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Almost Canonical Inverse Images

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Generally, given a map from a manifold to a complex containing a subcomplex with a normal bundle, the transversal inverse image of the subcomplex can be quite arbitrary. In this note we give a universal homotopy model for such a map, and show that under a finiteness assumption on the model the map is homotopic to one such that the homotopy type of the inverse image and its complement agree with their models up to the middle dimension. Remarks are made on the simply-connected case and applications to codimension two homology classes.

1. Inverse Images

Suppose ξ is a bundle (vector, *PL* block, or topological) of dimension l over a complex Y, and suppose $X = X_0 \cup_{S\xi} D\xi$, where $S\xi$, $D\xi$ denote the sphere and disc bundles of ξ respectively. We will consider maps $f: M \to X$, M a compact m-manifold.

First we recall the *homotopy pullback* $E(f, Y)$. This is a homotopy commutative diagram

$$\begin{array}{ccc} E(f, Y) & \rightarrow & Y \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ f\ } & X \end{array}$$

which is universal with respect to such diagrams. E is constructed by making either M or Y a fibration over X, and pulling back over the other. More concretely one such E is $E(f, Y) = \{(\varphi, x, y) \mid x \in M, y \in Y, \varphi: [0, 1] \to X, \text{ and } \varphi(0) = f(x), \varphi(1) = y\}$.

The homotopy pullback setting for inverse images is also exploited in [6].

1.1 PROPOSITION. *M is homotopy equivalent to $E(f, X_0) \cup_{E(f, S\xi)} E(f, D\xi)$.*

Proof. Using the explicit path-space expression for E we actually get $M \simeq E(f, X) = E(f, X_0) \cup E(f, D\xi)$, and $E(f, X_0) \cap E(f, D\xi) = E(f, S\xi)$.

This establishes $E(f, Y)$ as a canonical 'homotopy' inverse image of Y. Since the transverse image of Y is a compact manifold, in order for it to agree with $E(f, Y)$ in low dimensions $E(f, Y)$ must have a finite skeleton. Our main result is that this condition is also sufficient below the middle dimension of $f^{-1}(Y)$. By a 'k-skeleton' we mean a k-connected map from a k-complex $K^k \to E$.

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1.2 DEFINITION. *Suppose $f: M^m \to X \supset Y$ is a map, M a manifold, Y a subcomplex with normal bundle of dimension l, and f transverse to Y. f is almost canonical if the induced maps $f^{-1}(Y) \to E(f, Y)$ and $M - f^{-1}(Y) \to E(f, X-Y)$ are $\left[\frac{m-l}{2}\right]$ and $\left[\frac{m}{2}\right]$ connected, respectively*

Here $[r]$ denotes the greatest integer less than equal to the real number r.

1.3 PROPOSITION. *If $f: M \to X$ as above is tranverse to Y and $E(f, Y)$, $E(f, X-Y)$ have finite $\left[\frac{m-l+1}{2}\right]$, $\left[\frac{m+1}{2}\right]$ skeleton respectively, then f is homotopic holding the boundary fixed to an almost canonical map.*

Note that this is independent of the behavior on the boundary of M.

Proof. The cases $m \leqslant 4$ are easily handled by special arguments, so we concentrate on the cases where the s-cobordism theorem is available. The proof is adapted from the usual proof of Browder's embedding theorem. The idea is to do surgery on $f^{-1}(Y)$, $M - f^{-1}(Y)$, to make them connected as desired. This gives a normal bordism of f with kernels only in the middle dimension. Surgery is performed to make this an s-cobordism, carefully to preserve the desired connectivities.

Replace $X \supset Y$ by $E(f, X) \supset E(f, Y)$, and note that $E(f, S\xi)$ is the sphere bundle of the pullback of ξ over $E(f, Y) \to Y$. This reduces us to the following problem: if $f: M \to X \supset Y$ is a homotopy equivalence, make $f^{-1}Y \to Y$ and $M - f^{-1}(Y) \to X - Y$ highly connected. We assume f is a homotopy equivalence from now on. Denote $f^{-1}(Y)$ by N.

