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Geodesics Satisfying General Boundary Conditions

by KARSTEN GROVE¹⁾

The existence of infinitely many geodesics joining orthogonal two submanifolds V_1 and V_2 of a Riemannian manifold M has been studied in Morse [4] and in Serre [5] under the assumptions V_1 compact and $V_1 \cap V_2 = \emptyset$. Existence of such geodesics is in that case clear. If however $V_1 \cap V_2 \neq \emptyset$ this is not in general the case (see §2). It is the purpose of this paper to examine that situation.

In §1 we shall study geodesics satisfying a general boundary condition. Special cases of this boundary condition are satisfied by $V_1 - V_2$ -connecting geodesics, closed geodesics and isometry-invariant geodesics (see Grove [2], [3]). In §2 we concentrate on $V_1 - V_2$ -connecting geodesics. A typical result in that section is that if V_1 and V_2 are compact and M is contractible then there exists $V_1 - V_2$ -connecting geodesics.

The main tools in our approach are critical point theory on infinite dimensional manifolds and elementary homotopy theory.

§1. N -Geodesics

Throughout this paper M will be a connected complete Riemannian manifold and $N \subset M \times M$ a closed submanifold of $M \times M$. We shall say that a geodesic $\gamma: [0,1] \rightarrow M$ is a N -geodesic if it satisfies the boundary condition

$$(\gamma(0), \gamma(1)) \in N \quad \text{and} \quad (\dot{\gamma}(0), -\dot{\gamma}(1)) \text{ is normal to } N, \quad (1.1)$$

where $\dot{\gamma}(t)$ denotes the velocity vector of γ at t and $M \times M$ is endowed with the product metric.

EXAMPLES. (1) $\Delta(M)$ -geodesics are closed geodesics on M .

(2) $\text{graph}(A)$ -geodesics, $A: M \rightarrow M$ an isometry, are A -invariant geodesics (see [2] and [3]).

(3) $V_1 \times V_2$ -geodesics are $V_1 - V_2$ -connecting geodesics.

Let $L_1^2(I, M)$ denote the complete Riemannian Hilbertmanifold consisting of absolutely continuous curves $\sigma: I = [0, 1] \rightarrow M$ with $\dot{\sigma}$ square integrable (see Flaschel and Klingenberg [1]). From the propositions I.1.4 and I.1.5 of [2] it follows easily that

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N-geodesics are in one to one correspondence with critical points for the energy integral $E: \Lambda_N(M) \rightarrow \mathbf{R}$, $\sigma \mapsto \frac{1}{2} \int_0^1 \|\dot{\sigma}\|^2$, where $\Lambda_N(M) = \{\sigma \in L^2_*(I, M) \mid (\sigma(0), \sigma(1)) \in N\}$ is a submanifold of $L^2_*(I, M)$. Furthermore $E: \Lambda_N(M) \rightarrow \mathbf{R}$ satisfies condition (C) of Palais and Smale if projection on the first, $P_1(N) \subset M$ or the second factor $P_2(N) \subset M$ is compact (Theorem I.2.4 of [2]). We shall therefore assume that e.g. $P_1(N) \subset M$ is compact. To make successful use of critical point theory for Hilbert-manifolds we assume furthermore that $N \cap \Delta$ is a union of closed submanifolds, – here Δ denotes the diagonal $\Delta(M)$ of M in $M \times M$.

With N as above we have,

LEMMA 1.2. *If there are no non-trivial N-geodesics on M, then the inclusion*

$$e: N \cap \Delta \rightarrow \Lambda_N(M), \quad e(x, x)(t) = x \quad \forall t \in I$$

is a homotopy equivalence.

Proof (Sketch). First we observe that each component of $N \cap \Delta$ is a compact non-degenerate critical submanifold of $\Lambda_N(M)$ of index 0. To see this we just note that the Hessian of $E: \Lambda_N(M) \rightarrow \mathbf{R}$ at a constant curve $\bar{p}: I \rightarrow M$, $\bar{p}(t) = p$ for all $t \in I$ is given by

