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The $\mathscr{A}(p)$ Cohomology of Some k Stage Postnikov Systems

by David Kraines

In this paper we compute the cohomology of a class of k stage Postnikov systems as a Hopf algebra over the mod p Steenrod algebra $\mathscr{A}(p)$. The results generalize and complete the mod 2 computations in [K3]. The techniques are similar, although considerably more care must be practiced to handle the odd prime results.

To derive the structure theorem, we introduce a few new facts about Dyer Lashof operations. These homology operations are applied to compute differentials in an Eilenberg Moore spectral sequence and also to obtain information about the coalgebra structure of classifying spaces. These techniques should have wide applicability.

This Postnikov system is the universal example for certain Massey products. Our results will be used in a later paper to prove the relation stated by Moore and Smith (Section 5 [M3]) between certain higher order differentials in an Eilenberg Moore spectral sequence. In this paper, (Theorem C) we solve a generalization of Conjecture 29 [S2] about these operations.

In [K5] we studied a generalized Eilenberg MacLane space with a twisted H structure

$$\mathbf{K}_k = K(Z_p, 2m) \times \cdots \times K(Z_p, 2mp^k).$$

We showed that the H structure was homotopy associative. In this paper, we show that $\mathbf{K}_k$ has a classifying space $\mathbf{E}_k$, and thus this H structure is strongly homotopy associative [S1]. Our technique will be to build up $\mathbf{E}_k$ as a Postnikov system.

Before describing this Postnikov system, we must fix some notation. Unless explicitly stated to the contrary, p will denote an odd prime. The fundamental class of $H^q(K(Z_p, q); Z_p)$ will be denoted by ι_q. Finally $\sigma: H^q(X; Z_p) \to H^{q-1}(\Omega X; Z_p)$ will denote the loop suspension homomorphism. Thus $\sigma\iota_q = \iota_{q-1}$.

THEOREM A. *There is a k stage Postnikov system*

$$\begin{array}{lcl}
K(Z_p; 2mp^k+1) \xrightarrow{j_k} & \mathbf{E}_k & \\
& \downarrow & \\
& \mathbf{E}_{k-1} & \xrightarrow{\kappa_{k-1}} K(Z_p, 2mp^k+2) \\
& \downarrow & \\
& \vdots & \\
& \downarrow & \\
K(Z_p, 2mp+1) \xrightarrow{j_1} & \mathbf{E}_1 & \xrightarrow{\kappa_1} K(Z_p, 2mp^2+2) \\
& \downarrow & \\
K(Z_p, 2m+1) = \xrightarrow{j_0} & \mathbf{E}_0 & \xrightarrow{\kappa_0} K(Z_p, 2mp+2)
\end{array}$$

satisfying the following conditions:

1. $j_r^* \kappa_r^*(\iota_{sp+2}) = -\beta\, \mathscr{P}^{p^r m}(\iota_{s+1})$ *where* $s = 2mp^r$ for $r = 0, \ldots, k-1$.
2. $\sigma\, \kappa_r^*(\iota) = 0$.
3. *There is an H space equivalence* $\theta: \Omega \mathbf{E}_k \to \mathbf{K}_k$.

This theorem is trivial if $k=0$. Also $\mathbf{E}_1$ is just the 2 stage Postnikov system with stable k invariant $\beta\mathscr{P}^m\, \iota_{2m+1}$. Thus $\mathbf{E}_0$ and $\mathbf{E}_1$ are in fact infinite loop spaces. It is probable that $\mathbf{E}_k$ is an infinite loop space as well. The strongest result that we can obtain at this time is the following.

THEOREM B. *There is an H space* $\mathbf{E}_k'$ *such that* $\Omega^{2p-4}\mathbf{E}_k' = \mathbf{E}_k$ *for each* k.

Let $\langle x \rangle^m$ denote the m fold restricted Massey product [K2]. This operation is a subset of the m fold Massey product $\langle x, \ldots, x \rangle$. Theorem 14 of [K2] states that if $x \in H^{2m+1}(X; Z_p)$ then $\langle x \rangle^p$ is defined as a single class and equals $-\beta\mathscr{P}^m x$.

As in [K4] and [K6], basis elements in the Milnor basis of $\mathscr{A}(p)$ will be denoted $\mathscr{P}(E, R)$ where $E = (\varepsilon_0, \varepsilon_1, \ldots)$, $R = (r_1, r_2, \ldots)$, $\varepsilon_i = 0$ or 1, $r_i \geqslant 0$, and $\sum \varepsilon_i + \sum r_i < \infty$. This element is dual to $\tau_0^{\varepsilon_1}\, \tau_1^{\varepsilon_1} \ldots\, \xi_1^{r_1}\, \xi_2^{r_2} \ldots$. Furthermore let $\mathscr{P}((0, \ldots, 1, 0,), (0, \ldots)) = Q_i$ as usual, let $\mathscr{P}((0, \ldots), (0, \ldots, r, \ldots)) = \mathscr{P}_j(r)$ (the basis element dual to ξ_j^r), and let $\mathscr{P}_j = \mathscr{P}_j(1)$.

Conjecture 29 of [S2] generalizes Theorem 14 of [K2] to p^2 fold Massey products and secondary Bocksteins. Theorem C below proves this generalization for p^{k+1} fold Massey products and $(k+1)$ order Bocksteins β_{k+1}.

THEOREM C. *Let* $u \in H^{2m+1}(\mathbf{E}_k; Z_p)$ *be the generator. Then the higher order cohomology operations* $\langle u \rangle^{p^{k+1}}$, $-\beta_{k+1}\, \mathscr{P}^{p^k m} \ldots \mathscr{P}^m u$ *and* $-\beta_{k+1}\, \mathscr{P}_{k+1}(m)\, u$ *are defined as subsets of* $H^s(\mathbf{E}_k; Z_p)$ *for* $s = 2mp^{k+1} + 2$. *Furthermore each operation contains* $\kappa_{k+1}^*(\iota_s)$, *at least modulo decomposables.*

Our method in this paper is to prove Theorems A, B and C by induction on k. Trivially $\mathbf{E}_0 = K(Z_p, 2m)$ satisfies Theorems A and B. Theorem C is immediate from $\mathscr{P}_1(m) = \mathscr{P}^m$ [M3] and $\langle u \rangle^p = -\beta\mathscr{P}^m u$ [K2]. The two stage system $\mathbf{E}_1$ has been studied in [H]. Theorems A and B are easy in this case. We will assume that $\mathbf{E}_k$ exists and satisfies Theorems A and B. By Theorem A part 3, $\Omega\mathbf{E}_k$ has the H type of $\mathbf{K}_k$.