$f: M \to X$ and therefore $f: N \to Y$ are normal maps in the sense of Browder [1]. A normal bordism of $N \to Y$ holding the boundary fixed to a $\left[\frac{m-l}{2}\right]$-connected map can be constructed by surgery blow the middle dimension (see Wall [11, 0:1 Theorem 1.2]). At this point we can be a little more specific about the exact inverse images possible.

1.4 IMPROVEMENT. In the situation of 1.3, if $l > 2$, or $l = 2$, and m is odd, any $N' \to Y$ which is $\left[\frac{m-l}{2}\right]$ connected and normally bordant to N by a bordism constant on the boundary can be realized as an almost canonical inverse image. If $l = 2$ and m is even, the $N' \to Y$ obtained by surgery below the middle dimension can be realized.

To resume the proof, surgery can be applied to a normal bordism of N to make the map of the bordism to $Y\left[\frac{m-l+1}{2}\right]$ connected without changing the boundary. Denote this bordism by $g: W \to Y$, with $\partial_0(W, g) = (N, f \mid N)$, and $\partial_1(W, g)$ the improved manifold.

The map of pairs $(W, \partial_1 W) \to (Y, Y)$ is $\left[\frac{m-l+1}{2}\right]$ connected, so the pair $(W, \partial_1 W$ is $\left[\frac{m-l+1}{2}\right]-1=\left[\frac{m-l-1}{2}\right]$-connected. If the bordism is obtained by surgery below the middle dimension and $m-l$ is even, then $(W, \partial_1 W)$ is actually $\frac{m-l}{2}$-connected. This can also be achieved by modifying an arbitrary bordism by surgery on the pair, as in [11, 0:1, addendum]. This modification changes $\partial_1 W$ by connected sum with a few copies of $S^j \times S^j, j=\frac{m-l}{2}$. This slight extra connectivity will be needed if $m-l \leqslant 2$.

Construct a normal bordism of M by identifying the disc bundle of the normal bundle of $N \subset M \times \{1\} \subset M \times I$ with $D(\partial_0 g^*\xi) \subset D(g^*\xi)$. The complement of $\partial_1 W$ in the new boundary, and the complement of W in the whole bordism can be changed as above to make the maps to $X-Y$ highly connected. Call this improved bordism $G: V \to X$. $G^{-1}(Y) = W$, and $G: V-W \to X-Y$, $(V-W, \partial_1 V - \partial_1 W)$ are $\left[\frac{m+1}{2}\right]$, $\left[\frac{m}{2}\right]+1$-connected, respectively. $\partial_0 V = M$.

Next note that $V \to X$, $(V, \partial_1 V)$ are also $\left[\frac{m+1}{2}\right]$, $\left[\frac{m}{2}\right]+1$-connected. This follows from homology sequences with $\mathbf{Z}[\pi, X]$ coefficients, noting that $(V, V-W)$ is the Thom space of $g^*\xi$. It also follows from the geometric arguments below.

We are now in the situation of Wall's 'important special case', [11, 1:4]. Wall tells us how to do middle dimensional surgery on $(V, \partial_1 V)$ to make it an s-cobordism. We will observe that these surgeries can be arranged so as not to destroy our carefully accumulated connectivity. Actually since when $\dim V = m+1$ is odd $(V, \partial_1 V)$ is 2-connected, we will be able to use an embedding theorem unavailable to Wall and so will not have to separate even and odd dimensional cases.

Let $m+1=2k$, or $2k-1$. Then the only nonvanishing $K_*(V, \partial_1 V)$ is $*=k$, which we may assume free, based, and the basis represented by maps $f_i: (D^k, S^{k-1}) \to (V, \partial_1 V)$.

Case 1, $l \geqslant 2$.