$$H(E)_{\bar{p}}(X, Y) = \int_0^1 \langle X'(t), Y'(t) \rangle_p dt$$

for all $X, Y \in T_{\bar{p}}\Lambda_N(M)$. From this and $(X(0), X(1)) \in TN$ for $X \in T\Lambda_N(M)$ it easily follows that each component of $N \cap \Delta$ is a non-degenerate critical submanifold of index 0. We can now argue exactly as in the proofs of Corollary II.3 and Lemma II.4 of [2], i.e. by the generalized Morse Lemma and condition (C) prove that there exists an $\varepsilon > 0$ such that $N \cap \Delta$ is a strong deformation retract of $\Lambda_N(M)^\varepsilon := \{\sigma \in \Lambda_N(M) \mid E(\sigma) < \varepsilon\}$. Assuming that there are no critical values > 0 (no non-trivial *N*-geodesics) we obtain from this, completeness of $\Lambda_N(M)$ and condition (C) that the inclusion $e: N \cap \Delta \rightarrow \Lambda_N(M)$ is a weak homotopy equivalence and hence a homotopy equivalence.

We are now ready to prove the main result of this section.

THEOREM 1.3. *If there are no non-trivial N-geodesics on M, then there is an exact sequence of homotopy groups*

$$\dots - \pi_{*+1}(N) \xrightarrow{(P_1)^{*+1} - (P_2)^{*+1}} \pi_{*+1}(M) \rightarrow \pi_*(N \cap \Delta) \xrightarrow{i_*} \pi_*(N), \quad * \geq 0.$$

Exactness at $\pi_{*-1}(N \cap \Delta)$ implies that

$$\forall [k] \in \pi_*(M) \exists [g] \in \pi_*(N) \exists [h] \in \pi_*(N, N \cap \Delta) \text{ s.t.}$$

$$[k] = ((P_1)_*([h]) - (P_2)_*([h])) - ((P_1)_*([g]) - (P_2)_*([g])) \quad * \geq 1.$$

(for $*=1$ read multiplicative).

Proof. The inclusion $\Lambda_N(M) \rightarrow C_N^0(M) := \{f \in C^0(I, M) \mid (f(0), f(1)) \in N\}$ is a homotopy equivalence (Theorem I.1.3 of [2]) so instead of $\Lambda_N(M)$ we consider $C_N^0(M)$. Consider now the commutative diagram,

$$\begin{array}{ccccccc} & & \rightarrow \pi_{*+1}(C_N^0 M, \Omega M) & & & & \\ & & \downarrow P_{*+1} \cong & \searrow \delta & & & \\ & \rightarrow \pi_{*+1}(N) & \xrightarrow{\partial} & \pi_*(\Omega M) & \xrightarrow{j_*} & \pi_*(C_N^0 M) & \xrightarrow{P_*} \pi_*(N) \\ & & \searrow H \circ \partial & \downarrow H \cong & & \uparrow e_* \cong & \nearrow i_* \\ & & & \pi_{*+1}(M) & \rightarrow & \pi_*(N \cap \Delta) & \end{array}$$

where the mid-sequence is the exact sequence for the fibration $P: C_N^0(M) \rightarrow N$, $P(f) = (f(0), f(1))$ with fiber the loop space ΩM of M (Serre [5]), where δ is the boundary map in the exact sequence for the pair $(C_N^0(M), \Omega M)$ and H is the Hurewicz map. We shall compute the maps in cupic homotopi.

Let $\alpha: I^q \times I \rightarrow C_N^0(M)$ represent an element $[\alpha]$ in $\pi_{*+1}(C_N^0 M, \Omega M)$. Evaluation of $\alpha_i: I^q \times \{0\} \rightarrow \Omega M$, $\hat{\alpha}_i$ represents $H \circ \partial[\alpha] = H \circ \partial(P_*[\alpha])$. From this we see that evaluation of $\alpha: I^{q+1} \rightarrow C_N^0 M$, is a homotopy between $-P_1 \circ P \circ \alpha + \hat{\alpha}_i + P_2 \circ P \circ \alpha$ and the constant map, thus $H \circ \partial([\beta]) = (P_1)_*([\beta]) - (P_2)_*([\beta])$ for all $[\beta] = P_*[\alpha] \in \pi_{*+1}(N)$. Since we assume that there are no non-trivial N -geodesics $e_*: \pi_*(N \cap \Delta) \rightarrow \pi_*(C_N^0 M)$ is an isomorphism by Lemma 1.2 and the lower sequence is the desired sequence.

We will now examine in detail what exactness at $\pi_*(N \cap \Delta)$ i.e. $e_*(\ker i_*) = \text{im } j_*$ means.