In section 1 we review some algebraic results about $H^*(\mathbf{K}; Z_p)$ derived in [K6]. An Eilenberg Moore spectral sequence converging to $H^*(\mathbf{E}_k; Z_p)$ is introduced and partially computed in section 2. By using Dyer Lashof operations, the non trivial differentials in this spectral sequence are computed in section 3.

To describe $H^*(\mathbf{E}_k; Z_p)$ as an $\mathscr{A}(p)$ Hopf algebra, we must study the effect of Steenrod operations on Massey products. This is done in section 4. Our results give information about a problem of Milgram (Problem 30 [S2]).

In section 5, we consider an exotic Hopf algebra in order to state a concise structure theorem for $H^*(\mathbf{E}_k; Z_p)$. We also indicate the structure of $H^*(B\mathbf{E}_k; Z_p)$. Finally in section 6 we complete the induction step by explicitly identifying the next k invariant.

Section 1

In [K6] a twisted H structure of $\mathbf{K}_k$ was introduced. If we let $\alpha_i \in H^{2mp^i}(\mathbf{K}_k; Z_p)$ be the image of ι_{2mp^i} under the projection $\mathbf{K}_k \to \mathbf{K}(Z_p, 2mp^i)$, for $i=0,\dots,k$, then the sub Hopf algebra A_k of $H^*(\mathbf{K}_k; Z_p)$ generated by $\alpha_0,\dots,\alpha_k$ is isomorphic as an algebra to

$$Z_p[\alpha_0,\dots,\alpha_k].$$

As a coalgebra, A_k is isomorphic to

$$\Gamma_{k+1}[\alpha_0,\dots,\alpha_0^{p^r},\dots],$$

the divided power coalgebra truncated at height p^{k+1} (see [K5] for a more complete description of A_k).

Let H be a graded, connected, biassociative and bicommutative Hopf algebra. Then by methods of Milnor and Moore we have an exact sequence connecting the primitives and indecomposables of H.

$$PH \xrightarrow{\xi} PH \xrightarrow{\nu} QH \xrightarrow{\lambda} Qh \tag{1.1}$$

where $\xi(c)=c^p$ is the Frobenius map, ν is the composite $PH \rightarrowtail H \twoheadrightarrow QH$, and λ is the dual of the Frobenius map on the dual of H. Note that ν is an isomorphism if $q \not\equiv 0 \pmod{2p}$.

DEFINITION 1.2. If $q \not\equiv 0 \pmod{2p}$ and $c \in H^q$, then $\langle c\rangle \in PH^q$ denotes the unique primitive class such that $\langle c\rangle - c$ is decomposable.

THEOREM 1.3. *$PH^*(\mathbf{K}; Z_p)$ is generated as an unstable left $\mathscr{A}(p)$ module by α_0, $\langle Q_i\alpha_r\rangle$ and $\langle \mathscr{P}_j\alpha_r\rangle$ for $i \geqslant 0, j \geqslant 1$ and $r=1,\dots,k$. The relations are generated by the following:*

$$Q_i\langle Q_j\alpha_r\rangle = -Q_j\langle Q_i\alpha_r\rangle \tag{1}$$
$$\mathscr{P}_i\langle \mathscr{P}_j\alpha_r\rangle = \mathscr{P}_j\langle \mathscr{P}_j\alpha_r\rangle \tag{2}$$
$$Q_i\langle \mathscr{P}_j\alpha_r\rangle = \mathscr{P}_j\langle Q_i\alpha_r\rangle \quad \text{if} \quad i>0 \tag{3}$$
$$Q_0\langle \mathscr{P}_j\alpha_r\rangle = \mathscr{P}_j\langle Q_0\alpha_r\rangle - \langle Q_j\alpha_r\rangle$$
$$(\mathscr{P}_j)^{p-1}\langle \mathscr{P}_j\alpha_r\rangle = \langle \mathscr{P}_j\alpha_{r-1}\rangle^p \tag{4}$$
$$\sum(-1)^i \mathscr{P}^{s-p_i}\langle Q_i\alpha_r\rangle = 0 \quad \text{if} \quad s \geqslant mp^r \tag{5}$$
$$\sum(-1)^{j+1} \mathscr{P}^{s-p_i}\langle \mathscr{P}_j\alpha_r\rangle = 0 \quad \text{if} \quad s > mp^r \quad \text{and} \quad s \not\equiv 0 \pmod{p} \tag{6}$$

where $p_i = 1+p+\dots+p^{i-1}$.

Proof. See Theorem 3.7 and 3.11 [K6].

DEFINITION 1.4. For $r \geqslant 0$, let L_r be the unstable left $\mathscr{A}(p)$ submodule of $PH^*(\mathbf{K}; Z_p)$ generated by $\langle Q_i \alpha_r \rangle$, $i \geqslant 0$. L_r^+ and L_r^- will denote the even and odd dimensional submodules respectively. We can also write $L_r^- = L_r^0 \oplus L_r^1$ where L_r^0 and L_r^1 are the Z_p modules generated by $\mathscr{P}(R)\langle Q_i \alpha_r \rangle$ and $\mathscr{P}(E, R)\langle Q_i \alpha_r \rangle$ with $E \neq 0$ respectively.

Let G_0 be the Z_p submodule of $PH^*(\mathbf{K}; Z_p)$ generated by $\mathscr{P}(R)\alpha_0$ and inductively define G_t to be the union of G_{t-1} and the Z_p submodule of $PH^*(\mathbf{K}; Z_p)$ generated by $\mathscr{P}(R)\langle \mathscr{P}_j \alpha_t \rangle$ for $j \geqslant 1$.

By Theorem 1.3 part 4, this is not a disjoint union. Thus we define $M_r = G_r / G_{r-1}$. Let $M_r \approx M_r^0 \oplus M_r^1$ where M_r^0 is generated by the image of $\mathscr{P}(pR)\langle \mathscr{P}_j \alpha_r \rangle$ and M_r^1 is generated by the image of $\mathscr{P}(R)\langle \mathscr{P}_j \alpha_r \rangle$ where $p \nmid R$, i.e. the image of $\mathscr{P}(R')\mathscr{P}_i \langle \mathscr{P}_j \alpha_r \rangle$ Thus in the Adem basis, $vL_r^0 \subset QH^*(\mathbf{K}_k; Z_p)$ is generated by admissible monomials $\beta^{\varepsilon_i}\mathscr{P}^{s_1} \cdots \mathscr{P}^{s_k}\beta^{\varepsilon_k}\alpha_r$ with $\sum \varepsilon_i = 1$, that is monomials involving exactly one Bockstein. Similarly vM_r^0 is generated by admissible monomials as above with $\sum \varepsilon_i = 0$ and $s_i \equiv 0 \pmod{p}$ for $i \neq t$ and $s_t \equiv 1 \pmod{p}$ for some t, that is monomials involving no Bocksteins and exactly one $\mathscr{A}(p)$ algebra generator $\mathscr{P}^1$.

