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Some Non-Linear Equivariant Sphere Bundles

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1. Introduction

Let $\varphi: G \rightarrow O_m$ be a real m -dimensional representation of a compact Lie group G . Assume that $\pi: T \rightarrow B$ is a smooth G bundle such that the action takes place on the fibres, and each fibre is equivariantly diffeomorphic to S^{m-1} where the action of G on S^{m-1} is given by the representation φ .

Is π smoothly equivalent to the sphere bundle of a G vector bundle with fibre representation φ ?

If yes, π is called linear, otherwise it is called *non-linear*. If π is linear it bounds a smooth equivariant disk bundle with fibre action induced by φ . Topologically, of course, π is always the boundary of a disk bundle with fibre action induced by φ : The mapping cylinder of π serves as the total space of the required equivariant disk bundle.

For G the trivial group, examples of non-linear sphere bundles over spheres were found by S. P. Novikov [15] and P. Antonelli, D. Burghilea, P. J. Kahn [1]. Let G be one of the groups O_n , U_n , Sp_n , and let ϱ_n be the standard representation of G , of real dimension n , $2n$, or $4n$, respectively. It is not difficult to show that any sphere bundle with fibre representation ϱ_n is linear. We consider sphere bundles with fibre representation $\varrho_n \oplus \varrho_n$. We prove that for G the orthogonal group O_n , $n \geq 3$, any G sphere bundle with fibre representation $\varrho_n \oplus \varrho_n$ is linear (Corollary 4.4). On the other hand, for G the unitary or symplectic group of n dimensions, $n \geq 3$, we will construct many non-linear G sphere bundles with fibre representation $\varrho_n \oplus \varrho_n$ and base space a sphere (Theorem 4.5). It is not clear whether or not these sphere bundles are smoothly linear if one forgets the action of G .

The methods used in this work are quite different from those of [15; 1]. The total space of an equivariant sphere bundle with action induced by $\varrho_n \oplus \varrho_n$ ($n \geq 3$) on the fibres, is a G manifold with two orbit types and orbit space a manifold with boundary. The construction of our non-linear bundles relies on the classification of these G manifolds by W. C. Hsiang and W. Y. Hsiang [10] and K. Jänich [11].

Our results have some consequences, naturally, concerning the homotopy type of the topological group of all equivariant self-diffeomorphisms of the unit sphere in the representation space of $\varrho_n \oplus \varrho_n$, $n \geq 3$. In the orthogonal case, this group has the homotopy type of O_2 (Theorem 4.3), whereas in the unitary case it does not have the homotopy type of a finite CW complex (Theorem 4.8).

We finally deal with the problem of classifying equivariantly the total spaces of the non-linear bundles over spheres constructed here. It turns out that in most cases these

total spaces are products of a homotopy sphere and the fibre (Theorem 5.2 and Proposition 5.4).

2. A_n manifolds over $\Sigma^k \times D^{d+1}$

As we simultaneously deal with orthogonal, unitary, and symplectic actions, the following notation will be convenient (cf. [7]). A_n is the orthogonal group O_n , the unitary group U_n , or the symplectic group Sp_n . ϱ_n is the corresponding standard representation of real dimension n , $2n$, or $4n$, respectively. Let $\pi: T \rightarrow B$ be a smooth A_n sphere bundle over B , with fibre action $\varrho_n \oplus \varrho_n$. The fibre is S^{2dn-1} where $d=1, 2$, or 4 depending on the group acting. π factors through the orbit map $T \rightarrow T'$, and we have a commutative diagram:

$$\begin{array}{ccc} T & \longrightarrow & T' = T/A_n \\ & \searrow \pi & \swarrow \\ & B & \end{array}$$

S^{2dn-1} and T are A_n manifolds with orbit types (A_{n-1}) and (A_{n-2}) , the slice representations corresponding to the orbit types are $\varrho_{n-1} \oplus$ trivial and trivial, respectively. The orbit space of S^{2dn-1} is D^{d+1} , hence $T' \rightarrow B$ is a D^{d+1} bundle. To find and distinguish bundles $\pi: T \rightarrow B$, it is therefore important to classify A_n manifolds with orbit space a D^{d+1} bundle over B such that over each fibre of this bundle we have S^{2dn-1} with action $\varrho_n \oplus \varrho_n$. We use [10] and [11] to do this for a special case.

THEOREM 2.1. *Let k be a positive integer; $k > 1$ if $A_n = O_n$ or Sp_n . Let Σ^k be a smooth manifold homeomorphic to S^k . For every $n \geq 3$, there is a 1-1 correspondence between equivariant diffeomorphism classes of smooth A_n manifolds over $\Sigma^k \times D^{d+1}$ satisfying the conditions*

- (i) *for each $p \in \Sigma^k$, the union of the orbits over $p \times D^{d+1}$ is equivariantly diffeomorphic to S^{2dn-1} with action induced by $\varrho_n \oplus \varrho_n$,*
- (ii) *the principal orbit bundle is trivial, and elements of $\text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1})$.*

G_{d+1} is the H -space of degree one mappings of S^d onto itself, and $\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1}$ is induced by inclusion. Later on we will see that a A_n manifold corresponding to a non-zero element of $\text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1})$ is the total space of a non-linear A_n sphere bundle over Σ^k .

