

# Computation of Lojasiewicz Exponent of ...( $x$ , $y$ )

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## Computation of Lojasiewicz Exponent of $f(x, y)$ <sup>1)</sup>

TZEE-CHAR KUO

Let  $f(x)$ ,  $x \in \mathbf{R}^n$ ,  $f(0)=0$ , be a real analytic function defined near 0.

First, suppose  $f(x) > 0$  for  $x \neq 0$ . Then Lojasiewicz asserts that there exist  $\varepsilon > 0$ ,  $\alpha > 0$  such that

$$f(x) \geq \varepsilon |x|^\alpha, \quad x \text{ near } 0.$$

Geometrically, this says that the graph of  $y = f(x)$  lies above the bowl-like graph of  $y = \varepsilon |x|^\alpha$ .

For general  $f(x)$ , let  $V_f$  denote the variety  $f(x) = 0$  in  $\mathbf{R}^n$ , then Lojasiewicz ([2], p. 85; [3]) asserts that

$$|f(x)| \geq \varepsilon d(x, V_f)^\alpha, \quad x \text{ near } 0,$$

where  $d(\cdot, \cdot)$  denotes the usual distance in  $\mathbf{R}^n$ .

This inequality, known as the Lojasiewicz inequality, is of fundamental importance in singularities theory.

**PROBLEM.** Determine the smallest value of  $\alpha$  in the Lojasiewicz inequality.

For instance, to determine the smallest integer  $r$  such that in the Taylor expansion of  $f$  near 0, all terms of degree  $> r$  can be omitted without changing the local topological type of  $f$  (i.e. the  $r$ -jet  $j^{(r)}(f)$  is  $C^0$ -sufficient) amounts to determining  $\alpha$  for  $|\text{Grad } f(x)|$  ([1], Theorem 0), where  $\text{Grad } f(x) \neq 0$  for  $x \neq 0$ .

Let  $\mathbf{R}_+^2$  denote the upper half plane  $\{(x, y): y \geq 0\}$ ;  $\mathbf{R}_-^2$  the lower half plane  $y \leq 0$ . All points are understood to be in a sufficiently small neighborhood of 0.

### §1. The Results

Let  $f(x, y)$  be a real analytic function of two real variables with  $f(0, 0) = 0$ . We may assume, without loss of generality, that the initial form of the Taylor expansion of  $f$  is not divisible by  $y$  (this amounts to saying that the  $x$ -axis is not a tangent of  $f(x, y) = 0$  at 0). Then, by Puiseux's Theorem ([4], p. 98),  $f$  can be factored (near 0)

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<sup>1)</sup> The purpose of this paper is to give a complete solution of the Problem for functions  $f(x, y)$  of two real variables. Our solution depends heavily on the use of Puiseux's Theorem.

into a product

$$f(x, y) = a \prod_{i=1}^m (x - z_i^*) \quad a \neq 0 \quad (1.1)$$

where  $m = O(f)$ , the order of  $f$ , and each  $z_i^*$  is a fractional power series in  $y$  with order  $O(z_i^*) \geq 1$ .

EXAMPLE (1.2).  $f(x, y) = x(x^2 + y^3 - y^4)$ .

Roots are  $z_1^* = 0$ ,  $z_2^*, z_3^* = \pm iy^{3/2}(1 - \frac{1}{2}y + \dots)$  with  $O(z_1^*) = \infty$ ,  $O(z_2^*) = O(z_3^*) = 3/2$ .

SOME DEFINITIONS. (i) Each  $z_i^*$  in (1.1) is called a root of  $f$ . A fractional power series is *real* if all coefficients are real. For a non-real  $z^*$ ,  $x = z^*$  has no locus in  $\mathbf{R}_+^2$ . This is because for  $y > 0$ , all fractional powers of  $y$  are real numbers, and so  $z^*$  is not a real number. In  $\mathbf{R}_-^2$ , however,  $x = z^*$  may, or may not, have locus. For the roots  $z_2^*, z_3^*$  in Example (1.2), there are loci in  $\mathbf{R}_-^2$ , the two arcs of a cusp. For the roots of  $x^2 + y^4 = 0$ , there is no locus in either half plane (except the single point 0).

