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Epimorphic Extensions of Non-Commutative Rings

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Introduction

Let R be a subring of the ring S. Following Isbell [9] we say, that an element $d \in S$ is *dominated* by R if $f(d) = g(d)$ for every pair of ring homomorphisms $f, g: S \to T$ having the property, that $f(r) = g(r)$ for all $r \in R$. The set of all elements dominated by R is called the *dominion* $\mathrm{Dom}(R, S)$. This is clearly a subring of S containing R. The inclusion map $R \subseteq S$ is an *epimorphism* in the category of rings if and only if $\mathrm{Dom}(R, S) = S$. If this is the case, then we will say, that S is an *epimorphic extension* of R. Expressed otherwise, this means, that every ring homomorphism with domain R can be extended to S in at most one way.

In this paper, various properties of the dominion and of epimorphic extensions are studied. In the first section, we state a number of criteria for an element to be dominated. In section two it is shown, that the dominion behaves nicely (i.e. in the expected way) under finite (though not under infinite) products, as well as under the formation of matrix rings and of rings of the form eRe, where e is a suitable idempotent. As a consequence, the property of having only trivial epimorphic extensions is Morita invariant.

The third section deals with flat epimorphic extensions. Flat epimorphisms recently have attracted considerable attention. We show, that flat epimorphic extensions behave nicely under the various constructions mentioned above; this is done by proving the corresponding results for flat ring extensions in general. In the fourth section, we characterize the flat epimorphic extensions of a left perfect ring as the endomorphism rings of certain two-sided ideals. Another result is, that a right perfect ring has no proper epimorphic extensions provided it contains a copy of every simple right module. In the last section, it is shown, that if R is a principal ideal domain contained in the center of S, then the inclusion of R in its dominion is a ring epimorphism. This generalizes a result by Bousfield and Kan [4].

All rings under consideration are associative and have a unit element; ring homomorphisms and modules are unitary. In particular, any subring R of a ring S contains the unit element of S. If M is an S-module and if we consider M as an R-module, we shall always mean the R-module structure induced by the inclusion map.

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1. Characterizations of the Dominion

The following definition, adapted from [9], will be useful. A *zigzag* for $s \in S$ over (R, S) is a representation

$$s = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j$$

where x_i, $y_j \in S$, $a_{ij} \in R$, subject to the conditions

$$\sum_{i=1}^{m} x_i a_{ij} \in R \quad \text{for} \quad 1 \leqslant j \leqslant n$$

$$\sum_{j=1}^{n} a_{ij} y_j \in R \quad \text{for} \quad 1 \leqslant i \leqslant m.$$

Sometimes it is convenient to write the zigzag in matrix form:

$$s = XAY$$

where $X = (x_i)$ is a row vector over S, $Y = (y_j)$ a column vector over S and $A = (a_{ij})$ a $m \times n$ matrix over R.

PROPOSITION 1.1. *Let R be a subring of S and let $d \in S$. Then the following statements are equivalent:*

(a) $d \in \mathrm{Dom}(R, S)$,

(b) *if M is any S-S-bimodule and if $m \in M$ has the property that $rm = mr$ for all $r \in R$, then $dm = md$,*

(c) $d \otimes 1 = 1 \otimes d$ *in* $S \otimes_R S$,

(d) *there exists a zigzag for d over (R, S).*

Proof. The equivalence of (a), (b) and (c) was essentially proved by Silver [16, 1.1] (see also [17, 13.5 and 13.6]), and (d) is due to Mazet [15, exposé 2], at least in the commutative case. The proof uses lemma 10 of [3, chap. I, §2, No. 11] and works equally well in the non-commutative case.

Another set of equivalent conditions is as follows:

PROPOSITION 1.2. *Let R, S and d be as above. Then the following statements are equivalent:*

(a) $d \otimes 1 = 1 \otimes d$ *in* $S \otimes_R S$,

(b) *if M is a right and L a left S-module, then $md \otimes x = m \otimes dx$ in $M \otimes_R L$ for all $m \in M$, $x \in L$,*

(c) *if $h: M \to N$ is an R-homomorphism of arbitrary right S-modules, then $h(md) = h(m)d$ for all $m \in M$.*

Proof. (a)⇒(b). Apply the homomorphism $p: M\otimes_S S\otimes_R S\otimes_S L\to M\otimes_R L$ sending $m\otimes s\otimes s'\otimes x$ to $ms\otimes s'x$ to the element $m\otimes d\otimes 1\otimes x = m\otimes 1\otimes d\otimes x$.

(b)⇒(c). Let q be the canonical homomorphism $N\otimes_R S\to N$ and apply $q(h\otimes 1_S)$ to the element $md\otimes 1 = m\otimes d\in M\otimes_R S$.

(c)⇒(a). The map $h: S\to S\otimes_R S$ given by $h(s)=s\otimes 1$ is a homomorphism of right R-modules. Thus $h(1)d = 1\otimes d = h(d) = d\otimes 1$.