The following coalgebra structure theorem is essentially Theorem 3.12 of [K6]. The submodules L_r^0, L_r^1, M_r^0, and M_r^1 do not appear explicitly in this theorem. They will, however, be necessary later in the description of $H^*(\mathbf{E}_k; Z_p)$.

THEOREM 1.5. *There is a coalgebra isomorphism*

$$H^*(\mathbf{K}_k; Z_p) \approx E(\bigoplus_r L_r^-) \otimes \Gamma_1(\bigoplus_r L_r^+) \otimes \Gamma_{k-r+1}(\bigoplus_r M_r)$$

where Γ_t is the divided power coalgebra truncated at height p^t.

Section 2

Our induction hypothesis insures that $\mathbf{E}_k$ exists and that $H^*(\mathbf{E}_k; Z_p)$ is a biassociative, bicommutative Hopf algebra. Thus we have a spectral sequence of Hopf algebras with

$$\varepsilon_2 \approx \mathrm{Cotor}_{H^*(\mathbf{K}; Z_p)}(Z_p, Z_p)$$

which converges as algebras to $H^*(\mathbf{E}; Z_p)$. (see [K7] and [M4]). This is the dual spectral sequence to the homology Eilenberg Moore spectral sequences with

$$\varepsilon^2 \approx \mathrm{Tor}^{H_*(\mathbf{K}; Z_p)}(Z_p, Z_p) \Rightarrow H_*(\mathbf{E}; Z_p).$$

If $B \subset H^*(X; Z_p)$ is a submodule of odd dimensional cohomology classes, we define a submodule $\beta\mathcal{P}B$ of even dimensional cohomology classes to be generated by $\beta\mathcal{P}^s x$ where $x \in B$ has dimension $2s+1$ (see [M4]).

The computation of $\mathrm{Tor}^{H^*}(Z_p, Z_p)$ is well known if H^* is a bicommutative bi-associative Hopf algebra (see for example [C]). Since Cotor is the dual functor we can easily derive the following Hopf algebra isomorphisms.

$$\begin{aligned} &\mathrm{Cotor}_{E(x)}(Z_p, Z_p) \approx Z_p[sx] \\ &\mathrm{Cotor}_{\Gamma_t(y)}(Z_p, Z_p) \approx E(sy) \otimes Z_p[\mu_t y] \end{aligned} \tag{2.1}$$

as Hopf algebras where x is odd dimensional, y is even dimensional and

$$\begin{aligned} &\mathrm{bi}\deg sz = (1, \deg z) \\ &\mathrm{bi}\deg \mu_t y = (2, p^t \deg y). \end{aligned}$$

PROPOSITION 2.2. *The generator*

$$\mu_t x \in \mathrm{Cotor}\, \Gamma_t(y) \approx H^*(\mathfrak{F}\Gamma_t(y))$$

represents the restricted p^t fold Massey product $\langle sx\rangle^{p^t}$.

Proof. The element $\mu_t x$ is represented in the cobar construction $\mathfrak{F}\Gamma_t(x)$ by

$$\sum_{i+j=p^t} [\gamma_i][\gamma_j].$$

where γ_i is the ith divided power of $x=\gamma_1$ (see [K7] and [D]).

$$\begin{aligned} \text{Since } d[\gamma_i] &= [\bar{\psi}\gamma_i] \\ &= \sum_{j=1}^{i=1} [\gamma_j][\gamma_{i-j}] \end{aligned}$$

in the cobar construction, we have that (γ_i) forms a defining system for $\langle\{x\}\rangle^{p^t} \subset H^* \times (\mathfrak{F}\Gamma_t(x))$. The result follows by observing that $sx \in \mathrm{Cotor}_{\Gamma_t(x)}$ corresponds to $\{x\} \in H^*(\mathfrak{F}\Gamma_t(x))$.

Since Cotor commutes with $\otimes$ for cocommutative coalgebras we can immediately give the ε_2 term of the Eilenberg Moore spectral sequence. The theorem below follows from the above algebra and the observation in [K2], [K7] and [M4] that $\langle sx\rangle^p = -\beta\mathcal{P}sx$.

THEOREM 2.3. *As bigraded Hopf algebras*

$$\begin{aligned} \varepsilon_2 \approx\; &Z_p[\oplus\, sL_r^-] \otimes E[\oplus\, sL_r^+] \otimes Z_p[\oplus\, \beta\mathcal{P}sL_r^+] \\ &\otimes E[\oplus\, sM_r] \otimes Z_p[\oplus\, \mu_{k-r+1}M_r]. \end{aligned}$$

When $p=2$, it was shown in [K3] that $\varepsilon_2=\varepsilon_\infty$. For odd primes the situation is somewhat more complicated. To determine the differentials. we must make full use of the algebraic structure of ε_*.

THEOREM 2.4. *The differentials are determined algebraically by*

$$d_r:\varepsilon_r^{1,2s}\to\varepsilon_r^{r+1,2s-r+1}.$$

Furthermore d_r *is* 0 *unless* $r=p^q-1$ *and* $2s=2tp^q$ *or* $r=2p^q-1$ *and* $2s=(2tp+2)p^q$ *for some* $q\geqslant 1$ *and* $t\geqslant 1$.

Proof. Since ε_r is a spectral sequence of Hopf algebras, d_r is algebraically determined by its action on the indecomposables, which are concentrated in primary degrees 1 and 2. Also ε_2 is primitively generated and d_r maps primitives to primitives. The only primitives of primary degree greater than 2 are the p^qth powers of even dimensional primitives, and these have bidegree $(p^q, (2t+1)p^q)$ and $(2p^q, 2tp^{q+1})$ by (2.1). These primitives have even total degree, and so if they are in the image of d_r, then a simple counting argument shows that r and s must be as in the theorem. See also [C] and [M4] for similar arguments.

An element x $PH^t(\mathbf{K}_k; Z_p)$ determines an element $sx=[x]\in\varepsilon_2^{1,t}$. If $[x]$ is an infinite cycle, then $[x]$ represents an element in the associated graded of $H^{t+1}(\mathbf{E}_k; Z_p)$. If $t\not\equiv 1\,(\mathrm{mod}\,2p)$ then $\sigma QH^{t+1}(\mathbf{E}_k; Z_p)\to PH^t(\mathbf{K}_k; Z_p)$ is a monomorphism [C]. In this case we will denote by $[x]$ the unique element of $QH^{t+1}(\mathbf{E}_k; Z_p)$ satisfying $\sigma[x]=x$.

The classifying space $\boldsymbol{E}_k$ can be constructed from $\boldsymbol{K}_k$ using the geometric analogue of the bar construction [M2]. Thus, by naturality, the differentials in the associated spectral sequence commute with Steenrod operations in the graded sense. In particular if $[x]\in\varepsilon_2^{1,*}$ is an infinite cycle, so is $[\mathscr{P}(E,R)x]\in\varepsilon_2^{1,*}$.