(ii) Let  $z^* = \sum_{i=1}^{\infty} a_i y^{n_i}$   $a_i \neq 0$ ,  $1 \leq n_1 < n_2 < \dots$  be a given non-real series,  $a_s$  its first non real coefficient. We define the *real springboard* of  $z^*$  to be

$$z^*(t) = \sum_{i \leq s-1} a_i y^{n_i} + t y^{n_s}$$

where  $t$  is a generic real number.

(1.3) Call  $n_s$  the complex order of  $z^*$ .

Now we put

$$e(f, z^*) = O(f(z^*(t), y))$$

and

$$\delta(f, z^*) = \begin{cases} \max \{O(z^*(t) - z_i^*)\}, & \text{where } z_i^* \text{ runs through all real roots} \\ 1 & \text{if there is no real root.} \end{cases}$$

EXAMPLE (1.4). For  $f(x, y)$  as in Example (1.2),

$$e(f, z_2^*) = O(t(t^2 + 1)y^{9/2} + \dots) = 9/2$$

for generic values of  $t$ . Notice that for some special value of  $t$ , such as  $t = 0$ ,  $O(f(z_2^*(t), y)) > 9/2$ .

We may call  $e(f, z^*)$  the *climbing-exponent* of  $f$  along  $z^*$  and  $\delta(f, z^*)$  the *distance-exponent* of  $f$  along  $z^*$ .

**THEOREM.** *The number*

$$L_+(f) = \text{Max}_j \left\{ \frac{e(f, z_j^*)}{\delta(f, z_j^*)}, O(f) \right\}$$

where  $j$  runs through all indices for which  $z_j^*$  is a non real root, has the property that in any given sufficiently small compact neighborhood  $U$  of  $0$ , there exists a constant  $\varepsilon > 0$  such that

$$(L_+) \quad |f(x, y)| \geq \varepsilon d((x, y), V_f)^{L_+(f)}, \quad (x, y) \in U \cap \mathbf{R}_+^2,$$

where  $d(\cdot, \cdot)$  denotes the usual distance in  $\mathbf{R}^2$ ,  $V_f$  the real curve  $f(x, y) = 0$ . Moreover,  $L_+(f)$  is the smallest number with this property.

We put  $L(f) \equiv \text{Max}\{L_+(f), L_-(f)\}$ , where  $L_-(f) = L_+(f(x, -y))$ , and call  $L(f)$  the Lojasiewicz exponent of  $f$ .

**COROLLARY 1.** *The Lojasiewicz exponent  $L(f)$  is the smallest number having the property that in any given sufficiently small compact neighborhood  $U$  of  $0$  in  $\mathbf{R}^2$ , there exists a constant  $\varepsilon > 0$  such that*

$$(L) \quad |f(x, y)| \geq \varepsilon d((x, y), V_f)^{L(f)}, \quad (x, y) \in U.$$

**COROLLARY 2.** *If all roots of  $f(x, y)$  (respectively  $f(x, -y)$ ) are real, then  $L_+(f) = O(f)$  (respectively  $L_-(f) = O(f)$ ).*

**COROLLARY 3.** *If all roots of  $f(x, y)$  are non-real, then  $L_+(f) = \text{Max}_{1 \leq i \leq m} \{e(f, z_i^*)\}$ .*

**EXAMPLE.**  $f(x, y) = x^2 + y^3$ . Both roots of  $f(x, y)$  are non-real,  $L_+(f) = 3$ . Both roots of  $f(x, -y)$  are real,  $L_-(f) = 2$ . Hence  $L(f) = 3$ .

**COROLLARY 4.**  $L(f), L_+(f), L_-(f)$  are rational numbers.

## §2. Proof of Theorem

*Notations.* For two real-valued functions  $A(x_1, \dots, x_n) > 0$  and  $B(x_1, \dots, x_n) > 0$  defined for  $(x_1, \dots, x_n)$  in a domain  $D$  in  $\mathbf{R}^n$  with  $0 \in \bar{D} - D$ , we write

$$A \gtrsim B \tag{2.1}$$

if there exists a constant  $k > 0$  such that  $kA \geq B$  for all  $(x_1, \dots, x_n)$  in  $D$  near  $0$ . If

$A \gtrsim B \gtrsim A$ , then we write  $A \sim B$ ; this is the case if, and only if  $A/B$  lies between two positive constants,  $x$  near 0.