It follows immediately, that a ring homomorphism $R\to S$ is an epimorphism if and only if the canonical map $S\otimes_R S\to S$ is an isomorphism. Epimorphisms and dominions in the category of commutative rings are also described by (1.1) and (1.2). See e.g. [17, p. 76].

PROPOSITION 1.3. (a) *Let* $R\subseteq S$ *and* $d\in \mathrm{Dom}(R, S)$. *If* $s\in S$ *has the property, that* $rs=sr$ *for all* $r\in R$, *then* $ds=sd$.

(b) *If R is commutative, then so is* $\mathrm{Dom}(R, S)$. *In particular, all epimorphic images of a commutative ring are commutative.*

Proof. (a) Apply (1.1, b) to $M=S$. (b) follows by using (a) twice: R commutes elementwise with $\mathrm{Dom}(R, S)$, thus the latter commutes elementwise with itself. Compare [16,1.2].

Isbell [10, p. 268] has given an example of a finite ring having an infinite epimorphic extension. This cannot happen in the commutative case [18, 5.9]; however, a finite commutative ring may have an infinite dominion: Let K be a finite field and let S be the K-algebra with basis elements

$$1 = t^0, t^1, t^2, \dots u_0, u_1, u_2, \dots$$

and multiplication given by

$$t^i t^j = t^{i+j},\ u_i u_j = 0,\ u_i t^j = t^j u_i = u_{i+j} \quad \text{for all} \quad i, j.$$

The subalgebra R spanned by 1, u_0 and u_1 is finite. Now $u_2 = tu_0t$ is a zigzag over (R, S), whence $u_2\in \mathrm{Dom}(R, S)$. Since $u_3 = tu_1t$ is a zigzag over $(\mathrm{Dom}(R, S), S)$ it follows, that $u_3\in\mathrm{Dom}(R, S)$. Continuing in this fashion, we see, that all the u_i are in $\mathrm{Dom}(R, S)$.

Finally, we recall two definitions [9, 18]. A ring R is *dominant* (or absolutely closed) if $\mathrm{Dom}(R, S) = R$ for all rings S containing R, and a ring is *saturated* if it has no proper epimorphic extensions. A dominant ring is saturated, the converse does not hold in general. Among the dominant rings are the pure rings [5, 10, 18], a class of rings, which includes the von Neumann regular and the self-injective rings. Commutative Artinian rings are saturated [18, 5.9]. Some necessary conditions are given in [18, 19].

2. Invariance Properties of the Dominion

In this section, we study the behavior of the dominion under various constructions.

PROPOSITION 2.1. *Let R be the product of two rings R_1, R_2 with unit elements e_1, e_2 and let $R \subseteq S$. Then* $\mathrm{Dom}(R, S) = D_1 \times D_2$, *where* $D_i = \mathrm{Dom}(R_i, e_iSe_i)$ *for* $i = 1, 2$.

Proof. Let $e_1d_1e_1 + e_2d_2e_2 \in D_1 \times D_2$ and assume, that $f, g: S \to T$ are ring homomorphisms coinciding on R. These maps may be restricted to e_iSe_i $(i=1, 2)$ to yield unitary ring homomorphisms $e_iSe_i \to f(e_i)\, Tf(e_i)$ coinciding on R_i, whence $f(e_id_ie_i) = g(e_id_ie_i)$. This shows, that $D_1 \times D_2 \subseteq \mathrm{Dom}(R, S)$. On the other hand, if $d \in \mathrm{Dom}(R, S)$, then $d \otimes 1 = 1 \otimes d$ in $S \otimes_R S$. Since by (1.3) the e_i commute with the elements of $\mathrm{Dom}(R, S)$, we have $e_id = de_i = e_ide_i$ $(i=1, 2)$, thus $d = e_1de_1 + e_2de_2$. Now $S \otimes_R S \cong Se_1 \otimes_{R_1} e_1S \oplus Se_2 \otimes_{R_2} e_2S$ by an isomorphism sending $s \otimes t$ to $(se_1 \otimes e_1t, se_2 \otimes e_2t)$, and we have relations $de_i \otimes e_i = e_i \otimes e_id$. By the remark above $e_ide_i \otimes e_i = e_i \otimes e_ide_i$, but this relation holds already in $e_iSe_i \otimes_{R_i} e_iSe_i$, since e_iSe_i is a direct summand of the R_i-modules Se_i and e_iS. Thus $e_ide_i \in D_i$ $(i=1, 2)$.

COROLLARY 2.2. *In the situation of* (2.1) *if $R \subseteq S$ is epimorphic then $S = e_1Se_1 \times \times e_2Se_2$ and $R_i \subseteq e_iSe_i$ is epimorphic for $i=1, 2$. If $R_i \subseteq S_i$ is epimorphic for $i=1, 2$, then $R_1 \times R_2 \subseteq S_1 \times S_2$ is epimorphic.*

Of course similar results hold for any finite number of factors.