The transverse inverse image $f_i^{-1}(W)$ has dimension $k-l$. If $l \geqslant 3$, or m is odd, $\left[\frac{m-l-1}{2}\right] \geqslant k-l$, so $f^{-1}(W)$ can be deformed into $\partial_1 W$. If $l=2$ and m is even, then

$\dfrac{m-l}{2}=k-l$, and we can do the deformation with the extra connectivity mentioned above. This deformation can be extended to a homotopy of f_i to a map which with image in $(V-g^*\xi)\cup\partial_1 V$. Now choose a small k-disc in $D^k - f_i^{-1}(\partial_1 g^*\xi)$. The complement of this k-disc is $(k-1)$-dimensional, so since $(V-W, \partial_1 V-\partial_1 W)$ is $(k-1)$-connected can be deformed mod $f_i^{-1}(\partial_1 g^*\xi)-f_i^{-1}(V)$ into $\partial_1 V-\partial_1 W$. This gives a homotopy of f_i to a map into $(V-W, \partial_1 V-\partial_1 W)$.

The intersections of these maps are points or circles and arcs depending on whether $m+1$ is even or odd. Since $\pi_1(\partial_1 V-\partial_1 W)\simeq\pi_1(V-W)$, and if $m+1$ is odd $\pi_2(\partial_1 V-\partial_1 W)\simeq\pi_2(V-W)$, the maps are homotopic to disjoint framed embeddings (by the 'Whitney trick').

We have represented a basis for $K_k(V, \partial V)$ by embeddings in $V-W$. Now surgery on these converts V into an s-cobordism, by [11, 1:4], but the connectivity of $V-W$ is not worsened, and W is not changed at all. Thus if $l\geqslant 2$ we are finished.

Case 2. $l=1$.

In this case the inverse image $f_i^{-1}(W)\subset D^k$ has dimension $k-1=\left[\dfrac{m-l-1}{2}\right]+1$. First deforming f_i so that the disc $D^{k-1}_{1/2}$ lies in $f_i^{-1}(W)$, f_i can then be deformed to take the complement of the disc in $f_i^{-1}(W)$ into $\partial_1 W$. Now consider $D^k_{1/2}\subset D^k$. The components of $(D^k-D^k_{1/2},\ f^{-1}(W)-D^{k-1}_{1/2}\cup S^{k-1})$ have dimension less than or equal to $k-1$, so can be deformed into $\partial_1 V-\partial_1 W$ in the complement of $W-\partial_1 W$, holding $f^{-1}(W)-D^{k-1}_{1/2}$ fixed. This constructs a homotopy of each f_i to f_i with $f_i^{-1}(W)= =D^{k-1}\subset D^k$.

As in the previous case, the high connectivity of $(W, \partial_1 W)$ $(V-W, \partial_1 V-\partial_1 W)$ allows intersection points and arcs to be removed from these discs giving disjoint framed embeddings f_i with $f_i^{-1}(W)=D^{k-1}$. Surgery on these discs makes V an s-cobordism, and changes W by surgery on a middle-dimensional disc embedded in $(W, \partial_1 W)$. This surgery on W does not change the connectivity of $\partial_1 W\to Y$. Again an isomorphism of the s-cobordism with $M\times I$ produces the required homotopy to an almost canonical map. This completes the proof of 1.3.

1.5 COROLLARY. *Any two almost canonical inverse images are almost h-cobordant.*

Proof. 'Almost h-cobordant' for a k-manifold here means a bordism $(W; \partial_0 W, \partial_1 W)$ with $\pi_j(W, \partial_i W)=0$ for $j\leqslant\left[\dfrac{k-1}{2}\right]$. This implies the only nonvanishing relative homology groups are in the middle dimension. This follows from applying the proposition to the homotopy between the two almost canonical maps, holding the boundaries fixed.