For a fractional power series  $y^*$  in  $y$ , and for  $d > 0, w > 0$ , a *horn neighborhood* of  $y^*$  of degree  $d$  and width  $w$  is the point set

$$H_d(y^*; w) = \{(x, y) : |x - \bar{y}^*| \leq w |y|^d\},$$

where  $\bar{y}^*$  is  $y^*$  with all terms of degree  $> d$  omitted. We often write  $H_d(y^*)$  instead of  $H_d(y^*; w)$ . This is a horn-shaped set with vertex 0, containing the point set  $x - y^* = 0$ , except the origin, in its interior. The definition given here is slightly different from that in [1] in that this new horn neighborhood is the closure of the old one; in particular, the origin is contained in the new but not in the old.

$$\text{For } (x, y) \in H_d(y^*; w), |x - y^*| > w |y|^d. \tag{2.2}$$

For two fractional power series  $y^*, z^*$  let  $H(y^*, z^*)$  denote the horn neighborhood  $H_d(y^*; w)$  where  $d = O(y^* - z^*)$ ,  $w$  a sufficiently small number. Call  $H(y^*, z^*)$  the horn neighborhood of  $y^*$  against  $z^*$ .

$$H(y^*, z^*) \cap H(z^*, y^*) = \{0\}. \tag{2.3}$$

LEMMA (2.4). *Let  $z^*$  be a given fractional power series (in  $y$ ). For a finite set of fractional power series  $\{y_1^*, \dots, y_s^*\}$  and  $d > 0$ , there is a finite subset  $\Sigma = \Sigma_d(y_1^*, \dots, y_s^*)$  of  $\mathbf{R}$  such that for  $t \notin \Sigma$ ,*

$$O((z^* + ty^d) - y_i^*) \leq d, \quad 1 \leq i \leq s. \tag{2.5}$$

The case  $s = 1$  is quite obvious, the rest of the proof is by an easy induction on  $s$ .

Now consider all real springboards  $z_i^*(t_i)$  of the non-real roots  $z_i^*$  of  $f$ . Let  $d_i$  denote the complex order (see (1.3)) of  $z_i^*$ . By a repeated application of Lemma (2.4), we can choose specified real values for  $t_i$  in  $z^*(t_i)$  such that

$$O(z_i^*(t_i) - z^*) \leq d_i, \quad O(z_i^*(t_i) - z_j^*(t_j)) \leq d_i \quad i \neq j, \tag{2.6}$$

where  $z^*$  is any real root.

EXAMPLE (2.7).  $f(x, y) = (x - y^2)(x^2 + y^4)^2, z_1^* = y^2, z_2^* = z_3^* = iy^2, z_4^* = z_5^* = -iy^2$ .

We may choose any values for  $t_i, 2 \leq i \leq 5$ , provided that  $t_i \neq 1$ , and  $t_i \neq t_j$  for  $i \neq j$ . Note that  $t_2 = t_3$  or  $t_4 = t_5$  are not allowed.

LEMMA (2.8). *Let  $y^* = z^*(t)$  be the real springboard of a non-real fractional*

power series  $z^*$  with complex order  $\leq d$ . Then for  $(x, y) \notin H_d(y^*; w)$ ,  $(x, y) \in R_+^2$ ,

$$|x - y^*| \sim |x - z^*|. \tag{2.9}$$

*Proof.* Let  $c$  be the first non real coefficient of  $z^*$ . First, suppose  $d = O(y^* - z^*) =$  complex order of  $z^*$ . Then

$$|x - z^*| \leq |x - y^*| + |y^* - z^*| \leq |x - y^*| + 2|c - t| |y|^d.$$

By (2.2),  $|x - y^*| > w|y|^d$ , hence

$$|x - z^*| \leq |x - y^*| + (2|c - t|/w) |x - y^*|.$$

Hence

$$|x - z^*| \lesssim |x - y^*|.$$

Now,

$$|x - y^*| \leq |x - z^*| + |z^* - y^*| \leq |x - z^*| + 2|c - t| |y|^d.$$

For  $y > 0$ ,  $y^d$  is real, while  $cy^d$  is non-real. Hence

$$|x - z^*| \geq \frac{1}{2}|c'| |y|^d,$$

where  $c' = \text{Im}(c)$ , and so,

$$\begin{aligned} |x - y^*| &\leq |x - z^*| + (4|c - t|/|c'|) |x - z^*|, \\ |x - y^*| &\lesssim |x - z^*|, \quad \text{proving Lemma (2.8)}. \end{aligned}$$