Proof of Theorem 2.1. Let T be a A_n manifold over $\Sigma^k \times D^{d+1}$ with the properties stated in the theorem. T is a so-called special A_n manifold [11], also [10], and is classified by an equivalence class of pairs (P, σ) . Our notation follows [11; 12; 7]. P is the compactified principal bundle of the principal orbit bundle of T , i.e. $\Sigma^k \times D^{d+1} \times A_2 \rightarrow \Sigma^k \times D^{d+1}$ by (ii). σ is a reduction of the structure group A_2 of ∂P to the sub-

group $A_1 \times A_1$ (cf. [7, 3.2]), i.e. a cross-section $\sigma: \Sigma^k \times S^d \rightarrow \Sigma^k \times S^d \times (A_2/A_1 \times A_1)$ of the bundle $\partial P/A_1 \times A_1$. As $A_2/A_1 \times A_1$ is diffeomorphic to S^d , σ is given by a map $f: \Sigma^k \times S^d \rightarrow S^d (= A_2/A_1 \times A_1)$. Because of condition (i), $f|_{p \times S^d}$ has degree ± 1 [7, 3.2]. Thus f is a fibre homotopy trivialization of the trivial d -sphere bundle over Σ^k . By taking a suitable identification of $A_2/A_1 \times A_1$ with S^d , f becomes an oriented fibre homotopy trivialization. On the other hand, by Jänich's construction, any such fibre homotopy trivialization gives rise to a A_n manifold as in Theorem 2.1. Now $f: \Sigma^k \times S^d \rightarrow S^d$ is nothing but a map $\Sigma^k \rightarrow G_{d+1}$ which we also denote by f . It is the class represented by f in $\text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1})$ which corresponds to the equivariant diffeomorphism class of T . To prove the 1-1 correspondence we have to analyze Jänich's equivalence relation of pairs (P, σ) in our particular case. Two pairs (P, σ) and (P', σ') are equivalent (i.e. the corresponding A_n manifolds equivariantly diffeomorphic over $\Sigma^k \times D^{d+1}$) if and only if there is a bundle isomorphism of P and P' carrying σ to σ' [11, 3.1]. If P and P' are identified with the trivial bundle $\Sigma^k \times D^{d+1} \times A_2 \rightarrow \Sigma^k \times D^{d+1}$, (P, σ) and (P, σ') are equivalent if and only if there is a bundle automorphism of the above trivial bundle carrying σ to σ' . Such a bundle automorphism is given by

$$H: \Sigma^k \times D^{d+1} \times A_2 \rightarrow \Sigma^k \times D^{d+1} \times A_2$$

$$H(x, y, z) = (x, y, z\eta(x, y))$$

where $\eta: \Sigma^k \times D^{d+1} \rightarrow A_2$. (P is a right principal bundle.) Therefore equivalence of (P, σ) and (P, σ') means the existence of a commutative diagram

$$\begin{array}{ccc}
 \Sigma^k \times S^d \times A_2 & \xrightarrow{H| \dots} & \Sigma^k \times S^d \times A_2 \\
 \downarrow & & \downarrow \\
 \Sigma^k \times S^d \times S^d & \xrightarrow{h} & \Sigma^k \times S^d \times S^d \\
 \swarrow \sigma & & \searrow \sigma' \\
 & \Sigma^k \times S^d &
 \end{array}
 \tag{*}$$

where H is defined by $\eta: \Sigma^k \times D^{d+1} \rightarrow A_2$ as above and h is induced by H via the identification of $A_2/A_1 \times A_1$ with S^d . We shall need two facts: If σ and σ' are homotopic reductions, then (P, σ) and (P, σ') are equivalent [9, p. 23]. $A_2/A_1 \times A_1$ can be identified with S^d in such a way that the action of A_2 on $A_2/A_1 \times A_1$ corresponds to the orthogonal action of A_2 on S^d via a homomorphism $\tau: A_2 \rightarrow O_{d+1}$ with kernel the center of A_2 . This is well-known (e.g. [2]).

Now suppose (P, σ) is equivalent to (P, σ') , the equivalence given by $\eta: \Sigma^k \times D^{d+1} \rightarrow A_2$. Let σ (σ') be given by a map f (f'): $\Sigma^k \rightarrow G_{d+1}$. Change η by a homo-

topology such that it is constant on all disks $p \times D^{d+1}$, $p \in \Sigma^k$. This changes σ' by a homotopy, using the diagram (*), and changes neither the equivalence class of (P, σ') nor the homotopy class of f' . So we may assume that we have a diagram (*) with η a map from Σ^k to Λ_2 . $\sigma' = h\sigma$, h being defined by $\eta: \Sigma^k \rightarrow \Lambda_2$. If $\bar{\eta}: \Sigma^k \rightarrow O_{d+1}$ is the composition of η with the above homomorphism $\tau: \Lambda_2 \rightarrow O_{d+1}$, this means $f' = \bar{\eta} \cdot f$, or $[f'] - [f] = [\bar{\eta}]$. Thus $[f], [f'] \in \pi_k G_{d+1}$ differ by an element in the image of $\pi_k SO_{d+1}$.

Conversely, given $f, f': \Sigma^k \rightarrow G_{d+1}$, defining reductions σ, σ' , assume there is $\bar{\eta}: \Sigma^k \rightarrow SO_{d+1}$ such that $[f'] - [f] = [\bar{\eta}]$ in $\pi_k G_{d+1}$. $\bar{\eta}$ can be lifted to $\eta: \Sigma^k \rightarrow \Lambda_2$. (Here, if $\Lambda_n = O_n$ or Sp_n , $k > 1$ is used; see Remark 1 below.) This shows that the reduction defined by $\bar{\eta} \cdot f$ can be obtained from σ by an automorphism of P (namely the one defined by η). As $\bar{\eta} \cdot f$ and f' are homotopic, (P, σ) and (P, σ') are equivalent, and the proof is complete.