COROLLARY 2.3. *A finite product of rings is dominant [saturated] if and only if each factor is dominant [saturated].*

An infinite product of epimorphisms need not be an epimorphism, however. In order to give an example, we introduce the following definition. Let $R \subseteq S$ and let $d \in \mathrm{Dom}(R, S)$. Then d has a zigzag XAY and by inserting zeros, if necessary, we may assume, that A is a square $m \times m$ matrix. The *rank* of d is defined to be the smallest m occurring in all such zigzags for d. If R is a finite ring and if the ranks of the elements of an epimorphic extension S of R have a finite maximum, then S is necessarily finite. To see this, we note that if $s = XAY$, $s' = X'A'Y'$ are zigzags with $XA = X'A'$, $A = A'$, $AY = A'Y'$, then $s = XAY = X'A'Y = X'AY = X'A'Y' = s'$. Thus s is completely determined by the matrices and vectors XA, A and AY over R, and the assumption on R implies, that there are only finitely many of them. Hence S is finite. (Incidentally, the argument shows, that the dominion of a finite ring is at most countable, and that the dominion of an infinite ring R has the same cardinality as R. Compare [9,1.5])

Thus if R is a finite ring with an infinite epimorphic extension S (see [10, p. 268] for an example), then S must have elements of arbitrarily high rank. Let now R^* (resp. S^*) be a countably infinite product of copies of R (resp. S). If $s_i \in S (1 \leqslant i < \infty)$ is a sequence of elements of strictly increasing rank, then $s = (s_i) \in S^*$ cannot have a zigzag over (R^*, S^*), for if it had one of rank p, say, projection onto the factors would yield that rank $s_i \leqslant p$ for all i. This shows, that $R^* \subseteq S^*$ is not epimorphic.

We now turn to matrix rings. The $n \times n$ matrix ring over R will be denoted by $R_{(n)}$. There is a canonical embedding of R in $R_{(n)}$ sending $r \in R$ to the diagonal matrix $\operatorname{diag}(r, \ldots, r) = \Delta(r)$. If $f: R \to S$ is a ring homomorphism, there is an obvious ring homomorphisms $f_{(n)}: R_{(n)} \to S_{(n)}$. In particular, if $R \subseteq S$, then $R_{(n)} \subseteq S_{(n)}$. Moreover, every ring T_0 between $R_{(n)}$ and $S_{(n)}$ is of the form $T_{(n)}$, where T is the set of all coefficients of the matrices in T_0. It follows, that $\operatorname{Dom}(R_{(n)}, S_{(n)}) = D_{(n)}$ for some ring D, $R \subseteq D \subseteq S$.

PROPOSITION 2.4. *With the notation above,* $D = \operatorname{Dom}(R, S)$.

Proof. If $d \in \operatorname{Dom}(R, S)$ and if $f, g: S_{(n)} \to T$ coincide on $R_{(n)}$, then $f\Delta(d) = g\Delta(d)$, thus $\Delta(d) \in \operatorname{Dom}(R_{(n)}, S_{(n)})$, whence $d \in D$. Conversely, if $d \in D$ and if $f, g: S \to T$ coincide on R, then $f_{(n)}\Delta(d) = g_{(n)}\Delta(d)$. This implies $f(d) = g(d)$ and $d \in \operatorname{Dom}(R, S)$.

COROLLARY 2.5. *R is dominant [saturated] if and only if $R_{(n)}$ is dominant [saturated].*

Proof. The "if" part follows directly from (2.4). To prove the "only if" part, note that whenever $R_{(n)}$ is a subring of a ring S_0, then $S_0 = S_{(n)}$, where S is the subring of S_0 consisting of all elements commuting with the matrix units [11, p. 52]. R can be identified with a subring of S and (2.4) applies.

PROPOSITION 2.6. *Suppose e is an idempotent of R such that $ReR = R$. Let $R \subseteq S$ and let $D = \operatorname{Dom}(R, S)$. Then $eDe = \operatorname{Dom}(eRe, eSe)$.*

Proof. Let $ede \in \operatorname{Dom}(eRe, eSe)$. Any two ring homomorphisms $f, g: S \to T$ coinciding on R may be restricted to (unitary) ring homomorphisms $eSe \to f(e)\,Tf(e)$ coinciding on eRe. Consequently $f(ede) = g(ede)$, whence $ede \in eDe$. The assumption on e has not been used.

To prove the other inclusion, assume that $\sum p_j e q_j = 1$ $(p_j, q_j \in R)$ and consider the homomorphism $h: S \otimes_R S \to eSe \otimes_{eRe} eSe$ of Abelian groups defined by $h(s \otimes t) = \sum esp_j e \otimes eq_j te$. One has to check, that $h(sr \otimes t) = h(s \otimes rt)$:

$$
\begin{aligned}
h(sr \otimes t) &= \sum_j esrp_j e \otimes eq_j te \\
&= \sum_{j,k} esp_k eq_k rp_j e \otimes eq_j te = \sum_{j,k} esp_k e \otimes eq_k rp_j eq_j te \\
&= \sum_k esp_k e \otimes eq_k rte = h(s \otimes rt).
\end{aligned}
$$

If $d \in \operatorname{Dom}(R, S)$, then $d \otimes 1 = 1 \otimes d$. Now $h(d \otimes 1) = ede \otimes e$ and similarly $h(1 \otimes d) = e \otimes ede$. Thus $ede \in \operatorname{Dom}(eRe, eSe)$.

