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On Spaces of Kleinian Groups

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Let G be a finitely generated Kleinian group and $M(G)$ the space of Beltrami coefficients for G (all definitions will be repeated in the body of this paper). For each $\mu \in M(G)$, let w^μ be the unique normalized μ-quasiconformal automorphism of the complex sphere $\hat{\mathbf{C}}$. Two Beltrami coefficients μ and $\nu \in M(G)$ are called *equivalent* (Bers [6]), if w^μ and w^ν agree on the limit set Λ of G; they are called *strongly equivalent* (Bers [7]), if in addition w^μ is homotopic to w^ν on each component D of the region of discontinuity Ω of G, modulo the ideal boundary of D. The *quasiconformal deformation space* $T(G)$ is the set of equivalence classes of $M(G)$, and the *strong quasiconformal deformation space* $\tilde{T}(G)$ is the set of strong equivalence classes. We shall prove that $T(G)$ is a complex analytic manifold (a result previously obtained by Maskit [17], and under some restrictive assumptions by Bers [6]), and $\tilde{T}(G)$ is its holomorphic universal covering space.

Our method differs considerably from those of Maskit [17] or Bers [6]. We rely instead on results of Bers [7]. With Bers and Maskit, we identify $\tilde{T}(G)$ with a product of Teichmüller spaces for the *Fuchsian model of* G, and show that $T(G)$ is $\tilde{T}(G)$ factored by a fixed point free group $\Gamma(G)$ that operates discontinuously on $\tilde{T}(G)$. Furthermore, our methods lead to more precise information about the fundamental group $\Gamma(G)$ of $T(G)$, and show that if $G_1, \ldots, G_n$ are a maximal set of inequivalent stability subgroups of the components of Ω, then

$$T(G) \cong T_0(G_1) \times \cdots \times T_0(G_n),$$

where $T_0(G_j)$ is a certain subspace of $T(G_j)$, $j = 1, \ldots, n$.

The above decomposition of $T(G)$ is apparently new. To obtain it we need the full strength of Maskit's Identity Theorem of [17]. However, we recover all other results of [17], on the basis of a much weaker theorem (Th. 1 of Bers [7]). In our development we also obtain characterizations of the trivial and strongly trivial Beltrami coefficients.

Finally, Bers [6] embeds $T(G)$ holomorphically into a certain algebraic variety and shows that if G satisfies a stability condition, then the image is a manifold. We show how to recover this result by our methods.

§1. Denote by $\mathcal{M}$ the *Möbius group*; that is, the group of conformal automorphisms

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of the Riemann sphere $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$. Denote by $\mathscr{R}$ the *real Möbius group*; that is, the group of conformal automorphism of the upper half plane $U=\{z\in\mathbf{C};\ \operatorname{Im} z>0\}$.

Let $G\subset\mathscr{M}$ be a Kleinian group, and let Δ be an invariant union of connected components of $\Omega=\Omega(G)$, the region of discontinuity of G. Denote by $\Lambda=\Lambda(G)$, the limit set of G.

§**2.** A *Beltrami coefficient* μ for G is a measurable function satisfying

i) $\mu(gz)\overline{g'(z)}/g'(z)=\mu(z)$, $g\in G$, a.e. $z\in\Omega$,

ii) $\mu \mid \Lambda=0$,

iii) ess $\sup|\mu|<1$.

The Beltrami coefficients form the open unit ball of the Banach space (with respect to the supremum norm, $\|\cdot\|$) of bounded measurable functions on Ω/G. The space of Beltrami coefficients for G with support in Δ is denoted by $M(G, \Delta)$. It is well known (Ahlfors-Bers [3]) that for every $\mu\in M(G, \Delta)$ there is a unique quasiconformal automorphism w^μ of $\hat{\mathbf{C}}$ satisfying the Beltrami equation

$$\frac{\partial w^\mu}{\partial \bar{z}}=\mu\frac{\partial w^\mu}{\partial z},$$

and $w^\mu(0)=0$, $w^\mu(1)=1$, $w^\mu(\infty)=\infty$.

For fixed $g\in G$ and $\mu\in M(G, \Delta)$,

$$w^\mu\circ g\circ(w^\mu)^{-1}\in\mathscr{M}. \tag{1}$$

Thus each μ determines an isomorphism (called a *quasiconformal deformation* of G)

$$\chi(\mu): G\to\mathscr{M}, \tag{2}$$

where $\chi(\mu)(g)$ is given by (1). We call $\mu\in M(G, \Delta)$ *trivial* if (2) is the identity isomorphism. The set of trivial Beltrami coefficients for G with support in Δ is denoted by $M_0(G, \Delta)$. Consequently, μ and $\nu\in M(G, \Delta)$ are called *equivalent* if $\chi(\mu)=\chi(\nu)$.

LEMMA 1. *A Beltrami coefficient $\mu\in M(G, \Delta)$ is trivial if and only if $w^\mu \mid \Lambda$ is the identity.*

Proof. If $\mu\in M_0(G, \Delta)$, then

$$w^\mu\circ g=g\circ w^\mu, \quad g\in G.$$

Let $x\in\Lambda$ be the attracting fixed point of an element $g\in G$. Then for all but (possibly) four values $z\in\hat{\mathbf{C}}$,

$$w^\mu(x)=\lim_{n\to\infty} w^\mu\circ g^n(z)=\lim_{n\to\infty} g^n\circ w^\mu(z)=x.$$

Since such points are dense in Λ, $w^\mu \mid \Lambda$ is the identity. Conversely, let $\mu \in M(G, \Delta)$ be such that $w^\mu \mid \Lambda = \text{identity}$. Since $g(\Lambda) = \Lambda$, for $g \in G$, and

$$w^\mu \circ g \circ (w^\mu)^{-1} = g^\mu, \quad g \in G,$$

we conclude that

$$g^\mu(x) = g(x), \quad \text{for } x \in \Lambda.$$

Since $g^\mu \in \mathcal{M}$ and Λ has more than two points, $g = g^\mu$.