Clearly $[\alpha_0]\in\varepsilon_2^{1,2m}$ is an infinite cycle representing u. Furthermore, by Theorem 3.4, $[\langle Q_i\,\alpha_r\rangle]$ and $[\langle\mathscr{P}_i\mathscr{P}_j\alpha_r\rangle]$ are infinite cycles for dimension reasons. Thus we have the following results.

PROPOSITION 2.5. *The elements of* sL_r, sM_0, *and* sM_r^1 *for* $r=0,\ldots,k$ *in* $\varepsilon_2^{1,*}$ *are infinite cycles.*

Thus it remains to compute the differentials on sM_r^0 for $r=1,\ldots,k$. This we will do in the next section.

Section 3

It turns out that d_{p-1} is nontrivial on $sM_r^0\subset\varepsilon_p^{1,*}$. We can determine this differential and simultaneously determine $\lambda:QH^q(\mathbf{E};Z_p)\to QH^{q/p}(\mathbf{E};Z_p)$ see (1.1) by looking at Dyer Lashof operations.

Recall the following facts about these operations ([M1], [N])

THEOREM 3.1. *If X is an n fold loop space, then there are natural homology operations*

$$\beta: H_s(X; Z_p) \to H_{s-1}(X; Z_p)$$

and

$$\mathbf{Q}^j: H_s(X; Z_p) \to H_{s+2j(p-1)}(X; Z_p)$$

for $2j(p-1) < n+2sp$ *satisfying*

$$\sigma_* \mathbf{Q}^j = \mathbf{Q}^j \sigma_* \tag{1}$$

where $\sigma_*: H_{q-1}(\Omega X) \to H_q(X)$ *is the loop homomorphism.*

$$\mathbf{Q}^j x = x^p \quad \text{if} \quad \dim x = 2j. \tag{2}$$

$$\mathbf{Q}^j x = 0 \quad \text{if} \quad \dim x > 2j. \tag{3}$$

$$\mathscr{P}_{i+1*} \mathbf{Q}^{j+p^i} = \mathbf{Q}^j \mathscr{P}_{i*} - \mathscr{P}_{i*} \mathbf{Q}^j$$

$$Q_{i+1*} \mathbf{Q}^{j+p^i} = \mathbf{Q}^j Q_{i*} - Q_i \mathbf{Q}_*^j \tag{4}$$

Proof. The first three properties can be found in [M1]. The last one can be derived from the Nishida relations [N]. A full exposition of these relations will be given in a later paper.

PROPOSITION 3.2. *For* $i \geqslant 0$ *and* $r \geqslant 0$,

$$\lambda[Q_{i+1}\alpha_{r+1}] = [Q_i \alpha_r]$$

in $QH^*(\mathbf{E}; Z_p)$.

Proof. With $s = mp^r + p^i$, the composite $\sigma_* v$

$$PH_{2s-1}(\mathbf{K}; Z_p) \to QH_{2s-1}(\mathbf{K}; Z_p) \to PH_{2s}(\mathbf{E}; Z_p)$$

is an isomorphism. Let $y \in PH_{2s}(\mathbf{E}; Z_p)$ and $x = (\sigma_* v)^{-1} y$. The following chain of equalities uses the adjointness properties of homology and cohomology, Theorem 3.1, and the facts that $\langle x, u \rangle = 0$ if u is decomposable and $\mathbf{Q}^s x$ is primitive if x is.

$$\begin{aligned}
\langle y, \lambda[Q_{i+1}\alpha_{r+1}] \rangle &= \langle y^p, [Q_{i+1}\alpha_{r+1}] \rangle \\
&= \langle \sigma_* \mathbf{Q}^s x, [Q_{i+1}\alpha_{r+1}] \rangle \\
&= \langle \mathbf{Q}^s x, \langle Q_{i+1}\alpha_{r+1} \rangle \rangle \\
&= \langle \mathbf{Q}^s x, Q_{i+1}\alpha_{r+1} \rangle \\
&= \langle \mathbf{Q}^{s-p^i} Q_{i*} x, \alpha_{r+1} \rangle
\end{aligned}$$

$$= \langle Q_{i*}x, \lambda\alpha_{r+1}\rangle$$
$$= \langle x, \langle Q_i\alpha_r\rangle\rangle$$
$$= \langle y, [Q_i\alpha_r]\rangle.$$

The theorem follows immediately by the duality between PH_* and QH^*.

THEOREM 3.3. *In the spectral sequence* ε_r

$$d_{p-1}[\mathscr{P}_{i+1}\alpha_{r+1}] = [Q_i\alpha_r| \overset{p}{\cdots} |Q_i\alpha_r].$$

Proof. Let $x \in PH_{2s-1}(\mathbf{K}; Z_p)$ where $s = mp^r + p^i$. By [K1]$-\beta\mathbf{Q}^s x = \langle x\rangle^p$ in $H_{2sp-2}(\mathbf{K}; Z_p)$, where $\langle x\rangle^p$ is the restricted Massey product defined with the Pontrjagin product in $H_*(\mathbf{K}; Z_p)$. By [K7], the differentials in the homology spectral sequence

$$\varepsilon^2 = \mathrm{Tor}^{H_*(\mathbf{K}; Z_p)}(Z_p, Z_p) \Rightarrow H_*(\mathbf{E}; Z_p)$$

are determined by Massey products. In particular

$$d^{p-1}[x| \overset{p}{\cdots} |x] = [\langle x\rangle^p].$$

Using the previous theorem and the equations $\lambda\mathscr{P}_{i+1} = 0$ and

$$\beta\mathscr{P}_{i+1} = \mathscr{P}_{i+1}\beta - Q_{i+1}$$

we have

$$\langle \beta\mathbf{Q}^s x, \mathscr{P}_{i+1}\alpha_{r+1}\rangle = \langle \mathbf{Q}^s x, (\mathscr{P}_{i+1}\beta - Q_{i+1})\,\alpha_{r+1}\rangle$$
$$= \langle y^p, \mathscr{P}_{i+1}[\beta\alpha_{r+1}] - [Q_{i+1}\alpha_{r+1}]\rangle$$
$$= \langle y, -[Q_i\alpha_r]\rangle$$
$$= \langle x, -Q_i\alpha_r\rangle.$$

The theorem now follows from the duality between the spectral sequences ε^r and ε_r.

Remark 3.4. This theorem could also have been proven directly, that is without reference to Dyer Lashof operations, using the techniques of [K7].

Note that in the Serre spectral sequence of the fibration

$$K(Z_p, 2mp + 1) \to \mathbf{E}_1 \to K(Z_p, 2m + 1)$$

we have $\tau\iota_{2mp+1} = \beta\mathscr{P}^m\iota_{2m+1}$ and $\tau\mathscr{P}_{j+1}\iota_{2mp+1} = (Q_j\iota_{2m+1})^p$ by naturality of the transgression τ and Adem relations.