COROLLARY 2.7. *Let R and e be as above and suppose eRe is dominant [saturated]. Then R is dominant [saturated].*

Proof. If $R \subseteq S$ is epimorphic, then $eRe \subseteq eSe$ is epimorphic, and thus $eRe = eSe$. Since $ReR = R$, this yields $R = S$. A similar argument works for the dominion.

An example to be presented later (3.8) will show, that some condition on e is necessary in (2.6) and (2.7). The converse of (2.7) is also true. This will follow from the result below. Two rings R, R' are said to be *Morita equivalent*, if the categories of right (or equivalently of left) R-modules and R'-modules are equivalent. This holds if and only if $R' \cong eR_{(n)}e$ for some n, where e is an idempotent of $R_{(n)}$ such that $R_{(n)}eR_{(n)} = R_{(n)}$ (see [6, p. 47]). A property of rings is said to be *Morita invariant* if it is shared by all rings in a Morita equivalence class.

PROPOSITION 2.8. *The properties of being dominant or saturated are Morita invariant.*

Proof. It is sufficient to show, that if R and R' are Morita equivalent and if R is not saturated (or not dominant), then so is R'. This follows immediately from (2.5), (2.7) and the formula for R' quoted above.

3. Flat Epimorphic Extensions

The ring homomorphism $h: R \to S$ is called a *right flat epimorphism* if h is a ring epimorphism and if S is a flat left R-module for the R-module structure induced by h. Flat epimorphisms have been characterized by Popescu and Spircu [14]; the reader is also referred to the notes by Stenström [17, Thm. 13.10]. Specialized to extensions, their result is as follows:

PROPOSITION 3.1. *If $R \subseteq S$ is a ring extension, then the following statements are equivalent*:

(a) *$R \subseteq S$ is a right flat epimorphism,*

(b) *$(s^{-1}R)S = S$ for all $s \in S$,*

(c) *the set $\mathfrak{F}$ of right ideals J of R such that $JS = S$ is a topology (also called an idempotent topologizing set) and there is an isomorphism g from S to $Q_{\mathfrak{F}}(R)$, the quotient ring of R relative to $\mathfrak{F}$, such that the restriction of g to R is the canonical map $R \to Q_{\mathfrak{F}}(R)$.*

For the theory of quotient rings, the reader may consult [17, §7]. By $s^{-1}R$ we mean the right ideal of all $a \in R$ such that $sa \in R$.

From (3.1, b) one sees, that every element s of S has a zigzag of the special form

$$s = s \sum_j a_j y_j \quad \text{with} \quad a_j, sa_j \in R \quad \text{and} \quad \sum_j a_j y_j = 1. \tag{3.2}$$

Thus one might be tempted to define, for an arbitrary extension $R \subseteq S$, the analog of the dominion as the set of all $s \in S$ having a zigzag of the form (3.2). This set is, however, not in general a ring, unless some additional conditions are imposed, such as commutativity of R or flatness of ${}_RS$. On the other hand, it follows from results of Morita [13, 1.2], that the set of all s such that sr has a zigzag of the form (3.2) for all $r \in R$ is indeed a subring of S.

We now wish to show, that flat epimorphisms behave nicely under the various constructions already studied in section two. We first prove three lemmas on flat extensions in general and we shall use freely and repeatedly the following well-known result [3, chap. I, §2, No. 7, Prop. 8]: If R, S are rings and if ${}_RM_S$, ${}_SN$ are (bi)modules such that ${}_RM$ and ${}_SN$ are flat, then $M \otimes_S N$ is a flat left R-module.

LEMMA 3.3. *Let $R_i \subseteq S_i$ $(1 \leqslant i \leqslant m)$ be rings and let $R = \prod R_i$, $S = \prod S_i$. Then S is a flat left R-module if and only if S_i is a flat left R_i-module for every i.*

Proof. Since direct sums and direct summands of flat modules are flat, it suffices to observe, that S_i is a flat R_i-module if and only if it is a flat R-module.

LEMMA 3.4. *If $R \subseteq S$ are rings, then S is a flat left R-module if and only if $S_{(n)}$ is a flat left $R_{(n)}$-module.*

Proof. If ${}_RS$ is flat, then so is the left $R_{(n)}$-module $S_{(n)} \cong R_{(n)} \otimes_R S$, since $R_{(n)}$ is a free R-module. Conversely, if $S_{(n)}$ is flat as $R_{(n)}$-module, then the left R-module $R^n \otimes_{R_{(n)}} S_{(n)}$ is flat, where R^n denotes the direct sum of n copies of R. Now $h: R^n \otimes_{R_{(n)}} S_{(n)} \to S^n$ given by $h((r_i) \otimes (s_{ij})) = (\sum r_i s_{ij})$ is an isomorphism of left R-modules, thus S^n and consequently S is flat.