Remark. The lemma shows that the definition of triviality agrees with the one given in the introduction.

§3. The set $M_0(G, \Delta)$ acts as a group of *right translations* on $M(G, \Delta)$ by

$$M(G, \Delta) \times M_0(G, \Delta) \ni (\nu, \mu) \mapsto \nu\mu \in M(G, \Delta), \tag{3}$$

where

$$w^{\nu\mu} = w^\nu \circ w^\mu.$$

In this manner, we view $M_0(G, \Delta)$ as a group of biholomorphic automorphisms of $M(G, \Delta)$. The *quasiconformal deformation space of G with support in Δ* is

$$T(G, \Delta) = M(G, \Delta)/M_0(G, \Delta),$$

endowed with the quotient topology.

Remark. If F is a Fuchsian group operating on U, then

$$T(F, U) = T(F)$$

is the usual *Teichmüller space* if and only if F is of the first kind. If F is of the second kind, $T(F)$ is the socalled *reduced* Teichmüller space. See, for example, Earle [8] and [9]. (Of course, we could have let Δ be an arbitrary G-invariant subset of $\Omega(G)$. With this convention $T(F, U)$ makes sense for Fuchsian groups of the second kind.)

In either case, $T(F)$ is a manifold. If F is of the first kind, it is a complex analytic manifold. If $\dim T(F) < \infty$ and F is of the first kind (if and only if F finitely generated of the first kind), then $T(F)$ is canonically representable as a bounded domain of holomorphy in $\mathbf{C}^n$, where n is the dimension of the space of cusp forms for F (with support in U) of weight -4. See Ahlfors [2] or Bers [4] and the literature quoted there.

§4. Let $\{\Delta_j\}_{j \in J}$ be a maximal collection of non-equivalent components of Δ. For each $j \in J$, let G_j be the subgroup of G that leaves Δ_j invariant. Let

$$h_j: U \to \Delta_j$$

be a holomorphic universal covering. Let K_j be the covering group of h_j; that is,

$$K_j = \{k \in \mathscr{R};\ h_j \circ k = h_j\}.$$

Let F_j be the *Fuchsian equivalent* of G_j; that is,

$$F_j = \{f \in \mathscr{R};\ \exists g \in G_j \text{ with } h_j \circ f = g \circ h_j\}.$$

Then there is an exact sequence

$$\{1\} \to K_j \to F_j \overset{\chi_j}{\to} G_j \to \{1\},$$

where χ_j is the *uniformizing homomorphism of* G_j *with respect to the cover* h_j. We shall call the collection $\{F_j\}_{j \in J}$, the *Fuchsian model of* G *on* Δ. Note that the pair (χ_j, h_j) satisfies the relation

$$h_j \circ f = \chi_j(f) \circ h_j, \qquad f \in F_j.$$

§5. We now recall some results of [13] and [14]. Let F be a Fuchsian group acting on the upper half plane U. The pair (χ, h) is a *deformation* of F if (i) χ is a homomorphism of F into $\mathscr{M}$, and h is a meromorphic local homeomorphism of U onto an (open) subset of $\hat{\mathbf{C}}$, such that $h \circ f = \chi(f) \circ h, f \in F$. For any deformation (χ, h) of F, the Schwarzian φ of h,

$$\varphi = \theta_2 h = (h''/h')' - (1/2)(h''/h')^2,$$

is a holomorphic 2-form (automorphic form of weight -4) for F on U; that is,

$$\varphi(fz) f'(z)^2 = \varphi(z), \qquad z \in U, \quad f \in F.$$

It is a *cusp form* if

$$\sup\{(2 \operatorname{Im} z)^2 |\varphi(z)|; z \in U\} < \infty.$$

In [13], we proved

PROPOSITION 2. *Let F be a finitely generated Fuchsian group of the first kind. If* (χ, h_1) *and* (χ, h_2) *are deformations of F, then* $h_1 = h_2$ *whenever* $\theta_2 h_1$ *and* $\theta_2 h_2$ *are cusp forms.*

PROPOSITION 3. *Let G be a Kleinian group with an invariant domain D. Let F be the Fuchsian equivalent of G with respect to the universal holomorphic covering* $h: U \to D$, *and* $\chi: F \to G$, *the corresponding uniformizing homomorphism. If D/G is of finite type, then* $\theta_2 h$ *is a cusp form (for F on U).*

Proof. See [14].

THEOREM 4. *Let G be a Kleinian group. Let Δ and D be invariant unions of components of its region of discontinuity. Assume that Δ/G is of finite type. Let $w:\Delta\to D$ be a conformal homeomorphism such that $w\circ g=g\circ w$, all $g\in G$. Then w is the identity map (in particular $\Delta=D$).*

Remark. This theorem is a special case of Maskit's Identity Theorem (the main theorem of [17]). Maskit shows that even if w is topological, quasiconformal or conformal, it can be extended to an automorphism of $\hat{\mathbf{C}}$ of the same type by setting w to be the identity on $\hat{\mathbf{C}}-\Delta$ (provided $\Delta=D$). Maskit proves first the topological theorem, then the quasiconformal theorem, and finally the conformal theorem. We present here a direct proof of the conformal case in the expectation that it will lead to a new proof of the quasiconformal case. (Also, all the result of [17] on spaces of Kleinian groups can be obtained without the full strength of Maskit's Identity Theorem. For some new results we shall, however, need this Identity Theorem.)