Next, suppose  $d < O(y^* - z^*)$ . Then

$$\lim_{y \rightarrow 0} |y^* - z^*| |y|^{-d} = 0.$$

Again, (2.9) follows from the triangle inequalities

$$|x - y^*| \leq |x - z^*| + |z^* - y^*|$$

and

$$|x - z^*| \leq |x - y^*| + |y^* - z^*|.$$

From now on,  $z_1^*, \dots, z_m^*$  denote the  $m$  roots of  $f$  (with multiplicity), and let  $y_1^*, \dots, y_m^*$  denote the real roots and the springboards of non-real roots, satisfying (2.6).

That is,  $y_i^* = z_i^*$  if  $z_i^*$  is real, and  $y_i^* = z_i^*(t_i)$  if  $z_i^*$  is not real. All fractional power series are understood to be in  $y$  with order  $\geq 1$ . All points  $(x, y)$  are understood to be in  $\mathbf{R}_+^2$ .

Thus  $H_d(y^*; w) \cap \mathbf{R}_+^2$  will be written simply as  $H_d(y^*; w)$ .

Let  $\mathfrak{F}$  denote the family of horn neighborhoods of the  $y_i^*$ 's against one another:  $\mathfrak{F} = \{H(y_i^*, y_j^*): i \neq j\}$ .

We divide a neighborhood of 0 (in  $\mathbf{R}_2^+$ ) into regions of three types. *Type 1*: Regions which are the smallest members of  $\mathfrak{F}$ . *Type 2*: Those of type  $H - \bigcup_\alpha H_\alpha$ ,  $H, H_\alpha \in \mathfrak{F}$ , where  $H_\alpha$  runs through all members of  $\mathfrak{F}$  contained in  $H$ . *Type 3*: The complements of the union of all members of  $\mathfrak{F}$ .

Inequality  $(L_+)$  will be established in regions of each type.

For a real fractional power series  $w^*$ , let  $V_{w^*}$  denote the point set  $x - w^* = 0$ .

LEMMA (2.10). *Let  $y^*$  be a real fractional power series. Then*

$$d((x, y), V_{y^*}) \sim |x - y^*|. \tag{2.11}$$

This is obvious.

LEMMA (2.12). *Let  $y^*, z^*$  be two real fractional power series. Then for  $(x, y) \notin H(y^*, z^*)$ ,*

$$d((x, y), V_{y^*}) \gtrsim d((x, y), V_{z^*}). \tag{2.13}$$

Consequently, for  $(x, y) \notin H(y^*, z^*) \cup H(z^*, y^*)$ ,

$$d((x, y), V_{y^*}) \sim d((x, y), V_{z^*}). \tag{2.14}$$

*Proof.* To show (2.13), let us first consider the special case  $y^* = 0$ , whence  $V_{y^*}$  is the  $y$ -axis. Write  $z^* = ay^d + \dots$ ,  $a \neq 0$ . Now, by (2.11),  $d((x, y), V_{y^*}) \sim x$ ,  $d((x, y), V_{z^*}) \sim |x - (ay^d + \dots)|$ . For  $(x, y) \notin H(y^*, z^*) = H_d(y^*)$ , where  $d = O(y^* - z^*)$ ,  $|x| > w|y|^d$  by (2.2). Hence

$$|(x - (ay^d + \dots))/x| = |1 - (ay^d + \dots)/x| \leq 1 + 2|a| |y|^d/w|y|^d = 1 + 2|a|/w,$$

and (2.13) follows.

For the general case, we can perform a  $C^1$ -coordinate transformation

$$X = x - y^*, \quad Y = y \tag{2.15}$$

(this transformation is  $C^1$  since  $O(y^*) \geq 1$ ). Then the general case is reduced to the above special case.