Recall that if $x \in H^{2m}(X; Z_p)$ satisfies $x^n = 0$, then the n fold transpotence $\varphi_n(x) \subset H^{2mn-2}(\Omega X; Z_p)$ is defined. By [D], the following is immediate.

COROLLARY 3.5. $\langle \mathscr{P}_{j+1}\alpha_{r+1}\rangle \in \varphi_p([Q_j\alpha_r]) \subset H^*(\mathbf{K}; Z_p)$.

By Theorems 2.5 and 3.3 the ε_p term of the spectral sequence is obtained from ε_2 by dividing out by $E(sM_r^0)$ for $r=1,\ldots,k$ and $Z_p[(sL_r^0)^p]$ for $r=0,\ldots,k-1$. For dimension reasons, we have already checked that no other differentials are possible.

Let $Z_p^{(1)}[x]=Z_p[x]/(x^p)$. Then we have proved the following structure theorem.

THEOREM 3.6. *As an algebra*

$$\begin{aligned} H^*(\mathbf{E}_k; Z_p) \approx\ & Z_p[\oplus\, sL_r^1] \otimes Z_p[sL_k^0] \\ & \otimes Z_p^{(1)}[\oplus_{r<k}\, sL_r^0] \otimes E[\oplus\, sL_r^+] \\ & \otimes Z_p[\oplus\, \beta\mathscr{P} sL_r^+] \otimes E(\oplus\, sM_r^1) \\ & \otimes E[sM_0^0] \otimes Z_p[\oplus\, \mu_{k-r+1} M_r]. \end{aligned}$$

Section 4

The submodule $\mu_{k-r+1}M_r$ of $\varepsilon_2^{2,*}$ consists entirely of infinite cycles by Theorem 2.4. These elements correspond to p^{k-r+1} fold restricted Massey products on the odd dimensional classes $[\mathscr{P}(R)\mathscr{P}_j\alpha_r]$ in $H^*(\mathbf{E}_k; Z_p)$ by Proposition 2.2 and the results of [K7]. In fact $\mathbf{E}_k$ is clearly the universal example for $\langle u\rangle^{p^{k+1}}$.

THEOREM 4.1. *Let* $v\in H^{2m+1}(X; Z_p)$. *Then* $\langle v\rangle^{p^{k+1}}$ *is defined if and only if there is a map*

$$f: X \to \mathbf{E}_k$$

such that $f^*u=v$ *and* $f^*\langle u\rangle^{p^{k+1}} \subset \langle v\rangle^{p^{k+1}}$.

Proof. If f exists, then $\langle v\rangle^{p^{k+1}}$ is defined by naturality. Conversely if $\langle v\rangle^{p^{k+1}}$ is defined, then $\langle v\rangle^{p^k}$ is defined and contains 0. By induction, on k and Theorem A there is a map $f': X\to \mathbf{E}_{k-1}$ such that $f'^*\kappa^*(\iota)=0$. Thus f' lifts to $f: X\to\mathbf{E}_k$.

Note that this shows that $\langle v\rangle^{p^{k+1}}$ is defined if and only if $\langle v\rangle^{p^k}$ is defined and contains 0. This situation is in sharp contrast to the definition of general higher order cohomology operations. Also the indeterminacy of $\langle u\rangle^{p^{k+1}}$ is more controllable.

PROPOSITION 4.2. *Let* $v\in H^{2n+1}(X; Z^p)$ *and suppose that* $\langle v\rangle^{p^{t+1}}$ *is defined for some* $t\geqslant 1$. *Then the indeterminacy of* $\langle v\rangle^{p^{t+1}}$ *contains* $\langle w\rangle^p = -\beta\mathscr{P}^{np^t}w$ *for all* $w\in H^{2np^t+1}(X; Z_p)$.

Proof. Let (a_i) be a defining system for $\langle v\rangle^{p^{t+1}}$ and let b a cocycle representative for w. Using the techniques of Theorem 14 and Lemma 16 of [K2], we can construct a defining system (a'_i) with $a'_i=a_i$ for $i<p^t$, $a'_{p^t}=a_{p^t}+b$, and such that the

difference in the related cocycles is cohomologous to a defining system for $\langle w\rangle^p$. The proposition follows.

By Theorem B, $H^*(\mathbf{E}_k, Z_p)$ is a bicommutative, biassociative Hopf algebra. This extra structure is useful in getting a better hold on the Massey products.

DEFINITION 4.3. Let $x \in PH^{2n+1}(\mathbf{E}_k; Z_p)$ and assume that $\langle x\rangle^{p^t}$ is defined and non zero in $QH^{2np^t+2}(\mathbf{E}_k; Z_p)$. Let $\mu_t(\sigma x)$ denote the set of all $w \in PH(\mathbf{E}_k; Z_p)$ such that $vw \in \langle x\rangle^{p^t}$.

It is probably true that $\mu_t \sigma x \subset \langle x\rangle^{p^t}$, although we have no proof of this as yet. We may consider μ_t to be a higher order operation from $PH^{2m}(\Omega X; Z_p)$ to $PH^{2mp^t+2}(X; Z_p)$ whenever X is an H space. The results of [D] and [K7] show that μ_t is dual to the transpotence.

Let $\mathbf{F} = K(Z_p, 2mp^k+1)$ and $j:\mathbf{F}\to\mathbf{E}_k$ be the fiber inclusion of Theorem A. We will show that j^* detects Massey products.

THEOREM 4.4. *If* $r=0$ *or* $p \nmid R$, *then*

$$j^*\mu_{k-r+1}\langle \mathscr{P}(R)\,\alpha_r\rangle = -\,\beta\mathscr{P}\,\mathscr{P}(p^{k-r}R)\,\iota\,.$$

Thus j^* restricted to $\mu_{k-r+1}M_r$ is a monomorphism.

Proof. First assume that $r=0$ and $R=0$.

Thus the equation reduces to

$$j^*\mu_{k+1}\alpha_0 = -\,\beta\mathscr{P}^{mp^k}\iota \quad \text{in} \quad PH^*(\mathbf{F}; Z_p)$$

or

$$j^*\langle u\rangle^{p^{k+1}} = \langle\iota\rangle^p \quad \text{in} \quad QH^*(\mathbf{F}; Z_p).$$

The map j and its loop suspension induce a map of spectral sequences

$$j^*:\varepsilon_r(\mathbf{K}_k)\to\varepsilon_r(\mathbf{F})\,.$$

It is easy to check that

$$j^*\Big(\sum_{i+j=p^{k+1}}[\gamma_i\alpha_0\,|\,\gamma_j\alpha_0]\Big) = \sum_{m+n=p}[\gamma_m\iota\,|\,\gamma_n\iota]$$

in $\varepsilon_2 \approx \mathrm{Cotor}_{H^*\Omega\mathbf{F},\,Z_p}(Z_p, Z_p) \approx H^*(\mathfrak{F}H^*(\Omega\mathbf{F}; Z_p))$. (see [K7]). The equation follows by Proposition 2.2.