Proof of Theorem. Let Δ_0 be a component of Δ and $G(\Delta_0)$ its stability subgroup. Since the hypothesis on the pair (G, Δ) is inherited by $(G(\Delta_0), \Delta_0)$, we may assume that Δ is connected. Thus we let F be the Fuchsian equivalent of G with respect to the universal holomorphic covering h and χ the corresponding uniformizing homomorphism. Clearly $\theta_2 h$ and $\theta_2(w\circ h)$ are cusp forms by Proposition 3, and since they both induce the same deformation, $h=w\circ h$ by Proposition 2. We have shown that w is the identity.

§6. We start now with a Fuchsian group F operating on the upper half plane U. For every $\mu\in M(F, U)$ there exists a unique quasiconformal automorphism w_μ of U that satisfies the Beltrami equation

$$\frac{\partial w_\mu}{\partial \bar{z}}=\mu\frac{\partial w_\mu}{\partial z},$$

and the normalization $w_\mu(0)=0$, $w_\mu(1)=1$, $w_\mu(\infty)=\infty$. Every $\mu\in M(F, U)$ determines an isomorphism

$$\psi(\mu): F\to\mathscr{R}$$

defined as follows:

$$\psi(\mu)f=w_\mu\circ f\circ w_\mu^{-1},\qquad f\in F.$$

If μ and $\nu\in M(F, U)$, then we say that μ and ν are *strongly equivalent* and write $\mu\sim\nu$ if $w_\mu\mid\mathbf{R}=w_\nu\mid\mathbf{R}$, where $\mathbf{R}$ is the real line.

For arbitrary G we consider the biholomorphic map (with the correct definition of product: $\mu=(\mu_j)_{j\in J}\in\prod_{j\in J} M(F_j, U)$ if $\mu_j\in M(F_j, U)$ and $\|\mu_j\|\leqslant k<1$, for all $j\in J$)

$$h^*: M(G, \Delta)\to \prod_{j\in J} M(F_j, U) \tag{4}$$

defined as follows: For $\mu\in M(G, \Delta)$,

$$h^*(\mu) = (h_j^*(\mu \mid \Delta_j))_{j\in J}, \tag{5}$$

where

$$h_j^*(\mu \mid \Delta_j)(z) = \mu(h_j z)\frac{\overline{h_j'(z)}}{h_j'(z)}, \qquad z\in U. \tag{6}$$

Two elements μ and $\nu\in M(G, \Delta)$ are called *strongly equivalent with respect to* Δ ($\mu\sim\nu$) if

$$w^\mu \mid \hat{\mathbf{C}} - \Delta = w^\nu \mid \hat{\mathbf{C}} - \Delta,$$

and

$$h_j^*(\mu \mid \Delta_j) \sim h_j^*(\nu \mid \Delta_j), \quad \text{all } j\in J.$$

An element $\mu\in M(G, \Delta)$ is called *strongly trivial with respect to* Δ if $\mu\sim 0$. The set of strongly trivial Beltrami coefficients is denoted by $\tilde{M}_0(G, \Delta)$. Since

$$\tilde{M}_0(G, \Delta) \subset M_0(G, \Delta),$$

we use (3) to define an action

$$M(G, \Delta)\times \tilde{M}_0(G, \Delta)\to M(G, \Delta).$$

The *strong quasiconformal deformation space of G with support in* Δ is

$$\tilde{T}(G, \Delta) = M(G, \Delta)/\tilde{M}_0(G, \Delta).$$

Remarks. (1) If F is a Fuchsian group operating on U, then

$$\tilde{T}(F, U) = \tilde{T}(F)$$

is the Teichmüller space as defined by Bers [4]. Thus for Fuchsian groups F of the first kind $\tilde{T}(F)=T(F)$. In all cases $\tilde{T}(F)$ is a complex analytic manifold.

(2) See Bers [7] for a proof that the definition of strong equivalence given in this paragraph agrees with the one given in the introduction.

§7. It is quite clear that if we define

$$\Gamma(G, \Delta) = M_0(G, \Delta)/\tilde{M}_0(G, \Delta),$$

then $\Gamma(G, \Delta)$ acts as a group of biholomorphic homeomorphisms on $\tilde{T}(G, \Delta)$ and

$$T(G, \Delta) \cong \tilde{T}(G, \Delta)/\Gamma(G, \Delta).$$

In order to understand better the action of $\Gamma(G, \Delta)$ on $\tilde{T}(G, \Delta)$ we note the meaning of the map h^*. Let $\mu \in M(G, \Delta)$ and write for fixed $j \in J$, $\mu_j = h_j^*(\mu \mid \Delta_j)$. Then for this j, there is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{w_{\mu_j}} & U \\ {\scriptstyle h_j}\downarrow & & \downarrow{\scriptstyle k_j} \\ \Delta_j & \xrightarrow{w^{\mu} \mid \Delta_j} & w^{\mu}(\Delta_j) \end{array}, \tag{7}$$

where k_j is some holomorphic universal covering of $w^{\mu}(\Delta_j)$.

Let G_j and $\hat{G}_j$ be the stability subgroups of Δ_j and $w^{\mu}(\Delta_j)$, respectively. Let F_j and $\hat{F}_j$ be their Fuchsian equivalents using the holomorphic universal coverings h_j and k_j, and χ_j and $\hat{\chi}_j$ be the corresponding uniformizing homomorphisms.