LEMMA (2.16). *Let  $y^*, z^*$  be two real fractional series. Then for  $(x, y) \in H(y^*, z^*)$ ,*  
 $d((x, y), V_{z^*}) \sim |y|^\alpha, \quad \alpha = O(y^* - z^*).$  (2.17)

*Proof.* By (2.3), (2.2),

$$|x - z^*| > w|y|^\alpha.$$

Now

$$\begin{aligned} H(y^*, z^*) &= H_\alpha(y^*), \\ |x - z^*| &\leq |x - y^*| + |y^* - z^*| \\ &\leq w|y|^\alpha + |y^* - z^*| \sim |y|^\alpha. \end{aligned}$$

Hence (2.17) follows.

*Type 1.* Consider a smallest member  $H$  of  $\mathfrak{F}$ . Say  $H = H(y_i^*, y_h^*)$ .

First, suppose  $y_i^*$  is the real springboard of a non-real root  $z_i^*$ ,  $y_i^* = z_i^*(t_i)$ .

Let  $d_i$  denote the complex order of  $z_i^*$  [(1.3)].

We claim that  $H = H_{d_i}(y_i^*)$ . Indeed, the complex conjugate  $\bar{z}_i^*$  of  $z_i^*$  is another root of  $f$ , since  $f$  is real. Say  $\bar{z}_i^* = z_k^*$ . Now  $O(z_k^* - z_i^*) = O(y_k^* - y_i^*) = d_i$ , where  $y_k^* = z_k^*(t_k)$ , and so  $H_{d_i}(y_i^*) = H(y_i^*, y_k^*)$  is a member of  $\mathfrak{F}$ . Since  $H$  is a smallest member, we must have  $H \subset H_{d_i}(y_i^*)$ . By (2.6),  $H_{d_i}(y_i^*)$  is a smallest member of  $\mathfrak{F}$ ; hence  $H = H_{d_i}(y_i^*)$ .

Now, for  $(x, y) \in H$ ,  $|x - z_i^*| \sim |y|^{d_i}$ ; moreover, for any  $j \neq i$ ,  $|x - y_j^*| \sim |y|^{\alpha_j}$ , where  $\alpha_j = O(y_j^* - y_i^*)$ , by (2.17). If  $y_j^*$ ,  $j \neq i$ , is the real springboard of a non-real root  $z_j^*$ , then we have  $|x - z_j^*| \sim |x - y_j^*|$ , by Lemma (2.8). Hence

$$|f(x, y)| = \prod_j |x - z_j^*| \sim |y|^{e(f, y^*)}, \tag{2.18}$$

since  $e(f, y_i^*) = d_i + \sum_{j \neq i} \alpha_j$ .

Now,  $V_f$  is defined by  $\prod_j (x - y_j^*) = 0$  where  $y_j^*$  runs through all real roots. Hence

$$d((x, y), V_f) = \text{Min}_j \{d((x, y), V_{y_j^*})\}.$$

For  $(x, y) \in H = H_{d_i}(y_i^*)$ ,

$$d((x, y), V_{y_j^*}) \sim |x - y_j^*| \sim |y|^{\alpha_j}$$

where

$$\alpha_j = O(y_i^* - y_j^*) = O(y_i^* - z_j^*)$$

(See (2.6)). Hence

$$d((x, y), V_f) \sim |y|^{\delta(f, y_i^*)}. \tag{2.19}$$

By (2.18), (2.19), we have, for  $(x, y) \in H_{d_i}(y_i^*)$

$$|f(x, y)| \sim d((x, y), V_f)^{\gamma_i} \tag{2.20}$$

where  $\gamma_i = e(f, y_i^*)/\delta(f, y_i^*)$ .

Next, suppose  $y_i^*$  is a real root. As  $y_i^*$  can be a multiple root, let  $\mu$  denote its multiplicity. Hence

$$|f(x, y)| \sim |x - y_i^*|^\mu \prod_j |x - z_j^*|, \tag{2.21}$$

where  $z_j^*$  runs through all roots other than  $y_i^*$ . For  $(x, y) \in H = H(y_i^*, y_h^*)$ , we claim that

$$d((x, y), V_f) \sim d((x, y), V_{y_i^*}) \sim |x - y_i^*|. \tag{2.22}$$

Indeed, since  $H$  is a smallest member,  $H$  is contained in  $H(y_i^*, y_s^*)$ , and is therefore disjoint from  $H(y_s^*, y_i^*)$ , where  $y_s^*$  is any real root other than  $y_i^*$ . Then the first  $\sim$  of (2.22) follows from (2.13), the second  $\sim$  follows from (2.11). Moreover, for  $(x, y) \in H$ , and for  $j \neq i$ ,

