated by all \mathfrak{q} -transvections is denoted by $\text{ESp}(\mathfrak{q})$. If $\sigma \in \text{Sp}(R)$, the formula

$$\sigma \tau(a, x) \sigma^{-1} = \tau(a, \sigma(x)) \tag{2.1}$$

shows that $\text{ESp}(\mathfrak{q})$, and in particular $\text{ESp}(R)$, is a normal subgroup of $\text{Sp}(R)$ for all ideals \mathfrak{q} . It is clear that $\text{ESp}(\mathfrak{q}) = \text{ESp}(\mathfrak{q}')$ only if $\mathfrak{q} = \mathfrak{q}'$.

(2.2) PROPOSITION. *The orbits of $\text{ESp}(\mathfrak{q})$ operating on the unimodular elements of M are the congruence classes mod $M \cdot \mathfrak{q}$. In particular, $\text{ESp}(R)$ operates transitively.*

Proof. Suppose x and y are unimodular elements of M congruent mod $M \cdot \mathfrak{q}$. Since (\cdot, \cdot) is locally hyperbolic, there exists a hyperbolic submodule N containing both x and y . Let $\{e_j, e^j\}_{j \in J}$ be a hyperbolic basis of N . It is sufficient to show that a fixed $e_i \in N$ can be mapped into any unimodular element $z \equiv e_i \pmod{M \cdot \mathfrak{q}}$ by an element of $\text{ESp}(\mathfrak{q})$. For then, applying this with $\mathfrak{q} = A$, we first see that $\beta(x) = e_i$ for some $\beta \in \text{ESp}(R)$. Since $\beta(y) \equiv e_i \pmod{M \cdot \mathfrak{q}}$, the same argument shows that $\gamma(e_i) = \beta(y)$ for some $\gamma \in \text{ESp}(\mathfrak{q})$. Therefore $\beta^{-1} \gamma \beta(x) = y$ and $\beta^{-1} \gamma \beta \in \text{ESp}(\mathfrak{q})$.

By enlarging N if necessary, we may assume that for a certain index $k \in J (k \neq i)$, both e_k and e^k occur with coefficient zero in z . Suppose

$$z = e_i(1 + q_i) + e^i q^i + \sum_{j \neq i} (e_j q_j + e^j q^j).$$

Since z is unimodular, there exists a relation

$$a_i(1 + q_i) + a^i q^i + \sum_{j \neq i} (a_j q_j + a^j q^j) = 1.$$

Let

$$\begin{aligned}\alpha_j &= \tau(-q_j - q^j - q_j q^j, e^i) \tau(-q^j, e^j) \tau(q^j, e^i + e^j) \tau(q_j, e^i + e_j) \\ \beta &= \tau(-q_i, e^k - e_i + e^i - e_k) \tau(q_i, e^k - e_i) \tau(q_i(1 + q^i), e^i - e_k) \tau(q^i, e^i) \\ \gamma_j &= \tau(q_i a_j, e^j + e^k) \tau(-q_i a_j, e^j) \tau(-q_i a^j, e_j + e^k) \tau(q_i a^j, e_j) \\ \delta &= \left(\prod_{j \neq k} \gamma_j \right) \beta \left(\prod_{j \neq i} \alpha_j \right)\end{aligned}$$

Then $\delta(e_i) = z$ since $\prod_{j \neq i} \alpha_j$ adds $\sum_{j \neq i} (e_j q_j + e^j q^j)$ to e_i , β adds $e_i q_i + e^i q^i$ at the expense of subtracting $e^k q_i$ and $\prod_{j \neq k} \gamma_j$ removes the $e^k q_i$. \parallel

(2.3) COROLLARY. *The natural homomorphism $\text{ESp}(R) \rightarrow \text{ESp}(R/(\mathfrak{q}))$ is surjective.*

Proof. Let $\tau(\bar{a}, \bar{x})$ be a transvection in $\text{ESp}(R/(\mathfrak{q}))$: \bar{x} is unimodular in $M \otimes_A A/\mathfrak{q}$, but x need not be unimodular in M . Suppose N is a hyperbolic submodule of M containing x with a hyperbolic basis $\{e_j, e^j\}_{j \in J}$. Applying (2.2) to $M \otimes_A A/\mathfrak{q}$, we see that $\bar{x} = \bar{\delta}(\bar{e}_i)$ for some $i \in J$ and $\bar{\delta}$ constructed as above; hence $\tau(\bar{a}, \bar{x}) = \bar{\delta} \tau(\bar{a}, \bar{e}_i) \bar{\delta}^{-1}$. However, each of the unimodular elements of $M \otimes_A A/\mathfrak{q}$ occurring in the transvections composing $\bar{\delta}$ clearly comes from a unimodular element of M . \parallel