2. Remarks and Examples

These are not deep results, but they unify and shorten the below-the-middle dimension starting points for several interesting investigations. Some examples are

Codimension 1. When $l=1$, 1.3 generalizes and shortens arguments used by Cappell in his splitting theorem [2].

Codimension 2. 1.4 generalizes an embedding theorem of [3] when m is odd, and provides a setting for the m even case. The arguments of Kato and Matsumoto [8] also fit into this framework, as will be explained in 2.1 below. See also [5].

Codimension $\geqslant 3$. 1.4 contains the usual codimension $\geqslant 3$ embedding theorems of [1], [11] as special cases.

Next some applications are given in more detail.

2.1 *Codimension* 1 *and* 2 *homology classes.* Suppose M is a closed manifold, $\xi \in H_{m-j}(M; \mathbf{Z}), j=1, 2$. Assume $\dim M \neq 5, 6$ if M has no PL structure. ξ is Poincaré dual to a cohomology class $\bar{\xi} \in H^j(M; \mathbf{Z})$, so is represented by a map $M \to S^1$, or $M \to \mathbf{C}P^\infty$. The transverse inverse image of a point, or $\mathbf{C}P^{\infty-1}$, gives a submanifold of codimension j whose fundamental class represents ξ. Bordism classes of such submanifolds, through bordisms embedded in $M \times I$, correspond bijectively by this construction with $H_{n-j}(M)$.

Consider the homotopy models for these inverse images. For $\xi: M \to S^1 \supset *$ it is the cover of M pulled back from the universal cover of S^1. This only rarely has finite skeleta. For example, if ξ is trivial, then $E(\xi, *)$ is a countable number of disjoint copies of M. If $E(\xi, *)$ is a finite complex, we are in the situation considered by Farrell [4], when one tries to fiber M over S^1. 1.3 gives a candidate for the fiber which is correct up to the middle dimension.

In the case $j=2$, the inclusion $\mathbf{C}P^{\infty-1} \to \mathbf{C}P^\infty$ is a homotopy equivalence, so the map $E(\xi, \mathbf{C}P^{\infty-1}) \to M$ is also. $\mathbf{C}P^\infty - \mathbf{C}P^{\infty-1}$ is contractible, so its inclusion has the homotopy type of the universal S^1 bundle over $\mathbf{C}P^\infty$. Denote this bundle by γ, then $\xi^*\gamma \to E(\xi, \mathbf{C}P^\infty - \mathbf{C}P^{\infty-1})$ is a homotopy equivalence. These all have finite skeleta, so applying 1.3 we get: every codimension two homology class is represented by an embedded submanifold $N \subset M$ so that $N \to M$ is $\left[\frac{m-2}{2}\right]$-connected, and $M - N \to \xi^*\gamma$ is $\left[\frac{m}{2}\right]$-connected. In this situation almost canonical is the same as the 'tautness' of [9].

Such almost canonical representatives for ξ are unique up to almost h-cobordism according to 1.5. If M is odd dimensional, any suitably connected manifold normally cobordant to a representative for ξ can be realized as a representative by 1.4. In the even dimensional case this is not true ([7], [9]). In the next section we will give a little more information about which ones can occur.

3. The Simply-Connected Case

In certain cases, algebraic information can be used to characterize the existence of finite skeleta for a space. For this see [10]. We will apply the simply-connected version due to J. Moore which states that if $\pi_1 Z = 1$, and $H_j Z$ is finitely generated for every j, then Z has finite skeleta.

3.1 PROPOSITION. *If $f: M \to X \supset Y$ are finite complexes and $X, M, E(f, Y)$ are connected and 1-connected, then $E(f, Y)$ has finite skeleta.*

Proof. By the result of Moore mentioned above, it is sufficient to show $E(f, Y)$ is $\mathscr{C}$-acyclic, where $\mathscr{C}$ is the Serre class of finitely generated abelian groups. By assumption, X, Y, M are $\mathscr{C}$-acyclic so by the Serre spectral sequence so is the fiber $F \to Y \to X$. $E(f, Y)$ is the pullback of this fibration over M, so there is a fibration $F \to E(f, Y) \to \to M$. Again by the Serre sequence $E(f, Y)$ is $\mathscr{C}$-acyclic.