$$|x - z_j^*| \sim |x - y_j^*| \gtrsim |x - y_i^*|, \tag{2.23}$$

the first relation follows from Lemma (2.8), and the last relation follows from (2.13). Now, by (2.21), (2.23) and (2.22),

$$|f(x, y)| \gtrsim |x - y_i^*|^m \sim d((x, y), V_f)^m, \quad m = O(f). \tag{2.24}$$

*Remark.* We may not replace  $\gtrsim$  by  $\sim$  in (2.24). See Example (3.6) in §3. However, for  $(x, y)$  in a sector  $S_\eta = \{(x, y) : |y| \leq \eta|x|\}$  we do have

$$|f(x, y)| \sim d((x, y), V_f)^m. \tag{2.25}$$

In fact, since the initial form of  $f$  is not divisible by  $y$ , for  $(x, y) \in S_\eta$ ,  $\eta$  sufficiently small,

$$|f(x, y)| \sim \varrho^m, \quad \varrho = (x^2 + y^2)^{1/2} \sim |x|,$$

and

$$d((x, y), V_f) \sim |x| \sim \varrho.$$

Hence we have (2.25).

*Type 2.* The region is of the form  $H - \bigcup_\alpha H_\alpha$ . Collect all  $y_i^*$  for which  $V_{y_i^*} \subset H$ . By permutting the indices, if necessary, we may assume they are  $y_1^*, \dots, y_k^*$ . Then

$V_{y_q^*} \not\subset H$ ,  $q > k$ . For  $i \leq k$ ,  $V_{y_i^*} \subset H$ , then  $H = H_{d_i}(y_i^*)$  for some  $d_i$ . Hence for  $q > k$ ,  $H(y_q^*, y_i^*) \cap H = \{0\}$ .

First, suppose none of the  $y_i^*$ 's,  $1 \leq i \leq k$ , is a real root. Choose a fixed  $y_i^*$ , say  $y_1^*$ . For any  $j$ ,  $1 < j \leq k$ ,  $H(y_1^*, y_j^*)$  and  $H(y_j^*, y_1^*)$  are disjoint proper subsets of  $H$ . Hence, by (2.11), (2.14),

$$|x - y_1^*| \sim |x - y_j^*|, \quad \text{for } (x, y) \in H - \bigcup_{\alpha} H_{\alpha}.$$

By (2.2),

$$|x - y_1^*| \gtrsim |y|^{\alpha_j}, \quad \alpha_j = O(y_1^* - y_j^*), \quad 1 < j \leq k.$$

Moreover, by (2.6),

$$O(y_1^* - y_j^*) = O(y_1^* - z_j^*) \quad 1 < j \leq k.$$

Hence, by Lemma (2.8),

$$\prod_{i=1}^k |x - z_i^*| \sim \prod_{i=1}^k |x - y_i^*| \gtrsim |y|^{\alpha},$$

where  $\alpha = \sum_{i=1}^k O(y_1^* - z_i^*)$ .

Now, for any  $q > k$ ,  $H(y_q^*, y_1^*) \cap H = \{0\}$ . Hence for  $(x, y) \in H$ ,

$$|x - z_q^*| \sim |x - y_q^*| \geq |y|^{\alpha_q}, \quad \alpha_q = O(y_1^* - y_q^*) = O(y_1^* - z_q^*).$$

Thus, for  $(x, y) \in H$ ,

$$|f(x, y)| \gtrsim |y|^{e(f, y_1^*)}. \tag{2.26}$$

We now show that

$$d((x, y), V_f) \sim |y|^{\delta(f, y_1^*)} \tag{2.27}$$

and then  $(L_+)$  follows.

In case  $f$  has no real root,  $V_f = \{0\}$ ,  $\delta(f, y_1^*) = 1$  and (2.27) is obvious.