LEMMA 3.5 *Let $R \subseteq S$ be rings and let e be an idempotent of R such that $ReR = R$. Then S is a flat left R-module if and only if eSe is a flat left eRe-module.*

Proof. If $ReR = R$, then the right R-module eR is a finitely generated projective generator ([2, II.4.4]) hence eR is finitely generated projective as left module over its endomorphism ring eRe and furthermore $Re \otimes_{eRe} eR \cong R$ (see [2, II.4.2 and II.3.5]). Of course all this can be proved directly. To prove e.g. that eR is projective over eRe, one observes, that if $\sum p_j e q_j = 1$, then $\{eq_j\}$ and $\{f_j\}$ form a pair of dual bases ([2, II.4.5]), where $f_j : eR \to eRe$ is given by $f_j(er) = erp_je$.

Suppose now, that ${}_RS$ is flat, then so is the left eRe-module $eR \otimes_R S \cong eS$, and eSe is a direct summand of eS. Conversely, since $Re \otimes_{eRe} eR \cong R$, there are isomorphisms

$$S \cong Re \otimes_{eRe} eR \otimes_R S \otimes_R Re \otimes_{eRe} eR \cong Re \otimes_{eRe} eSe \otimes_{eRe} eR$$

and if eSe is a flat eRe-module, then S is a flat R-module.

As an immediate consequence of the three preceding lemmas and the results in section two, we obtain the following statements:

PROPOSITION 3.6. (a) *If* $R_i \subseteq S_i$ $(1 \leqslant i \leqslant m)$ *are rings and if* $R = \prod R_i$, $S = \prod S_i$, *then* $R \subseteq S$ *is a right flat epimorphism if and only if* $R_i \subseteq S_i$ *is a right flat epimorphism for every i.*

(b) $R \subseteq S$ is a right flat epimorphism if and only if $R_{(n)} \subseteq S_{(n)}$ *is a right flat epimorphism.*

(c) *If e is an idempotent of R such that* $ReR = R$, *then* $R \subseteq S$ *is a right flat epimorphism if and only if* $eRe \subseteq eSe$ *is a right flat epimorphism.*

PROPOSITION 3.7. (a) *A finite product of rings has no proper right flat epimorphic extensions if and only if each factor has this property.*

(b) *The property of having no proper right flat epimorphic extensions is Morita invariant.*

Finally, we give two examples to show, that some condition on e is necessary for (2.6), (2.7), (3.5) and (3.6, c).

EXAMPLE 3.8. Let S be the full 4×4 matrix ring over a commutative field K and let R be the subring of S consisting of all matrices of the form

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta & \varepsilon & 0 & 0 \\ \gamma & 0 & \varepsilon & 0 \\ \delta & 0 & \zeta & \varepsilon \end{pmatrix}.$$

Then, for any $s \in S$

$$s = s \sum_{j=1}^{4} e_{j1} e_{1j}$$

is a zigzag of the form (3.2), where the e_{ij} are the matrix units [11, p. 52]. Thus $R \subseteq S$ is a right flat epimorphism by (3.1). This would also follow from the fact, that the simple ring S is the complete right quotient ring of R. If $e = e_{22} + e_{33} + e_{44}$, then $eRe \neq eSe$ and the inclusion is not an epimorphism, since eRe is a commutative local Artinian ring and hence saturated [18, 5.9]. Furthermore, eSe is not a flat left eRe-module.

EXAMPLE 3.9 Let S be the ring of all 3×3 matrices of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ \delta & \varepsilon & \alpha \end{pmatrix}$$

and let R be the subring consisting of all matrices in S with $\varepsilon=0$. Then S is not a flat left R-module, but for $e=e_{22}$ $eSe=eRe$.

4. Epimorphic Extensions of Perfect Rings

We recall a few definitions. A ring R is *left perfect* if it satisfies the minimum condition on principal right ideals. The Jacobson radical of R will be denoted by N. R is left perfect if and only if R/N is Artinian and N is left T-nilpotent [1]. A right Artinian ring is both left and right perfect. A left perfect ring is called *primary* (resp. *local*) if R/N is a simple ring (resp. a division ring). R is said to be *primary decomposable* if it is a finite product of primary rings. For a subset X of R, $l(X)$ and $r(X)$ denote the left and right annihilators; for any left perfect ring, $l(N)$ is the right socle.

We now describe the flat epimorphic extensions of a left perfect ring.

PROPOSITION 4.1. *Let R be left perfect and let I be a two-sided ideal of R such that*

(i) $I^2=I$,
(ii) $l(I)=0$,
(iii) *I is a finitely generated projective right R-module.*

Then the monomorphism $R\to\operatorname{Hom}_R(I_R, I_R)=S$ sending $r\in R$ to the homomorphism induced by left multiplication with r is a right flat epimorphism.