(2.4) PROPOSITION.

$$\text{ESp}(\mathfrak{q}) = [\text{ESp}(R), \text{ESp}(\mathfrak{q})].$$

Proof. In view of (2.2) and (2.1), it is sufficient to prove that all \mathfrak{q} -transvections $\tau(a, x)$ for some particular unimodular $x \in M$ are in $[\text{ESp}(R), \text{ESp}(\mathfrak{q})]$. Choose a hyperbolic submodule N with a hyperbolic basis $\{e_i, e^i\}_{1 \leq i \leq 3}$. The easily verified identity

$$\begin{aligned}\tau(-a, e_1 + e_2 + e_3) \tau(a, e_1 + e_2) \tau(a, e_1 + e_3) \tau(a, e_2 + e_3) \tau(-a, e_1) \\ \tau(-a, e_2) \tau(-a, e_3) = 1\end{aligned}$$

can be written in the form

$$\tau(a, e_1) = [\beta, \tau(-a, e_2 + e_3) \tau(a, e_3)] [\gamma, \tau(a, e_2)],$$

where $\beta = \tau(-1, e^3) \tau(1, e^3 + e_1)$ and $\gamma = \tau(-1, e^2) \tau(1, e^2 + e_1)$ are in $\text{ESp}(R)$ and have the effect, respectively, of sending e_3 to $e_3 + e_1$ and e_2 to $e_2 + e_1$. \parallel

(2.5) PROPOSITION.

$$[\text{ESp}(R), \text{Sp}'(\mathfrak{q})] = \text{ESp}(\mathfrak{q}).$$

Proof. We first show that $[\text{ESp}(R), \text{Sp}(\mathfrak{q})] \subset \text{ESp}(\mathfrak{q})$. If $\tau(a, x)$ is any transvection and $\sigma \in \text{Sp}(\mathfrak{q})$, then by (2.2) $\sigma(x) = \beta(x)$ for some $\beta \in \text{ESp}(\mathfrak{q})$. Hence $[\tau(a, x), \sigma] = \tau(a, x) \tau(-a, \sigma(x)) = [\tau(a, x), \beta] \in \text{ESp}(\mathfrak{q})$. Reducing mod \mathfrak{q} , we see that $[\text{ESp}(R), \text{Sp}'(\mathfrak{q})] \subset \text{Sp}(\mathfrak{q})$; therefore $[\text{ESp}(R), [\text{ESp}(R), \text{Sp}'(\mathfrak{q})]] \subset \text{ESp}(\mathfrak{q})$. The ‘‘3-subgroups’’ lemma [5, p. 59] now implies that $[[\text{ESp}(R), \text{ESp}(R)], \text{Sp}'(\mathfrak{q})] \subset \text{ESp}(\mathfrak{q})$. However, $[\text{ESp}(R), \text{ESp}(R)] = \text{ESp}(R)$ by (2.4) so that $[\text{ESp}(R), \text{Sp}'(\mathfrak{q})] \subset \text{ESp}(\mathfrak{q})$; the opposite inclusion follows from (2.4). \parallel

§3. The Main Theorem

From now on, we assume that $\frac{1}{2} \in A$. In the following propositions, G is a subgroup of $\text{Sp}(R)$ normalised by $\text{ESp}(R)$.

(3.1) PROPOSITION. *If $(x, \sigma(x)) = 0$ for all $\sigma \in G$ and all unimodular $x \in M$, then G is contained in the center of $\text{Sp}(R)$.*

Proof. Linearising the identity $(x, \sigma(x)) = 0$, we conclude that if x, y and $x + y$ are all unimodular, then $(x, \sigma(y)) + ((y, \sigma(x)) = 0$. Since every $x \in M$ can be written in the form $\sum e_i a_i$ for some basis $(e_i)_{i \in I}$ of M , we conclude that $(x, \sigma(x)) = \sum (e_i, \sigma(e_i)) a_i^2 + \sum_{i \neq j} (e_i, \sigma(e_j)) + (e_j, \sigma(e_i)) a_i a_j = 0$ for all $x \in M$.