Now let  $y_s^*$ ,  $s > k$ , be any real root. We claim that  $H \subset H(y_1^*, y_s^*)$ . Indeed,  $H$  is a horn neighborhood of  $y_1^*$ , say of degree  $d_1$ . If  $d_1 < O(y_1^* - y_s^*)$ , then we would have  $V_{y_s^*} \subset H$ , a contradiction. Therefore  $d_1 \geq O(y_1^* - y_s^*)$ ,  $H \subset H(y_1^*, y_s^*)$ . By Lemma (2.16),

$$d((x, y), V_{y_s^*}) \sim |y|^{\alpha_s}, \quad \alpha_s = O(y_1^* - y_s^*). \tag{2.28}$$

Since  $y_s^*$  is any real root, (2.27) follows.

Next, suppose some  $y_i^*$ ,  $1 \leq i \leq k$ , is a real root. Say  $i = 1$ . By (2.11) and (2.14),

$$|x - y_1^*| \sim |x - y_j^*| \quad 2 \leq j \leq k.$$

For  $q > k$ ,

$$|x - y_q^*| \gtrsim |x - y_1^*|$$

by (2.13). Therefore,

$$|f(x, y)| = \prod_{j=1}^m |x - z_j^*| \sim \prod_j |x - y_j^*| \gtrsim |x - y_1^*|^m \sim d((x, y), V_f)^m.$$

We have again established  $(L_+)$  in this case.

*Type 3.* First, suppose  $f$  has at least one real root. Say  $y_1^*$  is a real root. For  $(x, y) \notin \bigcup_{i,j} H(y_i^*, y_j^*)$ ,

$$|x - y_i^*| \sim |x - y_j^*|, \quad \text{for all } i, j. \quad (2.29)$$

Moreover,

$$|f(x, y)| = \prod_i |x - z_i^*| \sim \prod_i |x - y_i^*| \sim |x - y_1^*|^m.$$

Now,

$$d((x, y), V_f) \sim \min_i \{|x - y_i^*|\} \sim |x - y_1^*|$$

by (2.29), where  $y_i^*$  runs through all real roots.

Therefore we have

$$|f(x, y)| \sim d((x, y), V_f)^m. \quad (2.30)$$

Finally, suppose there is no real root. We still have (2.29). Since  $(x, y) \notin H(y_i^*, y_j^*)$  for all  $i, j$ ,

$$|x - y_i^*| \gtrsim |y|^{\alpha_j} \quad \text{where } \alpha_j = O(y_i^* - y_j^*), \quad j \neq i.$$

Hence

$$|f(x, y)| \gtrsim |y|^{e(y_i^*, f)}, \quad 1 \leq i \leq m.$$

Again, we have proved  $(L_+)$ .

To complete the proof of the theorem, it remains to show that  $L_+(f)$  is the smallest number having the property  $(L_+)$ . This follows from (2.20) and (2.25).

*Proof of the Corollaries.* Corollary 1 follows immediately. Corollary 2 is obvious. Corollary 4 follows from Puiseux's Theorem; the exponents of the roots  $z_i^*$  are rational numbers (with a same denominator). Corollary 3 follows from the fact that  $O(z_i^*) \geq 1$  for all  $i$ , and hence  $e(f, z_i^*) \geq O(f)$ .

**§3. Illustrative Examples**

For two arcs  $\gamma: x=y^*$ ,  $\beta: x=z^*$ , call  $d(\gamma, \beta) = O(y^* - z^*)$  the degree of contact of  $\gamma$  and  $\beta$ .

For a real arc  $\gamma: x=y^*$ , the Lojasiewicz exponent of  $f$  along  $\gamma$ ,  $l_f(\gamma)$ , is defined by

$$|f(y^*, y)| \sim d((y^*, y), V_f)^{l_f(\gamma)}.$$

In particular, if  $f(x, y)$  is positive definite, then

$$f(y^*, y) \sim |y|^{l_f(\gamma)}.$$

EXAMPLE (3.1).  $f(x, y) = x^2 + y^{10}$ . Both roots are non-real. The real springboards of the roots are  $\gamma_i: x = t_i y^5$ ,  $i = 1, 2$ .

For any real arc  $\beta$ ,

$$l_f(\beta) = \begin{cases} 10 & \text{if } d(\gamma_i, \beta) \geq 5 \\ 2d(\gamma_i, \beta) & \text{if } d(\gamma_i, \beta) \leq 5. \end{cases}$$

(3.2). As  $\beta$  varies so that  $d(\gamma_i, \beta)$  increases,  $l_f(\beta)$  increases.