LEMMA 5. *The quasiconformal homeomorphism w_{μ_j} conjugates F_j onto $\hat{F}_j$. Furthermore, the following is a commutative diagram of groups and homomorphisms:*

$$\begin{array}{ccc} F_j & \xrightarrow{\psi(\mu_j)} & \hat{F}_j \\ {\scriptstyle \chi_j}\downarrow & & \downarrow{\scriptstyle \hat{\chi}_j} \\ G_j & \xrightarrow{\chi(\mu) \mid G_j} & \hat{G}_j \end{array}.$$

Proof. Let $f \in F_j$. Abbreviate w_{μ_j} by w and $w^{\mu} \mid \Delta_j$ by W. Since $\mu_j \in M(F_j, U)$, w conjugates F_j onto a Fuchsian group. To show $\hat{f} = w \circ f \circ w^{-1} \in \hat{F}_j$ for $f \in F_j$, it suffices to show that for some $\gamma \in \hat{G}_j$, we have

$$k_j \circ \hat{f} = \gamma \circ k_j .$$

But for $\gamma = \chi_j(f) \in G_j$, we have

$$\begin{aligned} k_j \circ \hat{f} &= k_j \circ w \circ f \circ w^{-1} = W \circ h_j \circ f \circ w^{-1} \\ &= W \circ \gamma \circ h_j \circ w^{-1} = \chi(\mu)\gamma \circ W \circ h_j \circ w^{-1} \\ &= \chi(\mu)\gamma \circ k_j \circ w \circ w^{-1} = \chi(\mu)\gamma \circ k_j . \end{aligned}$$

The above calculation also shows that $\hat{\chi}_j \circ \psi(\mu_j) = \chi(\mu) \mid G_j \circ \chi_j$.

§8. For future use we define another operation on Beltrami coefficients. For ν and $\mu \in M(U)$, the Beltrami coefficients on the upper half plane for the trivial group, we define $\nu \otimes \mu \in M(U)$ by

$$w_{\nu \otimes \mu} = w_{\nu} \circ w_{\mu} ,$$

and note that this composition of Beltrami coefficients is different from the one introduced in §3. Furthermore, if F is a Fuchsian group, $\nu \in M(F, U)$, and $\mu \in M_0(F, U)$, then $\nu \otimes \mu \in M(F, U)$.

Remark. The above composition of Beltrami coefficients introduces another action of $M_0(F, U)$ (and $\tilde{M}_0(F, U)$) on $M(F, U)$. It is well known (see, for example, Ahlfors [2]) that both of these actions lead to the same Teichmüller space $T(F)$ (respectively, $\tilde{T}(F)$).

§9. From now on we make the assumption that for each j,

$$\Delta_j/G_j \text{ is of finite type}.$$

It follows from Ahlfors' finiteness theorem [1], that Ω/G is of finite type whenever G is finitely generated.

LEMMA 6. (Maskit [17]). *If $\mu \in M_0(G, \Delta)$, then w^μ maps each connected component of Δ onto itself.*

Proof. Suppose there is a component D of Δ with $w^\mu(D) \neq D$. Let H be the stability subgroup of D. Since each element of G commutes with w^μ, H is also the stability subgroup of $w^\mu(D)$. Since D/H is of finite type, H is a finitely generated Kleinian group with two invariant components. Hence, by Maskit [16], H is quasi-Fuchsian. Since w^μ interchanges two components of H and is the identity on $\Lambda(H)$, w^μ reverses orientation. This is a contradiction.

LEMMA 7. *Let $\mu \in M_0(G, \Delta)$ and $\nu \in M(G, \Delta)$. Then*

$$\nu\mu = (h^*)^{-1}[(k^*\nu) \otimes (h^*\mu)],$$

where we define

$$[(k^*\nu) \otimes (h^*\mu)]_j = (k^*\nu)_j \otimes (h^*\mu)_j, \qquad j \in J,$$

and the map k^* using the coverings k_j of the domains Δ_j given by (7).

Proof. For fixed $j \in J$, we have the following commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{w_{\mu_j}} & U & \xrightarrow{w_{\nu_j}} & U \\ \downarrow h_j & & \downarrow k_j & & \downarrow l_j \\ \Delta_j & \xrightarrow{w^\mu \mid \Delta_j} & \Delta_j & \xrightarrow{w^\nu \mid \Delta_j} & w^{\nu\mu}(\Delta_j) \end{array},$$

with holomorphic universal covering maps k_j and l_j. Thus

$$w_{\nu_j \otimes \mu_j} = w_{\nu_j} \circ w_{\mu_j}.$$

We have shown

$$h^*(\nu\mu) = k^*(\nu) \otimes h^*(\mu).$$

§10. Under the hypothesis of Lemma 7, for each j, k_j and h_j are holomorphic coverings of the same domain Δ_j. Thus we have for some $\beta_j \in \mathscr{R}$,

$$k_j = h_j \circ \beta_j .$$

This results in replacing the Fuchsian group F_j by

$$\hat{F}_j = \beta_j^{-1} F_j \beta_j .$$

LEMMA 8. *If* $\mu \in \tilde{M}_0(G, \Delta)$, *then* $h_j = k_j$ *for all* j.

Proof. Since $\mu_j \in \tilde{M}_0(F_j, U)$, $w_{\mu_j} \mid \mathbf{R}$ = identity. Thus $\psi(\mu_j)$ = identity $= \chi(\mu)$. We conclude from Lemma 5 that $F_j = \hat{F}_j$ and $\chi_j = \hat{\chi}_j$. By Propositions 2 and 3, $h_j = k_j$.

LEMMA 9. *If* $\mu \in M_0(G, \Delta)$ *and if there is a component* D *of* Δ *such that* $\mu \mid D = 0$, *then* $w^\mu \mid D$ = identity.

Proof. This is an immediate consequence of Theorem 4, since

$$\mu \Big| \bigcup_{g \in G} g(D) = 0 .$$

§11. As a result of the above facts, (4) projects to a well defined surjective holomorphic mapping

$$\tilde{T}(G, \Delta) \to \prod_{j \in J} \tilde{T}(F_j, U).$$

To see this let ν and $\sigma \in M(G, \Delta)$. We map these elements into $h^* \nu$ and $h^* \sigma \in \prod_{j \in J} M(F_j, U)$. Now if ν and σ are equivalent under $\tilde{M}_0(G, \Delta)$, then $\sigma = \nu\mu$ for some $\mu \in \tilde{M}_0(G, \Delta)$. Thus $h^*\sigma = (h^*\nu) \otimes (h^*\mu)$. But for each $j \in J$, $(h^*\mu)_j \sim 0$. Thus we have a well defined mapping into the product of the Teichmüller spaces (again with the correct definition of product).