Let $[f] \in \pi_q(\Omega M)$ be represented by $f: I^q \rightarrow \Omega M$. Since $j_*([f]) \in \text{im } e_*$ there is a homotopy

$$G_1: I^q \times I \rightarrow C_N^0(M)$$

with $G_1(\cdot, 0) = j \circ f$ and $G_1(\cdot, 1): I^q \rightarrow e(N \cap \Delta)$. Identifying $e(N \cap \Delta)$ with $N \cap \Delta$ we get from $[G_1(\cdot, 1)] \in e_*(\ker i_*)$ a homotopy

$$G_2: I^q \times I \rightarrow N$$

with $G_2(\cdot, 0) = G_1(\cdot, 1)$ and $G_2(\cdot, 1) = \text{base point}$. The homotopies $P \circ G_1: I^q \times I \rightarrow N$ and $G_2: I^q \times I \rightarrow N$ combines to an element $[g] \in \pi_{q+1}(N)$ and G_2 itself represents an element $[h] \in \pi_{q+1}(N, N \cap \Delta)$. Evaluation of $G_1, \hat{G}_1: I^q \times I \times I \rightarrow M$ give rise to a homotopy between $P_1 \circ g + H(f) - P_2 \circ g: I^{q+1} \rightarrow M$ and $P_1 \circ h - P_2 \circ h: I^{q+1} \rightarrow M$, note that $\hat{G}_1(\cdot, 0, \cdot) = H(f)$ i.e. $H[f] = (P_{1*+1}[h] - P_{2*+1}[h]) - (P_{1*+1}[g] - P_{2*+1}[g])$. Q.E.D.

Remark. From theorem 1.3 we can derive the following theorems. *If M is compact there exist closed geodesics on M . If V is a compact submanifold of M and if there are no $V - V$ -connecting geodesics then the inclusion $N \rightarrow M$ is a homotopy equivalence.* We can also obtain some partial results from [2] and [3] by this theorem.

§2. $V_1 - V_2$ -Connecting Geodesics.²⁾

In this paragraph $N = V_1 \times V_2$, where V_1 and V_2 are closed connected submanifold of M , V_1 is compact and $V_1 \cap V_2$ is a union of closed submanifolds of M (may be of different dimensions). As mentioned in §1 is a $V_1 \times V_2$ -geodesic $\gamma: I \rightarrow M$ with $\gamma(0) \in V_1, \gamma(1) \in V_2, \dot{\gamma}(0) \perp V_1$ and $\dot{\gamma}(1) \perp V_2$.

We shall derive all our conclusions from the exact sequence of Theorem 1.3, which in the case $N = V_1 \times V_2$ can be written as,

$$\rightarrow \pi_{*+1}(V_1 \times V_2) \xrightarrow{(i_1)_{*+1} - (i_2)_{*+1}} \pi_{*+1}(M) \rightarrow \pi_*(V_1 \cap V_2) \rightarrow \pi_*(V_1 \times V_2) \quad (2.1)$$

We get immediately

COROLLARY 2.2. *Suppose that $\dim(V_1 \cap V_2) = 0$ and that $\pi_1(M) = 0$ or $V_1 \cap V_2 = \{pt\}$. If there are no non-trivial $V_1 - V_2$ -connecting geodesics on M then all the homotopy groups of M are isomorphic to those of $V_1 \times V_2$, - in fact $(i_1)_* - (i_2)_* : \pi_*(V_1 \times V_2) \rightarrow \pi_*(M)$ is an isomorphism.*

The following example illustrates this corollary.

EXAMPLE. Let V and W be Riemannian manifolds and let $M = V \times W$ be endowed with the product metric. For a fixed $(v, w) \in M$ put $V_1 = V \times \{w\}$ and $V_2 = \{v\} \times W$, then there are no non-trivial $V_1 - V_2$ -connecting geodesics on M . Other immediate consequences of (2.1) are

COROLLARY 2.3. *If $V_1 \cap V_2$ is not connected and M is 1-connected then there exists non-trivial $V_1 - V_2$ -connecting geodesics.*

²⁾ Note that N -geodesics are in 1-1-correspondence with $N - \Delta(M)$ -connecting geodesics in $M \times M$ with product metric (see e.g. L. N. Pattersen, On the index theorem, Amer. J. Math. 85 (1963), 271-297).