The maximal value of  $l_f(\beta)$  is 10 and is taken when  $d(\gamma_i, \beta) \geq 5$ .

Observe that  $L(f) = 10$  by Corollary 1.

A phenomenon similar to (3.2) appears in the next example.

EXAMPLE (3.3).  $f(x, y) = (x^2 + y^{10})((x - y^3)^2 + y^{40})$ .

Consider the real springboard  $\gamma_1: x = ty^5$ , arising from the first factor, we have  $l_f(\gamma_1) = 16$ . Let us perturb  $\gamma_1$  to  $\beta: x = ty^5 + (sy^d + \text{terms of degree } > d)$ , where  $s \neq 0$ ,  $|s|$  small. For  $d = d(\gamma_1, \beta)$  varies in the range  $1 \leq d < \infty$ ,

$$l_f(\beta) = \begin{cases} 16 & \text{if } d \geq 5 \\ 2d + 6 & \text{if } 5 \geq d \geq 3 \\ 4d & \text{if } 3 \geq d \end{cases}$$

(3.4). As  $d(\gamma_1, \beta)$  increases,  $l_f(\beta)$  increases.

Observe that  $l_f(\gamma_1) = 16$  is a maximal value, which is reached when  $d \geq 5$ .

Now consider  $\gamma_2: x = y^3 + ty^{20}$ , the real springboard of a root of the second factor.

For any  $\beta : x = (y^3 + ty^{20}) + (sy^d + \dots)$ , we have

$$l_f(\beta) = \begin{cases} 46 & \text{if } d \geq 20 \\ 6 + 2d & \text{if } 20 \geq d \geq 3 \\ 4d & \text{if } 3 \geq d \end{cases}$$

(3.5). Again, as  $d = d(\gamma_2, \beta)$  increases,  $l_f(\beta)$  increases.

Observe that  $l_f(\gamma_2) = 46$  is a maximal value. Also, by Corollary 1,  $L(f) = 46$ .

When  $f(x, y)$  has real roots, the way  $l_f$  varies near a real root is quite different from that near a non-real root as in the last two examples.

EXAMPLE (3.6).  $f(x, y) = (x - y^2)(x^4 + y^{10})$ .

Consider the value of  $l_f(\beta)$ , where

$$\beta : x = y^2 + (sy^d + \dots), \quad s \neq 0.$$

We have, along  $\beta$ ,

$$|f(x, y)| = \begin{cases} |y|^{d+10} & d \geq 5/2 \\ |y|^{5d} & 5/2 \geq d, \end{cases}$$

$$d((x, y), V_f) \sim |y|^d.$$

Hence

$$l_f(\beta) = \begin{cases} 1 + 10/d & d \geq 5/2 \\ 5 & 5/2 \geq d. \end{cases}$$

Now observe that in contrast with (3.2), (3.4) and (3.5),  $l_f(\beta)$  decreases as  $d$  increases.

The maximal value of  $l_f$  over arcs of type  $\beta$  is 5. Note that  $O(f) = 5$ . However, the maximal value of  $l_f$  near a real springboard of either root of the second factor is 12; and  $L(f) = 12$ .

In this example,  $e(f, \gamma_i^*)/\delta(f, \gamma_i^*) = 12 > O(f) = 5$ .

It is not true, however, that for general  $f$ ,  $e/\delta \geq O(f)$ .

EXAMPLE (3.7).  $f(x, y) = x(x - y^q)(x^2 + y^{2p})$ ,  $p > q$ . Then  $e = p + q + 2p$ ,  $\delta = p$ ,  $O(f) = 4$ , and  $e/\delta < O(f)$ .

To close this section, we give an example due to Lojasiewicz, which shows that for a polynomial  $f$  of degree  $n$ , one can have  $L(f) > n$ .

EXAMPLE (3.8). ([2], p. 85).

$$f(x, y) = x^{2n} + (x - y^n)^2 = (x - y^n + ix^n)(x - y^n - ix^n).$$

The roots are  $z_1^*, z_2^* : x = y^n \pm iy^{n^2} + \dots$ . We have  $e(f, z_i^*) = 2n^2 = L(f)$ .

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