Remark 1. For $k = 1$, the proof shows what has to be modified if $\Lambda_n = O_n$ or Sp_n . In the symplectic case, one gets a 1 – 1 correspondence to the elements of $\pi_1 G_5$. In the orthogonal case, one gets a 1 – 1 correspondence with the elements of $\text{cok}(\pi_1 SO_2 \rightarrow \pi_1 G_2)$ where $\pi_1 SO_2 \rightarrow \pi_1 G_2$ is obtained by composing the double covering $SO_2 \rightarrow SO_2$ with the inclusion $SO_2 \subset G_2$.

Remark 2. The zero element of $\text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1})$ clearly corresponds to the ‘trivial’ Λ_n manifold $\Sigma^k \times S^{2dn-1}$ over $\Sigma^k \times D^{d+1}$.

Remark 3. The inclusion $SO_2 \subset G_2$ is a homotopy equivalence, so $\text{cok}(\pi_k SO_2 \rightarrow \pi_k G_2) = 0$. $\pi_k SO_3 \rightarrow \pi_k G_3$ is a monomorphism [14], so $\text{cok}(\pi_k SO_3 \rightarrow \pi_k G_3) \cong \pi_k(G_3, SO_3)$ which is isomorphic to $\pi_{k+2} S^2$ for $k \geq 3$.

3. The Orbit Space as a Bundle

It was proved in [7, 2.3] that the linear automorphisms of S^{2dn-1} compatible with the representation $\varrho_n \oplus \varrho_n$, form a group isomorphic to Λ_2 ($n \geq 3$). The action of this group on the orbit space $S^{2dn-1}/\Lambda_n \cong D^{d+1}$ is what one would expect:

PROPOSITION 3.1. *The action of the group Λ_2 of equivariant linear automorphisms of S^{2dn-1} ($n \geq 3$) induced on the orbit space $S^{2dn-1}/\Lambda_n \cong D^{d+1}$ is equivalent to the orthogonal action of Λ_2 on D^{d+1} given by a homomorphism $\tau: \Lambda_2 \rightarrow O_{d+1}$ with $\ker \tau = \text{center}(\Lambda_2)$.*

Proof. Let F be the real, complex, or quaternionic field, depending on whether the orthogonal, the unitary, or the symplectic group acts. Recall from [7, 2.4] how Λ_n and Λ_2 act on S^{2dn-1} . Write elements of S^{2dn-1} as n by 2 matrices over F . Then Λ_n acts by left multiplication, and Λ_2 acts by right multiplication. To prove Proposition 3.1 we may confine ourselves to the orbits of Λ_n over $B^{d+1} = \text{int} D^{d+1}$ (i.e. principal orbits). An n by 2 matrix is on a principal orbit if and only if the two columns are linearly

independent. If $\alpha = \sqrt{\frac{1}{2}}$, then the Λ_n orbit of the point

$$q = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

is a fixed point of the action of Λ_2 . We are going to determine the orbit type of non-fixed points of the Λ_2 action on B^{d+1} . We first need nice representatives of the points in the orbit space B^{d+1}/Λ_2 .

Clearly, any point of S^{2dn-1} over B^{d+1} is on a Λ_n orbit of a point of the form

$$\begin{bmatrix} r & t \\ 0 & s \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \tag{**}$$

$r \in \mathbb{R}, r > 0, s \neq 0$. Applying a suitable element of Λ_2 (i.e. without changing the Λ_2 orbit) makes t real non-negative. Then we make s real positive by applying an appropriate element of Λ_n . So far we have shown that any point in the orbit space B^{d+1}/Λ_2 has a representative of the form (**) with $r, s, t \in \mathbb{R}, r > 0, s > 0, t \geq 0$. The following lemma guarantees that we may even assume $t = 0$.

LEMMA 3.2. *Given real numbers $r \neq 0, s \neq 0, t$, there are orthogonal 2 by 2 matrices M, N such that*

$$M \begin{bmatrix} r & t \\ 0 & s \end{bmatrix} N$$

is a diagonal matrix.

(Lemma 3.2 is proved below.)

Assume $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in the isotropy group of a point of B^{d+1} represented by

$$\begin{bmatrix} r & 0 \\ 0 & s \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}. \quad \text{Then} \quad \begin{bmatrix} r & 0 \\ 0 & s \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ sc & sd \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

is on the same A_n orbit as

$$\begin{bmatrix} r & 0 \\ 0 & s \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

Therefore $r^2 = r^2|a|^2 + s^2|c|^2$ and $s^2 = r^2|b|^2 + s^2|d|^2$, so $r^2(1 - |a|^2) = r^2|c|^2 = s^2|c|^2$ and $s^2(1 - |d|^2) = s^2|b|^2 = r^2|b|^2$, i.e. $r = s$ or $c = 0$ and $b = 0$. For $r = s$ we have the fixed point q , otherwise the isotropy group is $A_1 \times A_1$. Thus the positive dimensional orbits of A_2 on B^{d+1} are spheres $A_2/A_1 \times A_1$ of dimension d , the orbit space B^{d+1}/A_2 is the half open interval $(0, \alpha]$, parametrized by r . Hence B^{d+1} is equivalent, as a A_2 space, to the representation space of τ composed with the standard orthogonal representation of O_2 , SO_3 , or SO_5 , respectively.