Therefore, for all $x, y \in M$, we have $(x, \sigma(y)) = -(y, \sigma(x)) = (\sigma(x), y) = (x, \sigma^{-1}(y))$. Since (\cdot, \cdot) is nondegenerate, we conclude that $\sigma = \sigma^{-1}$ for all $\sigma \in G$, i. e. G is an abelian group consisting of involutions.

If $\sigma \in G$ and $x \in M$ is unimodular, $[\sigma, \tau(1, x)] = \tau(1, \sigma(x)) \tau(-1, x) \in G$ and is therefore an involution. Moreover, $\tau(1, \sigma(x))$ commutes with $\tau(-1, x)$ since $(x, \sigma(x)) = 0$. We conclude that $\tau(2, \sigma(x)) = \tau(2, x)$ i. e. $2(y, \sigma(x)) \sigma(x) = 2(y, x) x$ for all $y \in M$. In view of (1.1), we can choose y such that $(y, \sigma(x)) = 1$; since $\frac{1}{2} \in A$, it follows that $\sigma(x) = x a_x$ for some $a_x \in A$. If $(e_i)_{i \in I}$ is a basis of M and $\sigma(e_i) = e_i a_i$, then $\sigma(e_i + e_j) = (e_i + e_j) a_{ij} = e_i a_i + e_j a_j$, so that $a_i = a_{ij} = a_j$ for $i \neq j$. Hence a_x is independent of x and σ is in the center. \parallel

(3.2) PROPOSITION. *If G is not contained in the center of $\text{Sp}(R)$, then G contains a transvection $\tau \neq 1$.*

Proof. By (3.1), $(x, \sigma(x)) = a \neq 0$ for some $\sigma \in G$ and some unimodular $x \in M$. Then $\sigma_1 = [\sigma, \tau(1, x)] = \tau(1, \sigma(x))$. $\tau(-1, x) \in G$. Let N be a hyperbolic submodule of M containing both x and $\sigma(x)$; suppose $\{e_j, e^j\}_{j \in J}$ is a hyperbolic basis of N . Enlarging N if necessary, we may assume that for some $k \in J$, both e_k and e^k occur with zero coefficient in x and $\sigma(x)$. Then G contains $\sigma_2 = [\tau(-1, \sigma(x)) \tau(1, e_k + \sigma(x)), \sigma_1] = \tau(1, \sigma(x)) \tau(-1, x + e_k a) \tau(1, x) \tau(-1, \sigma(x))$ and hence $\sigma_3 = \sigma_1^{-1} \sigma_2 \sigma_1 = \tau(1, x) \cdot \tau(-1, x + e_k a)$.

The construction of (1.1) produces an element $y \in N$ such that $(y, x) = 1$ and both

e_k and e^k occur with zero coefficient in y . Thus G contains $[\tau(1, y) \tau(-1, e_k + y), \sigma_3] = \tau(1, x + e_k) \tau(-1, x + (a+1)e_k) \tau(1, x + ae_k) \tau(-1, x) = \tau(-2a, e_k)$. \parallel

(3.3) PROPOSITION *If G contains a transvection $\tau \neq 1$, then $G \supset \text{ESp}(\mathfrak{q})$ for some $\mathfrak{q} \neq 0$.*

Proof. Suppose $\tau(a, x) \in G$ for some $a \neq 0$ and some unimodular $x \in M$. In view of (2.1), (2.2) implies that $\tau(a, x) \in G$ for all unimodular $x \in M$. To prove that $\text{ESp}(aA) \subset G$, it is therefore sufficient to show that $\tau(ab, x) \in G$ for a particular unimodular $x \in M$ and all $b \in A$.

Let N be a hyperbolic submodule of M with a hyperbolic basis $\{e_j, e^j\}_{1 \leq j \leq 3}$. As in the proof of (2.4), the identity

$$\begin{aligned} &\tau(-a, e_2 + e_3 + be_1) \tau(a, e_2 + be_1) \tau(a, e_3 + be_1) \tau(a, e_2 + e_3) \tau(-a, e_2) \cdot \\ &\tau(-a, e_3) \tau(-ab^2, e_1) = 1 \end{aligned}$$

can be written as

$$\tau(ab^2, e_1) = [\beta, \tau(-a, e_2 + e_3) \tau(a, e_3)] [\gamma, \tau(a, e_2)]$$

where $\beta = \tau(-b, e^3) \tau(b, e^3 + e_1)$ and $\gamma = \tau(-b, e^2) \tau(b, e^2 + e_1)$ are in $\text{ESp}(R)$ and $\tau(-a, e_2 + e_3) = \tau(a, e_2 + e_3)^{-1}$, $\tau(a, e_3)$ and $\tau(a, e_2)$ belong to G in view of the initial remarks. Hence $\tau(ab^2, e_1) \tau(ac^2, e_1)^{-1} = \tau(a(b^2 - c^2), e_1) \in G$ for all $a, b \in A$. Since $\frac{1}{2} \in A$, any element in A can be written in the form $b^2 - c^2$, proving the assertion. \parallel

We now come to our principal result.