From now on we assume (in order to minimize topological questions) that Δ/G is of finite type, and consists of n components.

LEMMA 10. *Let* $\mu \in M(G, \Delta)$ *and assume that*

$$h_j^*(\mu \mid \Delta_j) \sim 0, \quad j = 1, \ldots, n. \tag{8}$$

Then $\mu \in \tilde{M}_0(G, \Delta)$.

Proof. Consider the map

$$h^* : \tilde{M}_0(G, \Delta) \to \tilde{M}_0(F_1, U) \times \cdots \times \tilde{M}_0(F_n, U) = \mathscr{E} . \tag{9}$$

Since h^* is one-to-one (as the map given by (4)), and since (8) shows that $h^*\mu \in \mathscr{E}$, it suffices to show that the map h^* of (9) is surjective.

Let F be any Fuchsian group. First, $\tilde{M}_0(F, U)$ is connected if and only if $\tilde{T}(F)$ is simply connected (Earle and Eells [11]). Furthermore, $\tilde{T}(F)$ is contractible (therefore simply connected), when F is finitely generated (Keen [12], Earle [10]). We conclude that $\mathscr{E}$ is connected. Let

$$E = \{\nu \in \mathscr{E};\ \nu = h^*\mu \text{ with } \mu \in \tilde{M}_0(G, \Delta)\}.$$

Given a $\nu \in \mathscr{E}$, there is a $\mu \in M(G, \Delta)$ such that $h^*\mu = \nu$. We must show that $\mu \in \tilde{M}_0(G, \Delta)$. For $\nu \in \mathscr{E}$ with sufficiently small norm, this was done by Bers (Th. 1 of [7]). Thus E contains a neighborhood of the origin from $\mathscr{E}$. Next we show that E is open. Consider $\nu \in E$. Then there is a $\mu \in \tilde{M}_0(G, \Delta)$ such that $h^*\mu = \nu$. Consider σ of small norm, $\sigma \in \mathscr{E}$. Then $\sigma = h^*\varrho$, ϱ of small norm $\varrho \in \tilde{M}_0(G, \Delta)$, and

$$h^*(\mu\varrho) = h^*(\mu) \otimes h^*(\varrho) = \nu \otimes \sigma$$

(since $\varrho \in \tilde{M}_0(G, \Delta)$). Thus a neighborhood of ν is also in E. Finally, E is closed. For if

$$\nu_j \in E \quad \text{and} \quad \lim_{j \to \infty} \nu_j = \nu \in \mathscr{E},$$

then there is a $\mu_j \in \tilde{M}_0(G, \Delta)$, $\nu_j = h^*\mu_j$, with $\lim_{j\to\infty} \mu_j = \mu \in M(G, \Delta)$. Clearly, for $i = 1, \ldots, n$, $(h_i^*\mu)$ is equivalent to zero. We must only verify that $w^\mu \mid \mathbf{C} - \Delta =$identity. But for each $z \in \hat{\mathbf{C}} - \Delta$, we have

$$w^\mu(z) = \lim_{j \to \infty} w^{\mu_j}(z) = z.$$

Since E is open, closed and non-empty, and $\mathscr{E}$ is connected, $E = \mathscr{E}$. Thus (9) is indeed a surjective isomorphism.

We have just established

THEOREM 11. *Let Δ be an invariant union of components of a Kleinian group G with Δ/G of finite type. Let $\{F_1, F_2, \ldots, F_n\}$ be the Fuchsian model of G on Δ. Then*

$$\tilde{T}(G, \Delta) \cong \tilde{T}(F_1) \times \cdots \times \tilde{T}(F_n). \tag{10}$$

In particular, $\tilde{T}(G, \Delta)$ is a simply connected complex analytic manifold.

§12. We now turn to the action of $\Gamma(G, \Delta)$ on $\tilde{T}(F_1) \times \cdots \times \tilde{T}(F_n)$. Let F be an arbitrary Fuchsian group operating on U. Let w be an arbitrary (not necessarily normalized) quasiconformal automorphism of U, with

$$w^{-1}Fw = F.$$

Every such w induces a biholomorphic automorphism w^* of $M(F, U)$ defined as follows. For $\mu \in M(F, U)$,

$$w_{w^*\mu} = \alpha \circ w_\mu \circ w,$$

where $\alpha \in \mathscr{R}$ is chosen so that $\alpha \circ w_\mu \circ w$ fixes 0, 1 and ∞. It is trivial to verify that for $\mu, \nu \in M(F, U)$, μ is equivalent to ν under $\tilde{M}_0(F, U)$ if and only if $w^*\mu$ is equivalent to $w^*\nu$ under $\tilde{M}_0(F, U)$. Thus we view w^* has a biholomorphic self mapping of $\tilde{T}(F)$. The set of all such mappings forms $\Gamma(F)$, the *modular group* of $\tilde{T}(F)$. For finitely generated F of the first kind $\Gamma(F)$ acts discontinuously on $\tilde{T}(F)$ (Kravetz [15], Earle and Eells [11]).

We have seen in §7 and §10 that for each $\mu \in M_0(G, \Delta)$, w_{μ_j} (the j-th component of $h^*\mu$ is μ_j) conjugates F_j into $\beta_j^{-1} F \beta_j$. Thus $\beta_j \circ w_{\mu_j}$ conjugates F_j into itself and hence induces an element of $\Gamma(F_j)$.