Conversely, every right flat epimorphism $R\subseteq S$ is of this form.

Proof. We shall prove the first part in detail. From (i) it follows, that $S=\operatorname{Hom}_R(I_R, R_R)=I^*$, the dual of I.

Since I_R is finitely generated projective, there exist $x_i\in I$, $s_i\in S=I^*$ $(i=1,\cdots,n)$ such that

$$x=\sum_i x_i s_i(x)=\sum_i x_i s_i x$$

for all $x\in I$ (See [2, II.4.5]). Thus $(1-\sum x_i s_i)\,I=0$ and from (ii) we obtain $1=\sum x_i s_i$. Since $sx_i\in I$ for all $s\in S$, we get a zigzag of the form (3.2) for s over (R, S). Thus $R\subseteq S$ is a right flat epimorphism by (3.1). Actually ${}_RS$ is finitely generated projective, being the dual of the finitely generated projective module I_R. Furthermore S is again left perfect.

The converse follows by putting together several known results. From (3.1) we know, that S is (isomorphic over R to) the quotient ring of R relative to the topology $\mathfrak{F}=\{J\mid JS=S\}$. Since R is left perfect, it follows from [7], that the intersection I of all right ideals in $\mathfrak{F}$ is a two-sided idempotent ideal which lies also in $\mathfrak{F}$, whence

$IS=S$, and consequently $l(I)=0$. Furthermore, we have $S=\mathrm{Hom}_R(I_R, I_R)$ and $IS=S$ implies, again by [2, II.4.5], that I is finitely generated projective. See [20] for a special case of this.

If R is a right Artinian ring with right singular ideal zero, then the conditions of the proposition are satisfied for $I=l(N)$. See [17, 20] for more examples along these lines.

Since every flat right epimorphic extension of R is contained in the complete right quotient ring $Q(R)$ [12], a right rationally complete ring (i.e. a ring with $R=Q(R)$) has no proper right flat epimorphic extensions. The converse does not hold; indeed the ring R of [20, 7.2] is not right rationally complete, yet no ideal different from R satisfies (i), (ii) and (iii) of (4.1).

We now return to epimorphic extensions in general.

PROPOSITION 4.2. *Let R be a right perfect ring and suppose, that every simple right R-module is isomorphic to a minimal right ideal. Then R is saturated.*

Proof. Let $R\subseteq S$ be an epimorphic extension. Then $S\otimes_R S\cong S$ or, equivalently, $S/R\otimes_R S=0$. Thus $\mathrm{Hom}_S(S/R\otimes_R S, S)\cong\mathrm{Hom}_R(S/R, \mathrm{Hom}_S(S, S))=0$ where S is considered as R-S-bimodule. Since $\mathrm{Hom}_S(S, S)\cong S$ as right R-modules, we find that $\mathrm{Hom}_R(S/R, S)=0$. If $S/R\neq 0$, then S/R maps onto a simple right R-module [1], which is isomorphic to a submodule of S by assumption. This contradiction shows, that $R=S$.

Rings containing a copy of every simple right module are characterized by the condition, that every proper right ideal has a nonzero left annihilator [17, 18.1]. Note that this property is Morita invariant.

Among the left perfect rings, these rings are just those having the property, that $r(l(N))=N$ [20, 5.1]. We thus have proved the following result:

PROPOSITION 4.3. *A left and right perfect ring is saturated if $r(l(N))=N$ or if $l(r(N))=N$.*

COROLLARY 4.4. *A left and right perfect ring R is saturated if it satisfies one of the following conditions:*

(a) *R is primary decomposable,*

(b) $r(N)\subseteq l(N)$ [*or* $l(N)\subseteq r(N)$],

(c) *R is a quasi-Frobenius ring,*

(d) *N is in the center of R.*

Proof. All these rings satisfy the condition of (4.3) [20, 5.4].

The following result yields another proof of (4.4, a).

PROPOSITION 4.5 *Let R be right perfect and let $R\subseteq S$ be an epimorphic exten-*

sion. Suppose further, that S, as left R-module, maps onto every simple left R-module. Then $R=S$.

Proof. As in the proof of (4.2) we have $S/R\otimes_R S=0$. The assumption on S implies, that $S/R\otimes_R U=0$ for every simple left R-module U. Since R/N is a direct sum of simple left R-modules, it follows that $S/R\otimes_R R/N\cong(S/R)/(S/R)N=0$. But over a right perfect ring a generalized form of Nakayama's Lemma holds [1]: $MN=M$ implies $M=0$ for all right modules M. Thus $S/R=0$.

In order to obtain (4.4, a) from (4.5) we first note, that by (2.3) it suffices to consider primary rings (actually, by (2.5), local rings would do). Now a primary ring has only one isomorphism class of simple modules and S, as left module over a left perfect ring has to map onto some simple module [1], hence onto all simple modules.