Proof of Lemma 3.2. Left (right) multiplication by an orthogonal matrix does not change the inner product of the columns (rows) of a real 2 by 2 matrix. As the orthogonal group operates transitively on spheres, it is sufficient to find an orthogonal matrix

$$\begin{bmatrix} u & u' \\ -u' & u \end{bmatrix}, u' = \sqrt{1 - u^2},$$

such that the columns of

$$\begin{bmatrix} r & t \\ 0 & s \end{bmatrix} \begin{bmatrix} u & u' \\ -u' & u \end{bmatrix}$$

have inner product zero. This leads to an equation

$$u^4 - u^2 + \frac{r^2 t^2}{(r^2 - t^2 - s^2)^2 + 4r^2 t^2} = 0$$

which does have a solution u in the unit interval.

COROLLARY 3.3. *If an equivariant linear S^{2d_n-1} bundle $\pi: T \rightarrow B$ is defined by transition functions $t_j: X_j \rightarrow A_2$, then the orbit space T' is the total space of a D^{d+1} bundle over B with transition functions $\tau \circ t_j$.*

4. The Homotopy Type of the Equivariant Diffeomorphism Group of the Fibre

The following two (well-known) lemmas are used in the proof of the next theorem.

LEMMA 4.1. *The group of diffeomorphisms of D^2 is homotopy equivalent to O_2 .*

Proof. The group of diffeomorphisms of S^1 is homotopy equivalent to O_2 . This is

elementary. So the group of all diffeomorphisms of D^2 is homotopy equivalent to the group of all diffeomorphisms of D^2 being orthogonal on the boundary. The latter is homotopy equivalent to the product of O_2 and the group of all diffeomorphisms of D^2 leaving S^1 fixed. But the second factor is contractible [6, p. 132].

Lemma 4.2. *Let G be the group of equivariant diffeomorphisms of a manifold M with respect to some fixed smooth action of a Lie group on M . Then G has the homotopy type of a countable CW complex.*

Proof. If the action is trivial, the Lemma is obtained by combining [5, p. 277, 283] and [16, Theorem 14]. In [5], the diffeomorphisms close to the identity are identified with certain cross-sections of the tangent bundle of M . This gives the local structure of a locally convex topological vector space. Therefore it is sufficient to observe that the equivariant diffeomorphisms correspond to equivariant cross-sections, which form a linear subspace.

DEFINITION. The diffeomorphisms of S^{2dn-1} onto itself which are equivariant with respect to the diagonal action $\varrho_n \oplus \varrho_n$ of A_n , endowed with the C^∞ topology, form a topological group. We denote this group by $\text{Diff}(A_n, S^{2dn-1})$, or briefly $D_n(A)$.

The group of all linear equivariant diffeomorphisms of S^{2dn-1} is a subgroup of $D_n(A)$ which is isomorphic to A_2 [7, 2.3].

THEOREM 4.3. *For $n \geq 3$, the inclusion $j: O_2 \subset D_n(O)$ is a homotopy equivalence.*

Proof. Every equivariant self-diffeomorphism of S^{2n-1} is homotopic to a linear one [7, 6.1]. So j induces an isomorphism for π_0 . To prove that j induces isomorphisms for π_k , $k > 0$, we use that the equivalence classes of bundles over S^{k+1} with structure group G are classified by $\pi_k G$ modulo the action of $\pi_0 G$ [19, 18.5]. Let $\pi: T \rightarrow S^{k+1}$ be an equivariant S^{2n-1} bundle (with structure group $D_n(O)$). The orbit space T' is a D^2 bundle over S^{k+1} , with structure group O_2 (Lemma 4.1).

Assume $k > 1$. Then $T' \rightarrow S^{k+1}$ is a trivial bundle, so T is an O_n manifold over $S^{k+1} \times D^2$. The principal orbit bundle of T is a bundle over $S^{k+1} \times B^2$ with structure group O_2 , so is also trivial. Thus by Theorem 2.1, the O_n manifold T corresponds to an element of $\text{cok}(\pi_{k+1} S O_2 \rightarrow \pi_{k+1} G_2)$. As this cokernel is zero, T is the 'trivial' O_n manifold over $S^{k+1} \times D^2$, i.e. $S^{k+1} \times S^{2n-1}$. Therefore every equivariant S^{2n-1} bundle over S^{k+1} is trivial, which means $\pi_k D(O) = 0 = \pi_k O_2$ ($k > 1$).

The case $k = 1$ is slightly more complicated. The principal orbit bundle of T is a bundle over $\text{int}(T') \simeq S^2$. If $\pi: T \rightarrow S^2$ is given by an element $t \in \pi_1 D_n(O)$, the principal orbit bundle of T is given by some element $t_0 \in \pi_1 O_2$ such that $(j_* t_0)^{-1} t \in \pi_1 D_n(O)$ defines an equivariant S^{2n-1} bundle over S^2 with trivial principal orbit bundle. So we may assume that T already has trivial principal orbit bundle. If T' is a non-trivial D^2 bundle over S^2 , $\partial T'$ is a lens space $L(q)$ ($q \geq 1$). The principal bundle of the princi-

pal orbit bundle of T is $\text{int } T' \times O_2 \rightarrow \text{int } T'$, the reduction of the structure group to $O_1 \times O_1$ according to Jänich's classification is a cross-section of the bundle $\partial T' \times O_2/O_1 \times O_1 \rightarrow \partial T'$, which is of degree ± 1 on any fibre of $\partial T' \rightarrow S^2$. So it is given by a map

$$\begin{array}{ccc} \partial T' & \rightarrow & O_2/O_1 \times O_1 \\ \parallel & & \parallel \\ L(q) & \rightarrow & S^1 \end{array}$$