We now establish a homomorphism

$$\Theta: M_0(G, \Delta) \to \Gamma(F_1) \times \cdots \times \Gamma(F_n),$$

by defining for $\mu \in M_0(G, \Delta)$

$$\Theta\mu = (\beta_1 \circ w_{\mu_1}^*, \ldots, \beta_n \circ w_{\mu_n}^*).$$

The verification that for $\mu, \nu \in M_0(G, \Delta)$,

$$\Theta(\nu\mu) = \Theta\nu \circ \Theta\mu$$

is straightforward and is thus left to the reader.

LEMMA 12. *Under the hypothesis of Theorem* 11, $\Gamma(G, \Delta)$ *is isomorphic to a fixed point free subgroup of* $\Gamma(F_1) \times \ldots \times \Gamma(F_n)$. *Furthermore, the isomorphism* (10) *of Teichmüller spaces commutes with this isomorphism of modular groups.*

Proof. We have already remarked that every $\mu \in \tilde{M}_0(G, \Delta)$, induces the identity element of $\Gamma(F_1) \times \cdots \times \Gamma(F_n)$. We show next that if $\mu \in M_0(G, \Delta)$ and if the corresponding element of $\Gamma(F_j)$ has a fixed point, then $\mu_j \sim 0$. This will show that $\mu \in \tilde{M}_0(G, \Delta)$. And hence the two statements, that the correspondence we have established is one-to-one and that the image group acts fixed point freely, will follow at once. Let $w = \beta_j \circ w_{\mu j}$. Find $\nu_j \in M(F_j, U)$ such that for some $\alpha_j \in \mathscr{R}$,

$$\alpha_j \circ w_{\nu_j} \circ \beta_j \circ w_{\mu_j} \big| \mathbf{R} = w_{\nu_j} \big| \mathbf{R}.$$

Let σ_j be the Beltrami coefficient of $w_{\nu_j}^{-1} \circ \alpha_j \circ w_{\nu_j} \circ \beta_j \circ w_{\mu_j}$, then $\sigma_j \in \tilde{M}_0(F_j, U)$. Choose $\sigma \in \tilde{M}_0(G, \Delta)$ such that $(\sigma_1, \ldots, \sigma_n) = h^*\sigma$. Similarly, let $\varrho_j \in M(\hat{F}_j, U)$ be the Beltrami coefficient of $\alpha_j \circ w_{\nu_j} \circ \beta_j$, and let $(\varrho_j, \ldots, \varrho_n) = k^*\varrho$, and $(\nu_1, \ldots, \nu_n) = h^*\nu$, $\nu \in M(G, \Delta)$, $\varrho \in M(G, \Delta)$. Then

$$w^\varrho \circ w^\mu = w^\nu \circ w^\sigma.$$

But $\varrho_j = \beta_j^* \nu_j$ and $k_j^* = \beta_j^* \circ h_j^*$. Thus $\nu = \varrho$ and hence $\sigma = \mu$. In particular, $\mu \in \tilde{M}_0(G, \Delta)$.

The last statement of the theorem follows from the commutativity of the following diagram for $\theta\in\Gamma(G, \Delta)$ and the corresponding element $\Theta\theta\in\Gamma(F_1)\times\cdots\times\Gamma(F_n)$:

$$\begin{array}{ccc} \tilde{T}(G,\Delta) & \xrightarrow{\cong} & \tilde{T}(F_1)\times\cdots\times\tilde{T}(F_n) \\ \theta\downarrow & & \downarrow\Theta\theta \\ \tilde{T}(G,\Delta) & \xrightarrow{\cong} & \tilde{T}(F_1)\times\cdots\times\tilde{T}(F_n). \end{array}$$

§13. We summarize the above results in the following theorem.

THEOREM 13. *Under the hypothesis of Theorem* 11, $T(G, \Delta)$ *is a complex analytic manifold with* $\tilde{T}(G, \Delta)$ *as a holomorphic universal covering space. Furthermore, if each component of* Δ *is simply connected, then*

$$T(G,\Delta)=\tilde{T}(G,\Delta).$$

Proof. Since $\Gamma(G, \Delta)$ acts properly discontinuously and fixed point freely on $\tilde{T}(G, \Delta)$, we conclude that $T(G, \Delta)$ is a complex analytic manifold. Since $\tilde{T}(G, \Delta)$ is simply connected and the covering

$$\tilde{T}(G,\Delta)\to T(G,\Delta)$$

is Galois, $\tilde{T}(G, \Delta)$ is a holomorphic universal covering of $T(G, \Delta)$.

Now assume that each component of Δ is simply connected. Let $\mu\in M_0(G, \Delta)$, and let $\mu_j=h_j^*(\mu \mid \Delta_j)$. We use the results and notation of Lemma 5. Note that $\chi(\mu)$=identity, and hence

$$\psi(\mu_j)=\hat{\chi}_j^{-1}\circ\chi_j.$$

Hence

$$\begin{aligned}\psi(\mu_j)f &= k_j^{-1}\circ h_j\circ f\circ h_j^{-1}\circ k_j=\beta_j^{-1}\circ f\circ\beta_j\\ &= w_{\mu_j}\circ f\circ w_{\mu_j}^{-1}, \qquad f\in F.\end{aligned}$$

Hence $\beta_j\circ w_{\mu_j} \mid R$=identity, and in particular $\beta_j\circ w_{\mu_j}^*$ is the identity in $\Gamma(F_j)$. Thus $\Gamma(G, \Delta)$ is the trivial group.

Remark. The fact that $T(G, \Delta)$ is a complex analytic manifold has also been obtained by Maskit [17].