In the general situation, Theorem 4.1 implies that there is a map

$$f:\mathbf{E}_k\to\mathbf{E}'_{k-r}$$

where $\Omega \mathbf{E}'_{k-r} \approx \mathbf{K}'_{k-r} = K(Z_p, 2t) \times \cdots \times K(Z_p, 2tp^{k-r})$with the twisted H structure, $2t = \dim \mathscr{P}(R)\alpha_r$ and $f^*u' = [\mathscr{P}(R)\alpha_r]$. The proposition now follows by naturality and the observation that $\gamma_{p^{k-1}} \mathscr{P}(R)\alpha_r = \mathscr{P}(p^{k-r}R)\alpha_k$.

These results enable us to get our hands on the generators and relations of $H^*(\mathbf{E}_k; Z_p)$.

THEOREM 4.5. *If* $x \in PH^{2np^t+2}(\mathbf{E}_k; Z_p)$ *satisfies* $\sigma x = 0$ *and* $j^*x = j^*\mu_t y$, *then* $x \in \mu_t y$.

Proof. Theorem 3.6, the only primitive generators which are in $\operatorname{Ker}\sigma$ are of the form $\beta\mathscr{P}z$ and μy. The theorem follows from Proposition 4.2 and Theorem 4.4.

To make results on the $\mathscr{A}(p)$ structure of $H^*(\mathbf{E}_k; Z_p)$ easier to state, we need to introduce some abbreviations. Recall that

$$\lambda[Q_{i+1}\alpha_{r+1}] = [Q_i\alpha_r]$$

by Proposition 3.2. Also for dimension reasons $[\theta\alpha_r]$ is in $\operatorname{Ker}\lambda$ for $r=0$ or

$$\theta = \mathscr{P}_i\mathscr{P}_j,\ \mathscr{P}_iQ_j,\ Q_iQ_j, \quad \text{or} \quad Q_0.$$

Compare the following with (1.2) and (2.6).

Notation 4.6. Let $[\theta\alpha_r] = \langle[\langle\theta\alpha_r\rangle]\rangle$ in $PH^*(\mathbf{E}_k; Z_p)$ if θ is as above. Also let $[Q_i\alpha_r]$ denote the divided power

$$\begin{aligned} &\gamma_{p^i}[Q_0\alpha_{r-i}] && \text{if} \quad i \leqslant r \\ &\gamma_{p^r}(Q_{i-r}u) && \text{if} \quad i \geqslant r. \end{aligned}$$

in $H^*(\mathbf{E}_k; Z_p)$.

THEOREM 4.7. *Let* $p_i = 1 + \cdots + p^{i-1}$.
Then the following relations hold in $PH^*(\mathbf{E}_k; Z_p)$.

$$\sum(-1)^i \mathscr{P}^{mp^k - p_i}[Q_i\alpha_k] \in \mu_{k+1}\alpha_0 \tag{1}$$

$$0 \in Q_i\mu_{k+1}\alpha_0 \tag{2}$$

$$[Q_i\alpha_k]^p \in \mathscr{P}_{j+1}\mu_{k+1}\alpha_0 \tag{3}$$

$$\begin{aligned} \mathscr{P}^{p^t}\mu_{k+1}\alpha_0 &\subset \mu_{k+1}\mathscr{P}^{p^{t-k-1}}\alpha_0 && \text{if} \quad t > k \\ &\subset \mu_t\langle\mathscr{P}^1\alpha_{k-t+1}\rangle && \text{if} \quad 0 < t < k. \end{aligned} \tag{4}$$

Proof. By Proposition 3.10 of [K6] we have

$$\sum(-1)^i \mathscr{P}^{s-p_i}Q_i = \beta\mathscr{P}^s.$$

By [K4], $\operatorname{exc}\beta\mathscr{P}^s = 2s+1$. Equation 1 now follows directly from Theorems 4.4 and

4.5. To verify equation 2, use the above and the formula $\operatorname{exc} Q_i\beta\mathscr{P}^s = 2s+2$. Equation 3 is derived from the relation

$$\mathscr{P}_j\beta\mathscr{P}^s\iota_{2s+1} = (Q_j\iota_{2s+1})^p \quad \text{of} \quad [\text{M3}] \quad \text{and} \quad [\text{K4}],$$

while equation 4 follows from

$$\mathscr{P}^{p^t}\beta\mathscr{P}x = \beta\mathscr{P}\,\mathscr{P}^{p^{t-1}}x \quad \text{for}$$

an odd dimensional class x.

By naturality, these relations will generate all others involving $\mu_{k-r+1}M_r$. For completeness we record some of them.

COROLLARY 4.8. *If $r=0$ or $p\nmid R$, then*

$$\textstyle\sum(-1)^i\mathscr{P}(p^{k-r}R)\,\mathscr{P}^{mp^k-p^i}[Q_i\alpha_k] \in \mu_{k-r+1}\langle\mathscr{P}(R)\,\alpha_r\rangle. \tag{1}$$

$$0 \in Q_j\mu_{k-r+1}\langle\mathscr{P}(R)\,\alpha_r\rangle \tag{2}$$

$$[Q_j\mathscr{P}(p^{k-r}R)\,\alpha_k]^p \in \mathscr{P}_{j+1}\mu_{k-r+1}\langle\mathscr{P}(R)\,\alpha_r\rangle. \tag{3}$$

THEOREM 4.9. *$PH^*(\mathbf{E}_k; Z_p)$ is generated as an unstable left $\mathscr{A}(p)$ module by u, $[\mathscr{P}_i\mathscr{P}_j\alpha_r]$, $[Q_0\alpha_r]$, $\mathscr{P}_i[Q_j\alpha_r]$ and $Q_i[Q_j\alpha_r]$. for i, $j\geqslant 1$ and $r=1,\dots,k$.*