which is of degree ± 1 on any fibre of $L(q) \rightarrow S^1$. As every map $L(q) \rightarrow S^1$ is null homotopic, this is impossible. Therefore the bundle $T' \rightarrow S^2$ is trivial. Now we can apply Theorem 2.1. As $\text{cok}(\pi_2 SO_2 \rightarrow \pi_2 G_2) = 0$, π is equivalent to the trivial bundle $S^2 \times S^{2n-1}$ over S^2 . This proves that j induces a surjective map $\pi_1 O_2/\pi_0 O_2 \rightarrow \pi_1 D_n(O)/\pi_0 D_n(O)$. As the total spaces of two different linear equivariant S^{2n-1} bundles over S^2 have different principal orbit bundles, $\pi_1 O_2/\pi_0 O_2 \rightarrow \pi_1 D_n(O)/\pi_0 D_n(O)$ is injective. Then $j_*: \pi_1 O_2 \rightarrow \pi_1 D_n(O)$ is an isomorphism because $\pi_0 O_2 = \pi_0 D_n(O) = \mathbb{Z}_2$.

So far we have shown that j is a weak homotopy equivalence. But O_2 and $D_n(O)$ have the homotopy type of CW complexes (Lemma 4.2). Hence j is a homotopy equivalence [18, p. 405].

COROLLARY 4.4. *Any O_n equivariant S^{2n-1} bundle with fibre action $\varrho_n \oplus \varrho_n$, $n \geq 3$, is a linear bundle.*

THEOREM 4.5. *Let Λ_n be the group U_n or Sp_n , $n \geq 3$. Let $k \geq 3$. If T is a Λ_n manifold over $S^k \times D^{d+1}$ corresponding to a non-zero element of $\text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1})$ in the classification of Theorem 2.1, then $\pi: T \rightarrow S^k$ is a non-linear Λ_n equivariant S^{2dn-1} bundle with fibre action $\varrho_n \oplus \varrho_n$.*

$\pi: T \rightarrow S^k$ is of course the composition of the orbit map with the projection on the first factor. Note that in the orthogonal case, the above cokernel is always zero.

Proof. If $\pi: T \rightarrow S^k$ is a linear bundle, it is equivariantly trivial. This follows from Corollary 3.3 and the isomorphism $\tau_*: \pi_{k-1} \Lambda_2 \rightarrow \pi_{k-1} SO_{d+1}$. But then, by Remark 2 of section 2, T corresponds to zero in $\text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1})$. So we only have to make sure that $\pi: T \rightarrow S^k$ is a bundle, i.e. locally trivial. If $B = S^k$ -point, $\pi^{-1}B$ is a Λ_n manifold over $B \times D^{d+1}$. The reduction of a structure group occurring in the classification by the Hsiangs and Jänich, is a map $B \rightarrow G_{d+1}$, so is homotopic to a constant map. As homotopic reductions yield equivariantly equivalent Λ_n manifolds [9, p. 23], $\pi^{-1}B$ is equivariantly diffeomorphic over B to $B \times S^{2dn-1}$.

COROLLARY 4.6. *If $\text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1}) \neq 0$ for some $k \geq 3$, then $\text{cok}(\pi_{k-1} \Lambda_2 \rightarrow \pi_{k-1} D^n(\Lambda)) \neq 0$ for every $n \geq 3$.*

Proof. By Theorem 4.5, there is a bundle over S^k with structure group $D_n(\Lambda)$ which is non-linear, i.e. the structure group of which cannot be reduced to Λ_2 . So the corresponding element of $\pi_{k-1}D_n(\Lambda)$ is not in the image of $\pi_{k-1}\Lambda_2$.

COROLLARY 4.7. *Neither of the inclusions $U_2 \subset D_n(U)$, $Sp_2 \subset D_n(Sp)$ is a homotopy equivalence.*

Proof. This follows from $\text{cok}(\pi_3SO_3 \rightarrow \pi_3G_3) \cong \mathbb{Z}_2$ and $\text{cok}(\pi_6SO_5 \rightarrow \pi_6G_5) \cong \mathbb{Z}_2$.

THEOREM 4.8. *For any $n \geq 3$, $D_n(U) = \text{Diff}(U_n, S^{4n-1})$, the group of all self-diffeomorphisms of S^{4n-1} which are equivariant with respect to the action $\varrho_n \oplus \varrho_n$ of U_n , does not have the homotopy type of a finite CW complex.*

Proof. As $\pi_3(G_3, SO_3) \cong \text{cok}(\pi_3SO_3 \rightarrow \pi_3G_3) \cong \mathbb{Z}_2$, $\pi_2D_n(U)$ is non-zero. But according to [3, Theorem 6.11], a topological group of the homotopy type of a finite CW complex, has zero 2-dimensional homotopy group.

Remark. We do not know whether or not $D_n(Sp)$ has the homotopy type of a finite CW complex. The above method does not work in the symplectic case since $\text{cok}(\pi_3SO_5 \rightarrow \pi_3G_5) = 0$.

5. Classifying the Total Spaces

In view of the exact homotopy sequence

$$\cdots \rightarrow \pi_k SO_{d+1} \rightarrow \pi_k G_{d+1} \rightarrow \pi_k(G_{d+1}, SO_{d+1}) \xrightarrow{\partial} \pi_{k-1} SO_{d+1} \rightarrow \cdots,$$

$\text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1})$ is isomorphic to $\ker \partial \subset \pi_k(G_{d+1}, SO_{d+1})$. This kernel can be calculated to be non-zero in many cases, giving many examples of non-linear bundles by Theorem 4.5. It turns out, however, that the total spaces of these bundles in most cases are equivariantly diffeomorphic to a product of a homotopy sphere and S^{2dn-1} . Before going into this question, we prove a rather technical lemma.