§14. We can in a certain sense strengthen the above theorem. The deformation space of a Kleinian group recognizes only how the Riemann surfaces are represented by the Kleinian group and not how these Riemann surfaces are put together to form the group. Observe that if we define for $j=1,\ldots,n$, $\Gamma_j=\{\gamma\in\Gamma(G, \Delta)$; γ corresponds to an element $\mu \in M_0(G, \Delta)$ with μ supported in $D_j = \bigcup_{g \in G} g(\Delta_j)\}$, then we recognize at once that

$$\Gamma_j = \Gamma(G, D_j).$$

Furthermore, if $\mu_j \in M_0(G, D_j)$, $j = 1, \dots, n$, then (as a consequence of Lemma 9)

$$\mu_n \dots \mu_2 \mu_1 \in M_0(G, \Delta).$$

We thus have established (by Lemma 10) a monomorphism

$$\Gamma(G, D_1) \times \cdots \times \Gamma(G, D_n) \to \Gamma(G, \Delta).$$

The above map is an isomorphism (surjective), for if $\theta \in \Gamma(G, \Delta)$, then we represent θ by an element $\mu \in M_0(G, \Delta)$. Since w^μ induces the identity on G, so does

$$w_j = \begin{cases} w^\mu \text{ on } D_j, \\ \text{identity on } \hat{\mathbb{C}} - D_j, \end{cases}$$

$j = 1, \dots, n$. Letting $\mu_j = \mu \mid D_j$, we conclude by Maskit's Identity Theorem [17], that $w_j = w^{\mu_j}$, provided $\{0, 1, \infty\} \subset \Lambda(G)$. This latter condition involves no loss of generality.

It thus follows that

$$T(G, \Delta) \cong T(F_1)/\Gamma_0(F_1) \times \cdots \times T(F_n)/\Gamma_0(F_n),$$

where $\Gamma_0(F_j)$ is a certain fixed point free subgroup of $\Gamma(F_j)$, $j = 1, \dots, n$.

We thus have

THEOREM 14. *Let Δ be an invariant union of components of a Kleinian group G with Δ/G of finite type. Let $\Delta_1, \dots, \Delta_n$ be a maximal inequivalent set of components of Δ. Let $G_j = \{g \in G;\ g\Delta_j = \Delta_j\}$, $j = 1, \dots, n$. Then*

$$T(G, \Delta) \cong T(G_1, \Delta_1) \times \cdots \times T(G_n, \Delta_n).$$

Proof. We begin with the obvious isomorphism

$$\tilde{T}(G, \Delta) \cong \tilde{T}(G, D_1) \times \cdots \times \tilde{T}(G, D_n),$$

where

$$D_j = \bigcup_{g \in G} g(\Delta_j).$$

We have already shown that

$$\Gamma(G, \Delta) \cong \Gamma(G, D_1) \times \cdots \times \Gamma(G, D_n),$$

and hence

$$T(G, \Delta) \cong T(G, D_1) \times \cdots \times T(G, D_n).$$

To prove the theorem it suffices to show that for fixed j,

$$T(G_j, \Delta_j) \cong T(G, D_j). \tag{11}$$

We establish an isomorphism

$$*: M(G_j, \Delta_j) \xrightarrow{\cong} M(G, D_j)$$

as follows: Let $\mu \in M(G_j, \Delta_j)$. Set

$$*\mu \mid g^{-1}(\Delta_j) = g^*\mu \mid g^{-1}(\Delta_j), \quad g \in G,$$

and note that $*$ is well defined. For if $g^{-1}(\Delta_j) = g_1^{-1}(\Delta_j)$ for another $g_1 \in G$, then

$$\begin{aligned} *\mu \mid g^{-1}(\Delta_j) &= g^*\mu \mid g^{-1}(\Delta_j) = g^* \circ (g_1^{-1})^* \circ g_1{}^*\mu \mid g^{-1}(\Delta_j) \\ &= g_1{}^*\mu \mid g^{-1}(\Delta_j), \end{aligned}$$

since

$$g^* \circ (g_1^{-1})^* = (g_1^{-1} \circ g)^*, \quad \text{and } g_1^{-1} \circ g \in G_j.$$

It is a trivial consequence of Lemma 10, that

$$*: \tilde{M}_0(G_j, \Delta_j) \xrightarrow{\cong} \tilde{M}_0(G, D_j).$$

It thus remains to verify that

$$*: M_0(G_j, \Delta_j) \xrightarrow{\cong} M_0(G, D_j). \tag{12}$$

Since $\mu \in M_0(G_j, \Delta_j)$, by Lemma 6, $w^\mu(\Delta_j) = \Delta_j$. For the sake of convenience we assume that $0, 1, \infty \in \Lambda(G_j)$. We define a function w on $\Omega(G)$ as follows

$$w(z) = \begin{cases} g^{-1} \circ w^\mu \circ g(z), & z \in g^{-1}(\Delta_j), \quad g \in G, \\ z, & z \in \Omega(G) - D_j. \end{cases}$$

We note that w is well defined. For if $z \in g_1^{-1}(\Delta_j)$ with $g_1 \in G$, then on $g^{-1}(\Delta_j) = g_1^{-1}(\Delta_j)$ we have

$$\begin{aligned} g_1^{-1} \circ w^\mu \circ g_1 &= g_1^{-1} \circ (g \circ g_1^{-1})^{-1} \circ w^\mu \circ (g \circ g_1^{-1}) \circ g_1 \\ &= g^{-1} \circ w^\mu \circ g. \end{aligned}$$

The second of the above equalities hold since $g \circ g_1^{-1} \in G_j$ and w^μ commutes with each element of G_j. It is also clear that w commutes with each element of G. By the main theorem of Maskit [17], there is a global quasiconformal homeomorphism W such that

$$W \mid \Omega(G) = w.$$

In particular (since $W \mid \Lambda(G)$ is the identity), W is normalized. An obvious calculation shows that $W = w^{*\mu}$. We have thus verified that (12) is an isomorphism. To finish the proof of the isomorphism (11), it suffices to show that action defined by (3) commutes with the isomorphism*; that is,

$$^*(\nu\mu) = (^*\nu)\,(^*\mu), \quad \text{all } \mu \in M_0(G_j, \Delta_j), \quad \text{all } \nu \in M(G_j, \Delta_j).$$