The relations are generated by

$$Q_i[Q_j\alpha_r] = -\,Q_j[Q_i\alpha_r] \tag{1}$$

$$\mathscr{P}_i[\mathscr{P}_j\mathscr{P}_k\alpha_r] = \mathscr{P}_k[\mathscr{P}_i\mathscr{P}_j\alpha_r] \tag{2}$$

$$Q_i[\mathscr{P}_j\mathscr{P}_k\alpha_r] = \mathscr{P}_j\mathscr{P}_k[Q_i\alpha_r] \quad \text{if} \quad i>0 \tag{3}$$

$$\mathscr{P}_i\mathscr{P}_j[Q_0\alpha_r] = \mathscr{P}_j[Q_i\alpha_r] + \mathscr{P}_i[Q_j\alpha_r] + Q_0[\mathscr{P}_i\mathscr{P}_j\alpha_r]$$

$$(\mathscr{P}_j)^{p-2}[\mathscr{P}_j^2\alpha_r] = [\mathscr{P}^{mp^{r-1}+p^{j-1}}\mathscr{P}_j\alpha_{r-1}] \tag{4}$$

$$\sum_i(-1)^i\mathscr{P}^{s-p^i}[Q_i\alpha_r] = \mu_{k+1}\alpha_0 \quad \text{if} \quad s=mp^k \tag{5}$$

$$= 0 \qquad \text{if} \quad s>mp^k$$

$$\sum_j(-1)^{j+1}\mathscr{P}^{s-p^j}[\mathscr{P}_j\alpha_r] = 0 \qquad \text{if} \quad s>mp^k \tag{6}$$

$$\mathscr{P}^{mp^r+p^i}[Q_i\alpha_r] = [Q_i\alpha_r]^p = 0 \qquad \text{if} \quad r<k. \tag{7}$$

Proof. The sequence (1.1) and our observations on the action of σ imply that the set of generators is correct. That equations 1 through 6 are relations is immediate from Theorems 1.3, 4.3 and 4.4. Equation 7 follows from Theorem 3.3. The fact that equations 1 through 7 generate all relations also follows from Theorems 1.3 and 4.3 using the techniques of Section 3 [K6].

Note that the description of the generators and the relations of $PH^*(\mathbf{E}_k; Z_p)$ is considerably simpler when p is an odd prime than when $p=2$ (compare Theorem 12 [K3]). This is partly because the kernel of λ is larger in the odd prime case.

Section 5

All the information necessary for describing the Hopf algebra structure of $H^*(\mathbf{E}_k; Z_p)$ has now been developed. As in [K3] a Hopf algebra which starts off looking like a divided power Hopf algebra and ends up looking like a polynomial algebra is needed. This Hopf algebra was considered by W. Browder [B].

DEFINITION 5.1. Let x be an even dimensional class in a Z_p module. For $0 \leqslant t \leqslant \infty$, let $M_t(x)$ be the Hopf algebra, which as an algebra is generated by x, $\gamma_p x, \ldots, \gamma_{p^t} x$ with relations $(\gamma_{p^r} x)^p = 0$ for $r < t$. That is as algebras $M_t(x) = Z_p^{(1)}[x] \otimes \cdots \otimes Z_p^{(1)}[\gamma_{p^{t-1}} x] \otimes Z_p[\gamma_p t x]$. As a coalgebra, $\gamma_{p^r} x$ is the p^rth divided power of x and $(\gamma_{p^t} x)^{p^s}$ is primitive. That is as coalgebras $M_t(x) \approx \Gamma_t(x) \otimes \Gamma_1((\gamma_{p^t} x)^{p^s})$. As Hopf algebras $M_0(x) \approx Z_p[x]$, and $M_\infty(x) \approx \Gamma[x]$. If B is an even dimensional Z_p module with basis $\{x_i\}$, then $M_t(B) \approx \otimes M_t[x_i]$.

Let C be a graded Z_p module with C^+ and C^- the even and odd dimensional submodules. Then recall that $S(C)$ is the free commutative Hopf algebra on C. That is $S(C) \approx Z_p[C^+] \otimes E[C^-]$ as Hopf algebras.

To get the Hopf algebra structure theorem for $H^*(\mathbf{E}_k; Z_p)$, it simply remains to fix notation for various submodules of the primitives.

DEFINITION 5.2. Let $B_r \subset PH^*(\mathbf{E}_k; Z_p)$ be the Z_p module generated by $\mathscr{P}(R)Q_i u$ if $r=0$ and $\mathscr{P}(R)[Q_0 \alpha_r]$ and $\mathscr{P}(R)\mathscr{P}_j[Q_i \alpha_r]$ if $r=1, \ldots, k$. Let $C_r \subset PH^*(\mathbf{E}_k; Z_p)$ be the Z_p module generated by $\mathscr{P}(R)u$ if $r=0$ and $\mathscr{P}(R)[\mathscr{P}_i \mathscr{P}_j \alpha_r]$ if $r=1, \ldots, k$. Finally let D_r be generated by $\mathscr{P}(E, R)[Q_i \alpha_r]$ for $r=0, \ldots, k$, where $E \neq 0$.

THEOREM 5.3. *As a Hopf algebra*

$$H^*(E_k; Z_p) \approx \bigoplus_r M_{k-r}(B_r) \otimes S(\bigoplus_r C_r \oplus D_r)$$

$$H^*(\mathbf{E}_\infty; Z_p) \approx \oplus \Gamma(B_r) \otimes S(\bigoplus_r C_r \oplus D_r).$$

Roughly speaking $H^*(\mathbf{E}_\infty; Z_p)$ is a divided power coalgebra on elements involving one Bockstein tensor a free commutative algebra on elements involving 0 or more than one Bockstein.

It is possible to compute $H^*(B\mathbf{E}_k; Z_p)$ as a Hopf algebra over $\mathscr{A}(p)$ using very similar techniques. In fact it can be shown that

$$\lambda[[Q_{i+1}Q_{j+1}\alpha_{r+1}]] = [[Q_iQ_j\alpha_r]]$$

where

$$\sigma^2[[Q_iQ_j\alpha_r]] = \langle Q_iQ_j\alpha_r\rangle .$$

Moreover in the spectral sequence

$$\varepsilon_2 \approx \mathrm{Cotor}_{H^*(\mathbf{E};Z_p)}(Z_p, Z_p) \Rightarrow H^*(B\mathbf{E}; Z_p)$$

we have

$$d_{p-1}[[\mathscr{P}_{i+1}Q_{j+1}\alpha_{r+1}]] = [[Q_iQ_j\alpha_r] \overset{p}{\cdots} [Q_iQ_j\alpha_r]] .$$

Thus $H^*(B\mathbf{E}_\infty; Z_p)$ will be roughly a divided power coalgebra on elements involving two Bocksteins tensor with a free commutative algebra on elements involving 0 or more than two Bocksteins. In a later paper, the author plans to describe the cohomology of the iterated classifying spaces of K as well as the complete structure of $H^*(\mathbf{K}; Z_p)$ over the Dyer Lashof algebra.

Section 6

We now complete the induction hypothesis by identifying the k invariant of $\mathbf{E}_{k+1}$. By the induction hypothesis, there is an H space E_k' such that $\Omega^{2p-4}E_k'=\mathbf{E}_k$.