LEMMA 5.1. *Let Σ^k be a homotopy k -sphere, $k \geq 5$, $F: S^k \times D^{d+1} \rightarrow \Sigma^k \times D^{d+1}$ a diffeomorphism. Then F is strongly diffeotopic to a diffeomorphism G such that $G|_{\Delta \times D^{d+1}}: \Delta \times D^{d+1} \rightarrow \Delta' \times D^{d+1}$ has the form $G(x, y) = (g(x), y)$, where Δ, Δ' are k -disks in S^k, Σ^k , respectively, and $g: \Delta \rightarrow \Delta'$ is a diffeomorphism.*

Proof. If $p \in S^k, p' \in \Sigma^k$, then the map $F': p \times (D^{d+1}, S^d) \rightarrow \Sigma^k \times (D^{d+1}, S^d)$, defined by restricting F , is homotopic to a map $F'': p \times (D^{d+1}, S^d) \rightarrow \Sigma^k \times (D^{d+1}, S^d)$ such that $\text{im } F'' = p' \times (D^{d+1}, S^d)$ and $\pi_2 \circ F'' = \text{id}$. This homotopy may be assumed to be composed by two homotopies, the first one moving a neighborhood of the boundary close to the boundary and leaving the complement of a neighborhood of the boundary fixed, the second one moving only the complement of a neighborhood of the boundary in the complement of a neighborhood of the boundary.

As $F'' \mid p \times S^d: p \times S^d \rightarrow \Sigma^k \times S^d$ is $(k-1)$ -connected, we can replace the first homotopy by a strong diffeotopy of $\Sigma^k \times D^{d+1}$ which is the identity outside a neighborhood of the boundary. This is done using [8, p. 47] and the product structure of small neighborhoods of the boundary. To replace the second homotopy by a strong diffeotopy leaving a neighborhood of the boundary fixed, one has to extend Haefliger's existence theorem for diffeotopies [8, p. 47] to relative homotopies not affecting a neighborhood of the boundaries. Using the composition of the two diffeotopies, we have realized the homotopy between F' and F'' by a strong diffeotopy of $\Sigma^k \times D^{d+1}$. Now if Δ, Δ' are k -disks, $p \in \Delta \subset S^k, p' \in \Delta' \subset \Sigma^k, g: \Delta \rightarrow \Delta'$ a diffeomorphism such that $g(p) = p'$, then $F'' \mid \Delta \times D^{d+1}$ and $g \times id: \Delta \times D^{d+1} \rightarrow \Delta' \times D^{d+1}$ are tubular maps for $p' \times D^{d+1}$ in $\Sigma^k \times D^{d+1}$. (To be precise, we can give Δ and Δ' linear structures such that p is the origin in Δ and g is a linear isomorphism.) As D^{d+1} is contractible, there is another strong diffeotopy of $\Sigma^k \times D^{d+1}$ carrying $F \mid \Delta \times D^{d+1}$ to $g \times id$. Combining all the diffeotopies yields G .

Levine [13] constructed a homomorphism $\omega_3: \Theta^{k+d+1, k} \rightarrow \pi_k(G_{d+1}, SO_{d+1})$. ($\Theta^{m, k}$ is the group of k -dimensional knots which are homotopy spheres in $S^m, k \geq 5$.) $\omega_3(\kappa)$ is the obstruction for a knot κ to bound a framed manifold in S^{k+d+1} . $\omega_3(\kappa) \in \ker \partial$ if and only if κ has trivial normal bundle.

THEOREM 5.2. *Let Σ^k be a homotopy k -sphere, $k \geq 5$. Let T be a U_n or Sp_n manifold over $S^k \times D^{d+1}$, corresponding to an element $x \in \ker \partial \subset \pi_k(G_{d+1}, SO_{d+1})$. Then T is equivariantly diffeomorphic to $\Sigma^k \times S^{2dn-1}$ if and only if there is a knot κ diffeomorphic to Σ^k with $\omega_3(\kappa) = -x$.*

We first prove the following auxiliary

PROPOSITION 5.3. *Let κ be a knot diffeomorphic to Σ^k , of codimension $d+1$, with trivial normal bundle, and T the U_n or Sp_n manifold over $\Sigma^k \times D^{d+1}$ corresponding to $\omega_3(\kappa) \in \ker \partial \subset \pi_k(G_{d+1}, SO_{d+1})$. Then T is equivariantly diffeomorphic to $S^k \times S^{2dn-1}$.*

Proof. Recall how $\omega_3(\kappa)$ is defined if κ has trivial normal bundle [13, 3.1]. Let $h: \Sigma^k \times D^{d+1} \rightarrow X$ be a tubular map for κ . $S^{k+d+1} - \text{int } X$ is diffeomorphic to $D^{k+1} \times S^d$ by a diffeomorphism $g: D^{k+1} \times S^d \rightarrow S^{k+d+1} - \text{int } X$ such that $g \mid S^k \times S^d: S^k \times S^d \rightarrow \partial X$ extends to a diffeomorphism $h_0: S^k \times D^{d+1} \rightarrow X$ [17, Theorem 4.1]. If π_2 is the projection on the second factor, then $\pi_2 g^{-1} h: \Sigma^k \times S^d \rightarrow S^d$ defines an element of $\pi_k G_{d+1}$ whose image in $\pi_k(G_{d+1}, SO_{d+1})$ is $\omega_3(\kappa)$.