(3.4) THEOREM. *Suppose $\frac{1}{2} \in A$ and the form (\cdot, \cdot) is locally hyperbolic. The following assertions are equivalent:*

- (i) G is a subgroup of $\text{Sp}(R)$ normalised by $\text{ESp}(R)$.
- (ii) There exists a unique ideal \mathfrak{q} in A such that $\text{ESp}(\mathfrak{q}) \subset G \subset \text{Sp}'(\mathfrak{q})$.

Proof. Choose \mathfrak{q} maximal w.r.t. the property $\text{ESp}(\mathfrak{q}) \subset G$. Suppose $G \not\subset \text{Sp}'(\mathfrak{q})$; then the image \bar{G} of G in $\text{Sp}(R/(\mathfrak{q}))$ will not be in the center. Since the homomorphism $\text{ESp}(R) \rightarrow \text{ESp}(R/(\mathfrak{q}))$ is surjective by (2.3), we may apply (3.2) and (3.3) to \bar{G} and conclude that $\bar{G} \supset \text{ESp}(\mathfrak{q}'/\mathfrak{q})$ for some ideal $\mathfrak{q}' \not\supset \mathfrak{q}$; lifting to A , we have $\text{ESp}(\mathfrak{q}') \subset \text{Sp}(\mathfrak{q}) \cdot G$. Now by (2.4) and (2.5), $\text{ESp}(\mathfrak{q}') = [\text{ESp}(R), \text{ESp}(\mathfrak{q}')] \subset [\text{ESp}(R), \text{Sp}(\mathfrak{q}) \cdot G] \subset G$, contradicting the maximality of \mathfrak{q} . Therefore $G \subset \text{Sp}'(\mathfrak{q})$.

If $\text{ESp}(\mathfrak{q}) \subset G \subset \text{Sp}'(\mathfrak{q})$ then by (2.4) and (2.5) we have $\text{ESp}(\mathfrak{q}) = [\text{ESp}(R), \text{ESp}(\mathfrak{q})] \subset [\text{ESp}(R), G] \subset [\text{ESp}(R), \text{Sp}'(\mathfrak{q})] \subset \text{ESp}(\mathfrak{q}) \subset G$. This shows that \mathfrak{q} is unique and that (ii) \Rightarrow (i). \parallel

(3.5) COROLLARY. *The following are equivalent:*

- (i) G is a normal subgroup of $\text{ESp}(R)$.

(ii) *There exists a unique ideal \mathfrak{q} such that*

$$\mathrm{ESp}(\mathfrak{q}) \subset G \subset \mathrm{ESp}(R) \cap \mathrm{Sp}(\mathfrak{q}).$$

The groups $\delta(\mathfrak{q}) = \mathrm{ESp}(R) \cap \mathrm{Sp}(\mathfrak{q}) / \mathrm{ESp}(\mathfrak{q})$ are all abelian.

Proof. Suppose G is normal in $\mathrm{ESp}(R)$; (3.4) provides a unique ideal \mathfrak{q} such that $\mathrm{ESp}(\mathfrak{q}) \subset G \subset \mathrm{ESp}(R) \cap \mathrm{Sp}'(\mathfrak{q})$. To show (i) \Rightarrow (ii), it suffices to prove that $\mathrm{ESp}(R) \cap \mathrm{Sp}'(\mathfrak{q}) = \mathrm{ESp}(R) \cap \mathrm{Sp}(\mathfrak{q})$. However, it is easy to see that the center of $\mathrm{Sp}(R/\mathfrak{q})$ consists of homotheties, of which only 1 can lie in $\mathrm{ESp}(R/\mathfrak{q})$.

Both (ii) \Rightarrow (i) and the commutativity of $\delta(\mathfrak{q})$ are implied by (2.5). \parallel

(3.6) COROLLARY. *If \mathfrak{q} is a maximal ideal of A , the group $\mathrm{ESp}(R) / \mathrm{ESp}(R) \cap \mathrm{Sp}(\mathfrak{q})$ is simple. \parallel*

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Received January 7, 1972