Remark 4.6. For a different proof of (4.4, d) see Isbell [10]. In the same paper, Isbell proves that if a finite dimensional algebra R over a field K has proper epimorphic extension, then it has two orthogonal primitive idempotents contained in a copy of the 2×2 triangular matrices over K, but not contained in the full 2×2 matrix ring over K. This follows directly from (4.4, a). Indeed, if R is not saturated, then R is not primary decomposable. This is equivalent to the condition, that there are orthogonal primitive idempotents e, f such that $eRf\neq 0$ and eR is not isomorphic to fR. This in turn translates readily into Isbell's condition.

We now turn to dominant rings. It has already been noted, that self-injective rings, in particular quasi-Frobenius rings, are dominant. Thus a stronger result than (4.4, c) holds. We are now going to show, that there are left and right Artinian rings which are dominant, but not quasi-Frobenius.

If M is a right R-module, then it is a left R-module over its endomorphism ring $E=\mathrm{Hom}_R(M, M)$, and there is a canonical ring homomorphism $f: R\to \mathrm{Hom}_E(M, M)$. M is said to be *balanced* if f is surjective. The reader is referred to [8] for recent results on balanced modules and balanced rings (rings for which every module is balanced).

PROPOSITION 4.7. *If every faithful right R-module is balanced, then R is dominant.*

Proof. Let $R\subseteq S$ and let $d\in\mathrm{Dom}\,(R, S)$. Applying (1.2, c) to $M=S_R$ we see, that $h(md)=h(m)d$ for every $h\in\mathrm{Hom}_R(S, S)=E$. Thus $d\in\mathrm{Hom}_E(S, S)$, whence $d\in R$.

We can now give the promised example. The exceptional ring R constructed in [8, Lemma III.7.1] is balanced (thus left and right Artinian), but not self-injective. Indeed, one readily checks, that not every right R-module homomorphism from the radical of R to R is obtained by left multiplication with some element of R.

Other examples can be obtained as follows. A finite dimensional algebra R is called a QF-1 algebra if every finitely generated faithful R-module is balanced [21]. If $N^2=0$, then, according to a private communication from C. M. Ringel, it follows, that every faithful R-module is balanced. An example of a QF-1 algebra with $N^2=0$

which is not quasi-Frobenius is given in [21]. This example also shows, that the class of dominant algebras properly contains the class of pure algebras, for a (finite dimensional) pure algebra is quasi-Frobenius. The last statement may be proved using [5, Theorem 5.5].

As another application of (4.7) we have the following result, which has independently been proved by C. M. Ringel.

COROLLARY 4.8. *A commutative Noetherian ring is quasi-Frobenius if and only if every faithful module is balanced.*

Proof. It is well-known, that quasi-Frobenius rings have this property. Indeed, every faithful module is a generator and generators are balanced [8, 1.2.3]. Conversely, it was shown in [19], that a commutative Noetherian ring which is dominant is a quasi-Frobenius ring.

5. Epimorphic Extensions of Principal ideal Domains

In [4, 2.2] Bousfield and Kan proved, that for any commutative ring S the homomorphism $\mathbf{Z} \to \mathrm{Dom}\,(f(\mathbf{Z}), S)$ is a ring epmiorphism, where $f(\mathbf{Z})$ is the image of $\mathbf{Z}$ in S. Their proof does not seem to be applicable to the case where S is non-commutative. In this section, we shall present the following generalization of this result.

PROPOSITION 5.1. *Let R be a commutative principal ideal domain and let $f: R \to S$ be a ring homomorphism mapping R into the center of S. Then f induces a ring epimorphism $R \to \mathrm{Dom}(f(R), S)$.*

Proof. We first settle the case where f is not injective. Then $f(R)$ is a proper homomorphic image of R, hence self-injective (actually a quasi-Frobenius ring). This is readily proved by using Baer's criterion for injectivity. It then follows from [18, 5.4] that $f(R)$ is dominant, whence $f(R) = \mathrm{Dom}(f(R), S)$.

Thus we may assume, that R is a subring of the center of S. Let $d \in \mathrm{Dom}(R, S) = D$ with zigzag $d = XAY$. Now A is a $m \times n$ matrix over a principal ideal domain, thus there exist invertible $m \times m$ resp. $n \times n$ matrices P and Q over R such that PAQ is a diagonal matrix. Then

$$d = (XP^{-1})(PAQ)(Q^{-1}Y)$$

is also a zigzag for d. In terms of components, this new zigzag reads

$$d = \sum_i x_i a_i y_i$$

for suitable $x_i, y_i \in S$, $a_i \in R$ and each $d_i = x_i a_i y_i$ is a zigzag over (R, S). We have to show, that d_i has a zigzag over (R, D) and for this it is sufficient to find elements s_i, t_i

of D such that $x_i a_i = s_i a_i$ and $a_i y_i = a_i t_i$, for then $d_i = s_i a_i t_i$ will be the required zigzag. The existence of s_i (and similarly of t_i) now follows from the next lemma.