By the diffeomorphism $h^{-1} h_0: S^k \times D^{d+1} \rightarrow \Sigma^k \times D^{d+1}$, T can be lifted to a A_n manifold T' over $S^k \times D^{d+1}$, which can be detected by an element $y \in \pi_k(G_{d+1}, SO_{d+1})$ according to the classification in Theorem 2.1. As T' was obtained by lifting from T , y is represented by the composition $(\pi_2 g^{-1} h) \circ (h^{-1} h_0) = \pi_2$. So T' is equivariantly diffeomorphic to $S^k \times S^{2dn-1}$. But T is equivariantly diffeomorphic to T' .

Proof of Theorem 5.2. First assume the existence of κ diffeomorphic to Σ^k such

that $\omega_3(x) = -x$. By Proposition 5.3, the A_n manifold T' over $(-\Sigma^k) \times D^{d+1}$ corresponding to $x \in \pi_k(G_{d+1}, SO_{d+1})$, is equivariantly diffeomorphic to $S^k \times S^{2dn-1}$. Hence there is a commutative diagram

$$\begin{array}{ccc} S^k \times S^{2dn-1} & \xrightarrow{E} & T' \\ \downarrow & & \downarrow \\ S^k \times D^{d+1} & \xrightarrow{F} & (-\Sigma^k) \times D^{d+1} \end{array}$$

where the equivariant diffeomorphism E induces a diffeomorphism F of the orbit spaces. Let $S^k = D_+^k \cup_{id} D_-^k$, D_\pm^k k -disks, matching the boundaries by the identity of S^{k-1} , $-\Sigma^k = D_+^k \cup_s D_-^k$ matching the boundaries by an autodiffeomorphism s of S^{k-1} . Applying Lemma 5.1, F may be assumed to map $D_+^k \times D^{d+1}$ onto $D_+^k \times D^{d+1}$ by the identity. This means that the fibre homotopy trivialization $(-\Sigma^k) \times S^d \rightarrow S^d$ representing x (and defining the A_n manifold T') is just the second projection when restricted to $D_+^k \times S^d$. Now we cut our A_n manifolds $S^k \times S^{2dn-1}$ and T' in two pieces, according to the decomposition of S^k and $-\Sigma^k$ in two hemispheres. The two pieces are glued together after inserting a twist defined by the map s^{-1} on S^{k-1} . This defines a diagram

$$\begin{array}{ccc} \Sigma^k \times S^{2dn-1} & \xrightarrow{E'} & T'' \\ \downarrow & & \downarrow \\ \Sigma^k \times D^{d+1} & \xrightarrow{F'} & S^k \times D^{d+1} \end{array}$$

where E' is again an equivariant diffeomorphism. As the fibre homotopy trivializations that define the new A_n manifolds over $\Sigma^k \times D^{d+1}$ and $S^k \times D^{d+1}$ still are equal to the second projection when restricted to $D_+^k \times S^d$, we did not change the corresponding elements in $\pi_k(G_{d+1}, SO_{d+1})$. So we really have the product $\Sigma^k \times S^{2dn-1}$ on the left hand side (corresponding to $0 \in \pi_k(G_{d+1}, SO_{d+1})$), and a A_n manifold corresponding to x on the right hand side (i. e. T). Therefore T is equivariantly diffeomorphic to T'' , which is equivariantly diffeomorphic to $\Sigma^k \times S^{2dn-1}$ by E' .

Conversely, let T be equivariantly diffeomorphic to $\Sigma^k \times S^{2dn-1}$. As before, in the diagram

$$\begin{array}{ccc} T & \xrightarrow{E} & \Sigma^k \times S^{2dn-1} \\ \downarrow & & \downarrow \\ S^k \times D^{d+1} & \xrightarrow{F} & \Sigma^k \times D^{d+1} \end{array}$$

we may assume that $F|_{D_+^k \times D^{d+1}}$ is the identity (with respect to a decomposition $\Sigma^k = D_+^k \cup_t D_-^k$). Inserting an appropriate twist as above, we obtain a diagram

$$\begin{array}{ccc} T' & \xrightarrow{E'} & S^k \times S^{2dn-1} \\ \downarrow & & \downarrow \\ (-\Sigma^k) \times D^{d+1} & \xrightarrow{F'} & S^k \times D^{d+1} \end{array}$$

where T' is still a A_n manifold corresponding to $x \in \pi_k(G_{d+1}, SO_{d+1})$. Because of the above diagram, x is representable by $\pi_2 \circ F' | (-\Sigma^k) \times S^d$. On the other hand, according to our remark at the beginning of this proof, for the knot $-\kappa$ given by $F'((- \Sigma^k) \times 0) \subset S^k \times D^{d+1} \subset S^{k+d+1}$, $\omega_3(-\kappa)$ is also represented by $\pi_2 \circ F' | (-\Sigma^k) \times S^d$. Thus $\omega_3(\kappa) = -x$. This completes the proof of Theorem 5.2.

By Theorem 5.2, the problem of deciding whether the total spaces of the bundles constructed in Theorem 4.5 are equivariantly diffeomorphic to $\Sigma^k \times S^{2dn-1}$, is largely reduced to homotopy theory. As $\text{cok}(\pi_k SO_5 \rightarrow \pi_k G_5) = 0$ for $k=3,4$, there are no such non-linear symplectic bundles in these dimensions. We do not know whether our non-linear unitary bundles have familiar total spaces for $k=3,4$. Now assume $k \geq 5$. We have Levine's exact sequence [13]

$$\Theta^{k+d+1, k} \xrightarrow{\omega_3} \pi_k(G_{d+1}, SO_{d+1}) \rightarrow P_k \rightarrow \Theta^{k+d, k-1}.$$