COROLLARY 2.4. *Suppose that M is a $K(\pi, 1)$ (e.g. M has negative curvature) and that there are no non-trivial $V_1 - V_2$ -connecting geodesics on M . Then we have:*

(1) *If $\pi_1(V_1) = \pi_1(V_2) = 0$ then $\pi_1(V_1 \cap V_2) = 0$ and $\pi_1(M)$ is finite (impossible if M has neg. curvature).*

(2) *If $V_1 \cap V_2$ consists only of isolated points then $V_1 \times V_2$ is a $K(\pi', 1)$.*

COROLLARY 2.5. *Suppose that M is contractible. Then we have:*

(1) *If V_1 and V_2 are compact there exists non-trivial $V_1 - V_2$ -connecting geodesics. ($V_1 = V_2 = \{pt\}$ not included).*

(2) *If V_2 is contractible and there are no non-trivial $V_1 - V_2$ -connecting geodesics on M then $V_1 \subset V_2$.*

Proof. (1) From (2.1) we get that

$$i: V_1 \cap V_2 \rightarrow V_1 \times V_2, \quad i(p) = (p, p) \quad p \in V_1 \cap V_2$$

is a homotopy equivalence if there are no non-trivial $V_1 - V_2$ -connecting geodesics. Since now $V_1 \times V_2$ is compact we get especially that $\dim(V_1 \cap V_2) = \dim(V_1 \times V_2)$ which is impossible except for the case $V_1 = V_2 = \text{point}$.

(2) In the case V_2 contractible we obtain that

$$V_1 \cap V_2 \rightarrow V_1$$

is a homotopy equivalence if there are no non-trivial $V_1 - V_2$ -connecting geodesics on M . By compactness $\dim(V_1 \cap V_2) = \dim V_1$ which then implies that $V_1 \cap V_2 = V_1$ or equivalently $V_1 \subset V_2$.

Remark. $M = \mathbf{R}^3 \supset \mathbf{R}^2 = V_2 \supset V_1 = S^1$ gives an example of (2) in corollary 2.5. – It is clearly difficult to get more general results from (2.1). In concrete situations however where one knows more about homotopy groups of M , V_1 , and V_2 (2.1) is useful in deciding whether there exists $V_1 - V_2$ -connecting geodesics on M .

Let us finish with some remarks on the case where $M = \Sigma^n$ is a homotopy sphere.

COROLLARY 2.6. *Let $M = \Sigma^n$ be a homotopy sphere. Then*

(1) *If $V_1 \cap V_2$ consists only of isolated points there exists non-trivial $V_1 - V_2$ -connecting geodesics on M .*

(2) *If there is a $q < n - 1$ so that $\pi_q(V_1) \neq 0$ and $\pi_q(V_2) \neq 0$ then there exists non-trivial $V_1 - V_2$ -connecting geodesics on M (more general $q < l - 1$ if M is $(l - 1)$ -connected).*

Proof. Since $\max(\dim V_1, \dim V_2) < n$ we have that $\pi_q(V_1 \times V_2) \neq 0$ for some $q < n$. This together with corollary 2.2 proves (1). To prove (2) we see from (2.1) that $i_*: \pi_*(V_1 \cap V_2) \rightarrow \pi_*(V_1 \times V_2)$ is an isomorphism for $* < n - 1$ if there are no non-

trivial $V_1 - V_2$ -connecting geodesics. If there is a $q < n - 1$ with $\pi_q(V_1) \neq 0$ and $\pi_q(V_2) \neq 0$ this is clearly impossible.

To illustrate that no reasonable general existence results, besides those already mentioned, can be expected, let us give one more example on non-existence of $V_1 - V_2$ -connecting geodesics.

EXAMPLE. Let $M = S^3 \subset \mathbf{R}^4$ with standard metric of constant curvature 1 and similar $V_2 = S^2$ the equator of S^3 . Then for any $V_1 = S^1$ embedded in S^3 such that $V_1 \cap (-V_1) = \emptyset$ there are no non-trivial $V_1 - V_2$ -connecting geodesics. On the other hand we note that if $V_1 = S^k$, $V_2 = S^l$ and $V_i \setminus V_1 \cap V_2 \neq \emptyset$, $i = 1, 2$ then there exists non-trivial $V_1 - V_2$ -connecting geodesics on $M = S^n$.

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