LEMMA 5.2. *Suppose $xa = b$ with $x \in S$, $a, b \in R$. Then there exists $s \in D$ such that $xa = sa$.*

Proof. Let c be the greatest common divisor of a and b (determined up to a unit), then $c = aa' + bb'$ with $a', b' \in R$. Now xax is a zigzag for $x^2 a = xb$ over (R, S) (recall that R is central in S), thus $xb \in D$, and it follows, that $xc = xaa' + xbb' \in D$. Furthermore $x^2 c = xcx$ is a zigzag over (D, S) and this implies, that $x^2 c \in D$. By induction $x^n c = x^{n-1} cx$ is again a zigzag over (D, S), whence $x^n c \in D$ for all $n \geqslant 1$.

We claim, that there exist elements $q, r \in R$ and a natural number m such that $s = x^m cr + q \in D$ satisfies the desired relation $xa = sa$. To this end, let $a = a_1 c$, $b = b_1 c$. Here a_1 and b_1 are relatively prime. Let now c_1 be the product of all the prime factors of c, that divide a_1, and c_2 the product of all the others. Then a_1 and c_2 are relatively prime, and c_1 divides a suitable power of a_1, i.e. there exists $p \in R$ and a natural number m such that $c_1 p = a_1^{m-1}$ holds, as well as

$$cp = a_1^{m-1} c_2 . \tag{5.3}$$

Since a_1 and $b_1^{m-1} c_2$ are relatively prime, we have

$$1 = a_1 a'' + b_1^{m-1} c_2 c'' , \quad \text{with} \quad a'', c'' \in R . \tag{5.4}$$

If we put $u = c_2 c''$, $r = pc''$ and $q = b_1 a''$, then from (5.3) and (5.4) we obtain

$$cr = a_1^{m-1} u , \tag{5.5}$$

$$b_1 = q a_1 + b_1^m u . \tag{5.6}$$

From $xa_1 c = b_1 c$, we compute $x^2 a_1^2 c = xa_1 b_1 c = xab_1 = bb_1 = b_1^2 c$, and continuing in this fashion, we get

$$x^m a_1^m c = b_1^m c . \tag{5.7}$$

Then $sa = (x^m cr + q)\, a = x^m a_1^{m-1} uca_1 + qa_1 c = b_1^m cu + qa_1 c = b_1 c = b = xa$. The first equality holds by (5.5), the second by (5.7) and the third by (5.6). This completes the proof of the lemma and hence of proposition (5.1).

We note, that the conclusion of (5.1) remains valid, if R is an epimorphic extension of a principal ideal domain R_0. Indeed, in this case $\mathrm{Dom}(R, S) = \mathrm{Dom}(R_0, S)$ and since $R_0 \subseteq \mathrm{Dom}(R_0, S)$ is epimorphic by (5.1), so is $R \subseteq \mathrm{Dom}(R, S)$.

Another consequence of (5.1) is, that if the principal ideal domain R is central in S and if $d \in S$ is dominated by R in S, then d is already dominated in a commutative subring S_0 of S, viz. $\mathrm{Dom}(R, S)$. We do not know, whether such a statement holds for commutative rings in general, with or without the assumption of centrality.

Finally, we answer the following question: Which rings are *absolutely central* in the sense, that they are in the center of every ring containing them?

PROPOSITION 5.8. *The ring S is absolutely central if and only if there is a ring epimorphism* $g: \mathbf{Z} \to S$.

Proof. If g is not injective, then by the argument used in the proof of (5.1), $S \cong \mathbf{Z}/(m)$ for some non-zero integer m, and this ring is clearly absolutely central. If S is an epimorphic extension of $\mathbf{Z}$, and if T contains S, then $\mathbf{Z} \subseteq S \subseteq \mathrm{Dom}(\mathbf{Z}, T)$. Since $\mathbf{Z}$ is central in T, so is S by (1.3), thus S is absolutely central.

To prove the converse, consider an extension $R \subseteq S$ of commutative rings, which is not epimorphic. Let $d \notin \mathrm{Dom}(R, S)$, $M = S \otimes_R S$ and $z = 1 \otimes 1$. Let T be the semi-direct product of S and M, i.e. the direct sum $S \oplus M$ with product defined by $(s, m)(s', m') = (ss', sm' + ms')$. S can be considered as a subring of T in the obvious way, but as such it is not central in T since $(d, 0)(0, z) \neq (0, z)(d, 0)$. Thus if S is to be absolutely central, it must be an epimorphic extension of every subring, or, equivalently, of its smallest subring. This is either $\mathbf{Z}/(m)$, $m \neq 0$ (in this case $S = \mathbf{Z}/(m)$) or $\mathbf{Z}$. In either case, there is an epimorphism g as required.

In [4], Bousfield and Kan have given a description of all the epimorphic images of $\mathbf{Z}$ as certain direct limits.

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