THEOREM 6.1. *The iterated suspension homomorphism* $\sigma^{2p-4}: PH^{2mp^{k+1}+2p-2}(\mathbf{E}_k'; Z_p) \to QH^{2mp^{k+1}+2}(\mathbf{E}_k; Z_p)$ *is an isomorphism.*

Proof. The map $\nu: PH^q \to QH^q$ is an isomorphism if $q \not\equiv 0 (\mathrm{mod}\, 2p)$ by (1.1). The map $\sigma: QH^q \to PH^{q-1}$ is an isomorphism unless $q \equiv 2$ or $-1 (\mathrm{mod}\, 2p)$ by [C]. Just compose these isomorphisms.

Proof of Theorem B. Consider a class $\eta \in \langle u\rangle^{p^{k+1}} \subset QH^{2mp^{k+1}+2}(\mathbf{E}_k; Z_p)$.

By the previous theorem $\eta = \sigma^{2p-4}\eta'$ for $\eta' \in PH^*(\mathbf{E}_k')$. Let $\kappa': \mathbf{E}_k' \to K(Z_p, 2mp^{k+1}+2p-2)$ be an H map representing η' and let $\mathbf{E}_{k+1}'$ be the fiber space over $\mathbf{E}_k'$ induced by κ'. Then since η' is primitive, $\mathbf{E}_{k+1}'$ is an H space. Define $\mathbf{E}_{k+1} = \Omega^{2p-4}\mathbf{E}_{k+1}'$ and $\kappa: \mathbf{E}_k \to K(Z_p, 2mp+2)$ to be $\Omega^{2p-4}\kappa'$. Then clearly $\mathbf{E}_{k+1}$ is the fiber space induced by κ which proves Theorem B.

Proof of Theorem A. Part 1 follows from Proposition 4.2 and Part 2 follows since $\sigma\langle u\rangle^{p^{k+1}}=0$ [K2]. This implies immediately that $\Omega\mathbf{E}_{k+1}$ splits topologically into the product $\Omega\mathbf{E}_k \times K(Z_p, 2mp^{k+1})$ since the k invariant is null homotopic. We must show that there is an H map which induces the equivalence, where inductively $\Omega E_k \times K(Z_p, 2mp^{k+1}) \approx \mathbf{K}_{k+1}$.

This follows easily from [H] where $k=0$. The result can now be checked by naturality in the diagram

$$\begin{array}{ccccc} & & \bar{\mathbf{E}}_1 & \longrightarrow & \mathbf{E}_{k+1} \\ & & \downarrow & & \downarrow \\ K(Z_p, 2mp^k+1) & \longrightarrow & \mathbf{E}_k & \longrightarrow & K(Z_p, 2mp^{k+1}+2). \end{array}$$

Finally we must identify the k invariants in terms of p^kth order Bocksteins.

LEMMA 6.2. *If* $u \in H^{2m+1}(X; Z_p)$, *then* $\mathscr{P}_k(m)u = \mathscr{P}^{p^k m} \dots \mathscr{P}^m u$.

Proof. For $k=1$, this is well known. By Milnor's multiplication table [M3],

$$\mathscr{P}^{p^{k+1}} \mathscr{P}_k(m) = \sum_i \mathscr{P}(ip^k m, 0, \dots, i, m-i, 0, \dots).$$

The excess of the ith term is $2(ip^k m + m)$ by [K4]. Thus only the term $i=0$ contributes when applied to u. The lemma now follows by induction.

Proof of Theorem C. By construction $\kappa^*(\iota) \in \langle u \rangle^{p^{k+1}}$. This class is represented in the spectral sequence by

$$\sum_{i+j=p^{k+1}} [\gamma_i \mid \gamma_j] \in \mathfrak{F}H^*(\mathbf{K}; Z_p) \approx \varepsilon_1^{2,\, 2mp^{k+1}}.$$

[K7]. (see Proposition 2.2). Furthermore the element

$$[\alpha_0^{p^{k+1}}] \in \varepsilon_1^{1,\, 2mp^{k+1}}$$

represents $\mathscr{P}_k(m)u$ in $H^*(\mathbf{E}_k; Z_p)$.

Let $Z_{(p)}$ be the rationals with denominators prime to p. Then it suffices to show that

$$\psi\alpha_0^{p^{k+1}} = p^k \sum \gamma_i \otimes \gamma_j \,(\mathrm{mod}\, p^{k+1})$$

in

$$\mathfrak{F}H^*(\mathbf{K}; Z_{(p)}).$$

This is immediate from Theorem 11 [K5].

REFERENCES

[B] BROWDER, W., *Higher torsion in H spaces*, Trans. Amer. Math. Soc. *108* (1963) 353–375.

[C] CLARK, A., *Homotopy commutativity and the Moore spectral sequence*. Pacific J. Math. *60* (1954) 513–557.

[D] DRACHMAN, B. and KRAINES, D., *A duality between transpotence elements and Massey products*. Pacific J. Math. *39* (1971) 119–123.

[H] HARPER, J. and SCHOCHET, C., *Coalgebra extensions in two stage Postnikov systems*, Math. Scand *20* (1971) 232–236.

[K1] Kochman, S., *Symmetric Massey products and a Hirsch formula in homology*, Trans. Amer. Math. Soc. *163* (1972) 245–260.
[K2] Kraines, D., *Massey higher products*, Trans. Amer. Math. Soc. *124* (1966) 431–449.
[K3] ——, *The cohomology of some k stage Postnikov systems*, Proc. Adv. Study Inst. on Algebraic Topology, Vol. 1 Aarhus 1970.
[K4] ——, *On excess in the Milnor basis*, Bull. London. Math. Soc. *3* (1971) 363–365.
[K5] ——, *Approximations to self dual Hopf algebras*. Amer. J. Math. (to appear).
[K6] ——, *Twisted multiplications on generalized Eilenberg MacLane space*, Math. Scand. (to appear).
[K7] Kraines, D. and Schochet, C., *Differentials in the Eilenberg Moore spectral sequence*. J. Pure and Applied Algebra. 2 (1972) 131–148.
[M1] May, J. P., *A general algebraic approach to Steenrod operations. The Steenrod Algebra and its applications*. Springer Verlag Lecture Notes *168* (1970).
[M2] Milgram, R. J., *The bar construction and abelian H spaces*. Illinois J. Math. *11* (1967) 242–250.
[M3] Milnor, J., *The Steenrod algebra and its dual*. Ann. of Math. *67* (1958) 150–171.
[M4] Moore, J. C. and Smith, L., *Hopf algebras and multiplicative fibrations I*. Amer. J. Math. *90* (1968) 752–780.
[S1] Stasheff, J., *Homotopy associativity of H spaces*. Trans. Amer. Math. Soc. *108* (1963) 275–312.
[S2] ——, *Problems, Conference on Algebraic Topology*, University of Illinois at Chicago Circle, (1968) 288–293.

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