According to Theorem 5.2, the total spaces of all bundles constructed in Theorem 4.5 are equivariantly diffeomorphic to some $\Sigma^k \times S^{2dn-1}$ if and only if $\ker(\partial: \pi_k(G_{d+1}, SO_{d+1}) \rightarrow \pi_{k-1} SO_{d+1}) \cong \text{cok}(\pi_k SO_{d+1} \rightarrow \pi_k G_{d+1})$ is contained in $\text{im} \omega_3$. As $\pi_k(G_{d+1}, SO_{d+1})$ is finite for all $k \geq 5$, $d=2, 4$, ω_3 is certainly surjective unless $k \equiv 2 \pmod{4}$. In the latter case, $P_k = \mathbb{Z}_2$, and ω_3 is an epimorphism if and only if a codimension 2 knot in S^{k+1} with Arf invariant 1 remains non-trivial after $(d-1)$ -fold suspension. (Exactly then $P_k \rightarrow \Theta^{k+d, k-1}$ is injective.) As the Kervaire sphere is not diffeomorphic to the standard sphere in dimensions different from $2^r - 3$ [4, Corollary 2], ω_3 is surjective for all $k \neq 2^r - 2$. For $k=6, 14$, using [20], ω_3 can be computed to be surjective in the unitary case ($d=2$). For $k=6, d=4$ (symplectic action), $\Theta^{k+d, k-1}$ is zero for dimensional reasons [13]. As $\ker \partial = \pi_6(G_5, SO_5) = \mathbb{Z}_2$, ω_3 is not surjective in this case, and we have spotted a non-linear symplectic S^{8n-1} bundle over S^6 whose total space is not equivariantly diffeomorphic to $S^6 \times S^{8n-1}$. We summarize:

PROPOSITION 5.4. *If $k \geq 5$, $k \neq 2^r - 2$, then the total spaces of the nonlinear equivariant S^{2dn-1} bundles over S^k , constructed in Theorem 4.5, are equivariantly diffeomorphic to a product of a homotopy k -sphere with trivial action and S^{2dn-1} . This is also true for $k=6, 14$ in the unitary case. For $k=6$, there is a non-linear symplectic S^{8n-1} bundle over S^6 whose total space is not equivariantly diffeomorphic to a product of a homotopy sphere with trivial action and S^{8n-1} .*

REFERENCES

- [1] ANTONELLI, P., BURGHELEA, D., and KAHN, P. J., *Gromoll groups, Diff S^n , and bilinear constructions of exotic spheres*, Bull. Amer. Math. Soc. 76 (1970), 772-777.
- [2] BOREL, A., *Groupes d'homotopie des groupes de Lie I*, Séminaire H. Cartan 1949/50 No. 12.
- [3] BROWDER, W., *Torsion in H-spaces*, Ann. Math. 74 (1961), 24-51.

- [4] BROWDER, W., *The Kervaire invariant of framed manifolds and its generalization*, Ann. Math. 90 (1969), 157–186.
- [5] CERF, J., *Topologie de certains espaces de plongements*, Bull. Soc. math. France 89 (1961), 227–380.
- [6] CERF, J., *Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = O$)*, Springer Lecture Notes in Math. No. 53, Berlin 1968.
- [7] ERLE, D. and HSIANG, W. C., *On certain unitary and symplectic actions with three orbit types*, Amer. J. Math. 94 (1972), 289–308.
- [8] HAEFLIGER, A., *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. 36 (1961), 47–82.
- [9] HIRZEBRUCH, F. and MAYER, K. H., *O(n)-Mannigfaltigkeiten, exotische Spären und Singularitäten*, Springer Lecture Notes in Math. No. 57, Berlin 1968.
- [10] HSIANG, W. C. and HSIANG, W. Y., *Differentiable actions of compact connected classical groups I*, Amer. J. Math. 89 (1967), 705–786.
- [11] JÄNICH, K., *Differenzierbare Mannigfaltigkeiten mit Rand als Orbiträume differenzierbarer G-Mannigfaltigkeiten ohne Rand*, Topology 5 (1966), 301–320.
- [12] JÄNICH, K., *Differenzierbare G-Mannigfaltigkeiten*, Springer Lecture Notes in Math. No. 59, Berlin 1968.
- [13] LEVINE, J., *A classification of differentiable knots*, Ann. Math. 82 (1965), 15–50.
- [14] MASSEY, W. S., *On the normal bundle of a sphere imbedded in Euclidean space*, Proc. Amer. Math. Soc. 10 (1959), 959–964.
- [15] NOVIKOV, S. P., *Differentiable sphere bundles*, Amer. Math. Soc. Translations. Ser. 2. Vol. 63, p. 217–244. Amer. Math. Soc., Providence, R.I., 1967.
- [16] PALAIS, R. S., *Homotopy theory of infinite-dimensional manifolds*, Topology 5 (1966), 1–16.
- [17] SMALE, S., *On the structure of manifolds*, Amer. J. Math. 84 (1962), 387–399.
- [18] SPANIER, E. H., *Algebraic Topology*, McGraw-Hill Book Co. New York 1966.
- [19] STEENROD, N., *The Topology of Fibre Bundles*, Princeton Univ. Press. Princeton, N.J., 1951.
- [20] TODA, H., *Composition methods in homotopy groups of spheres*, Ann. Math. Studies No. 49. Princeton Univ. Press. Princeton, N.J., 1962.

Added in proof: G. Bredon has proved that Levine's homomorphism $P_k \rightarrow \theta^{k+2, k-1}$ is injective for all $k \equiv 2 \pmod{4}$ (Classification of regular actions of classical groups with three orbit types, preprint, Cor. 8.2). Thus in Proposition 5.4, we may drop the hypothesis " $k \neq 2^r - 2$ " in the unitary case.

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