Since $^*(\nu\mu)$ and $(^*\nu)\,(^*\mu) \in M_0(G, D_j)$, it suffices to show that

$$^*(\nu\mu) \mid \Delta_j = (^*\nu)\,(^*\mu) \mid \Delta_j.$$

But

$$^*(\nu\mu) \mid \Delta_j = \nu\mu \mid \Delta_j;$$

and if

$$w^\varrho = w^\sigma \circ w^\tau,$$

for any three Beltrami coefficients ϱ, σ, τ, then

$$\varrho(z) = \frac{\tau(z) + \sigma_1(z)}{1 + \bar{\tau}(z)\,\sigma_1(z)}, \qquad \sigma_1(z) = \sigma(w^\tau(z))\,\frac{\overline{\dfrac{\partial w^\tau}{\partial z}}}{\dfrac{\partial w^\tau}{\partial z}}.$$

Now since $\mu \in M_0(G_j, \Delta_j)$,

$$w^{*\mu} \mid \Delta_j = w^\mu.$$

Thus

$$(^*\nu)\,(^*\mu) \mid \Delta_j = \nu\mu \mid \Delta_j.$$

Remark. This is the first time we made full use of Maskit's Identity Theorem of [17].

§15. We characterize next the elements of $M_0(G, \Delta)$. In view of the last paragraph, it suffices to assume that Δ is an invariant component of the (finitely generated) Kleinian group G. Using notation that is standard by now, we have a commutative diagram (7), where we drop the index j. Letting K and $\hat{K}$ be the covering groups corresponding to the coverings h and k, we note from Lemma 5 that w_μ conjugates K into $\hat{K}$ (note $K = \chi^{-1}(1)$ and $\hat{K} = \hat{\chi}^{-1}(1)$). We thus rewrite the commutative diagram from Lemma 5 as

$$\begin{array}{ccc} F/K & \xrightarrow{\psi(\mu)} & \hat{F}/\hat{K} \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \hat{\chi}} \\ G & \xrightarrow{\chi(\mu)} & \hat{G} \end{array},$$

and note that all the homomorphisms are now isomorphisms. Next assume that $\mu \in M_0(G, \Delta)$. Let $w = \beta \circ w_{h*\mu}$, then w conjugates F and K into F and K respectively. Furthermore, since $\chi(\mu) = \text{identity}$

$$w \circ f \circ w^{-1} \circ f^{-1} \in K, \quad \text{all } f \in F. \tag{13}$$

THEOREM 15. *Let G be a finitely generated Kleinian group with an invariant component Δ. Let F be the Fuchsian equivalent of G with respect to the holomorphic universal covering h. Let K be the corresponding covering group. Then $\mu \in M_0(G, \Delta)$ if and only if there exists a $w \in \Gamma(F)$, $w = \beta \circ w_{h*\mu}$, $\beta \in \mathscr{R}$ such that* (13) *holds.*

Proof. The only if part has already been established. For the reverse implication consider the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{w_{h*\mu}} & U \\ h \big\downarrow & & \big\downarrow h \circ \beta \\ \Delta & \xrightarrow{W} & \Delta \end{array} .$$

From (13) we conclude that $w_{h*\mu}$ conjugates K into $\beta^{-1} K \beta$. Thus $w_{h*\mu}$ projects to a quasiconformal mapping W as above. The Beltrami coefficient of W is μ. From (13) once again, we see that

$$W \circ g = g \circ W, \quad \text{all } g \in G.$$

Thus by Maskit's Theorem [17], we may assume that W is the identity off Δ. It involves no loss of generality to assume $\{0, 1, \infty\} \subset \hat{\mathbf{C}} - \Delta$. Thus $\mu \in M_0(G, \Delta)$.

§16. In this last paragraph we show the connections between our and Maskit's [17] approaches with the one taken by Bers [6].

Assume that the Kleinian group G is generated by r elements; $\gamma_1, \ldots, \gamma_r$. A homomorphism $\chi : G \to \mathscr{M}$ naturally determines a point

$$(\chi(\gamma_1), \ldots, \chi(\gamma_r)) \in \mathscr{M}^r.$$

The set of all such homomorphisms, $\text{Hom}(G, \mathscr{M})$, forms an affine algebraic subvariety (see Bers [5], [6]) of $\mathscr{M}^r$. The group G is called (Bers [6]) *quasi-stable* if every quasiconformal deformation of G which is sufficiently close to the identity (in the topology of $\text{Hom}(G, \mathscr{M})$) can be induced by a quasiconformal automorphism of the Riemann sphere with a dilitation arbitrarily close to 1. The group G is called *strongly quasi-stable*, if for every quasiconformal deformation χ of G, $\chi(G)$ is quasi-stable. Let Δ be an invariant union of components of G. There is, of course, a map

$$\Phi : M(G, \Delta) \to \text{Hom}(G, \mathscr{M}) \subset \mathscr{M}^r \tag{14}$$

defined by $\Phi(\mu) = (\chi(\mu)\, \gamma_1, \ldots, \chi(\mu)\, \gamma_r)$.

THEOREM 16. (Bers [6]. *Let Δ be an invariant union of components of a Kleinian group G generated by r elements: $\gamma_1, \ldots, \gamma_r$. The mapping Φ defined by* (14) *is holomorphic. Furthermore, for strongly quasi-stable G, $\Phi(M(G, \Delta))$ is a submanifold of $\mathcal{M}^r$.*

Proof. It is almost obvious that Φ is holomorphic (see Bers [5]). Furthermore Φ projects to a well defined holomorphic map

$$\Phi: T(G, \Delta) \to \mathcal{M}^r. \tag{15}$$

The assumption of strong quasi-stability guarantees that the maps Φ of (14) and (15) are open. Since $T(G, \Delta)$ is a manifold, so is $\Phi(T(G, \Delta))$.

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