Another advantage of the simply-connected case is that surgery can be done very explicitly. We use this to improve 1.4.

3.2 PROPOSITION. *Suppose $X \supset Y$ is codimension 2 and $f: M^{2k} \to X$ is given and has $E(f, Y)$ 1-connected, then f is homotopic to an almost canonical map g with* $\operatorname{rank}(H_k(E(f, Y), g^{-1}(Y)) \leqslant \operatorname{rank}(H_k(E(f, Y), f^{-1}(Y)) + 2$ gen. torsion $\{H_{k-1}(E(f, Y), f^{-1}(Y))\}$.

Proof. By 1.4 it is only necessary to show that surgery can be done on $f^{-1}(Y) \to \to E(f, Y)$ below the middle dimension to make it $(k-2)$-connected, with the resulting situation satisfying the estimate.

Abbreviate the normal map by $N \to E$, with $\dim N = 2k-2$. To kill off $H_j(E, N)$, $j \leqslant k-1$, consider first a maximal free summand of the lowest nonvanishing one. Surgery on spheres representing a basis for this summand (see [1]) yields normal bordism of N to N' with

$$H_i(E, N') \simeq \begin{cases} H_i(E, N), & i \leqslant k, \quad i \neq j \\ H_j(E, N)/\text{summand}, & i = j \end{cases}.$$

The lowest nonzero H_j is now torsion. Again do surgery on generators. If $j < k-1$, this introduces a free summand in H_{j+1}, kills H_j, and does not otherwise change the homology below k. The new free summand may be removed as above.

Thus we can arrange that $H_i(E, N') = 0$, $i \leqslant k-2$, $H_{k-1}(E, N')$ is torsion $\{H_{k-1}(E, N')\}$, and $H_k(E, N') \simeq H_k(E, N)$. Surgery on generators of $H_{k-1}(E, N')$ is rationally like connected sum of N' with copies of $S^{k-1} \times S^{k-1}$: It kills off the torsion module $H_{k-1}(E, N')$ as above, and increases the ranks of $H_k(E, H')$ and $H_{k-1}(N')$ by twice the number of generators. This completes the argument.

3.3 COROLLARY. *If $N \subset M^{2k}$ is a codimension 2 submanifold, $\pi_1 M = 1$, then there is an almost canonical (= taut) representative W of the same homology class, with* rank $H_{k-1}(W)$ = rank $H_{k-1}(N)+2$ gen. torsion $\{H_{k-1}(M, N)\}$.

3.3 follows from 3.2 since by 2.1 the appropriate homotopy model for W is M.

Thomas and Wood in [9] have given a lower bound for rank $H_{k-1}(W)$ in the situation of 3.3. 3.3 implies that this lower bound applies to the sum rank $H_{k-1}(N)+2$ gen. tors. $\{H_{k-1}(M, N)\}$ for arbitrary submanifolds N representing a fixed homology class. Freedman [5] has results about the attainability of this lower bound.

Added in proof

To the right side of the inequality in 3.1 add 2 rank coker $\{H_{k-1}(f^{-1}Y) \to \to H_{k-1}(E(f, Y))\}$, in 3.3 add 2 rank coker $\{H_{k-1}N \to H_{k-1}M\}$. The argument concerning surgery on the free part of $H_j(E, N)$ is correct on the part which injects into $H_{j-1}N$. The remainder contributes to the rank just as the torsion does.

The dimensions of the homology groups should read as follows: rank $H_{k-1} \leqslant \leqslant$ rank $H_{k-1}+2$ gen torsion $\{H_{k-2}\}+2$ rank coker $\{H_{k-1}\}$.

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