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Poincaré Algebras Modulo an odd Prime

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§1. Introduction

Being given a closed oriented manifold M, of dimension n, and an odd prime p, the mod$p$ cohomology of M, $H^*(M; Z_p)$, forms the generic example of an n-dimensional Poincaré algebra over the mod$p$ Steenrod algebra A.

If M is the boundary of a compact oriented $(n+1)$-dimensional manifold with boundary V, one obtains an exact triangle

$$\begin{array}{ccc} H^*(V; Z_p) & \xrightarrow{i} & H^*(M; Z_p) \\ & {}^{j}\nwarrow \quad \swarrow{}^{\delta} & \\ & H^*(V, M; Z_p) & \end{array}$$

which is the generic example of a $(n+1)$-dimensional Lefschetz algebra over A. Abstracting the properties involved, one may form a cobordism group Ω_*^p, where Ω_n^p is a set of equivalence classes of n-dimensional Poincaré algebras over A in which the boundaries of $(n+1)$-dimensional Lefschetz algebras are zero.

The purpose of this note is to analyze Poincaré algebras over A and to determine Ω_*^p. In essence, this follows the work of J. F. Adams [1] who studied the characteristic ring of Poincaré algebras and Brown and Peterson [2] who studied Poincaré algebras over Z_2.

There is another approach to Poincaré algebras, completely unrelated to this one which may be found in A. S. Miščenko: *Homotopy invariants of nonsimply connected manifolds, I Rational Invariants*, Math USSR-Izvestia, **4** (1970), 506–519.

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§2. Axiomatics

Throughout this paper all algebras and modules will be graded, will be of finite type, and will have Z_p, p a fixed odd prime, as ground field. The mod$p$ Steenrod algebra will be denoted A, with β denoting the Bockstein and $\mathscr{P}^i$ the i-th reduced p-th power.

A left A module X is said to be unstable if $\mathscr{P}^i x=0$ for $x \in X^j$ and $2i>j$. X is a left algebra over the Hopf algebra A if X is a commutative algebra (in the graded sense) and a left A module so that the Cartan formulae hold; i.e.

$$\beta(xy) = \beta x \cdot y + (-1)^{\deg x} x \cdot \beta y$$

$$\mathscr{P}^i(xy) = \sum_{j=0}^{i} \mathscr{P}^j x \cdot \mathscr{P}^{i-j} y$$

(with $\mathscr{P}^0 x = x$). X is an unstable left algebra over A if it is an unstable left A module, a left algebra over the Hopf algebra A, and if $\mathscr{P}^i x = x^p$ if $x \in X^j$ and $2i = j$. Thus, an unstable left algebra over A is the algebraic analogue of the mod$p$ cohomology of a topological pair.

The analogue of the mod$p$ cohomology of a space is an unstable left algebra over A with unit. (*Note*: If the unit coincides with 0, the space is the empty set).

LEMMA 2.1. *If X is an unstable left algebra over A with unit, then X° has a basis over Z_p consisting of the minimal idempotents.*

Proof. $e \in X^\circ$ is an idempotent if $e^2 = e$. If e, f are idempotents, $e \leqslant f$ if $e = ef$, where ef is idempotent. More generally, if e, f are idempotents, so are ef and $e - ef$. By finite dimensionality of X°, one may write $1 = e_1 + \cdots + e_r$ where the e_i are the minimal idempotents, and are linearly independent. X then decomposes as the direct sum $X = e_1 X \oplus \cdots \oplus e_r X$ as algebras over A. To show that the e_i span X°, consider the case in which X has a unique minimal idempotent 1 (i.e. any $e_i X$ which has unique idempotent e_i which is its unit). If $u \in X^\circ$, $u = \mathscr{P}^\circ u = u^p$, and so $(u^{p-1})^2 = u^{p-2} \cdot u^p = u^{p-1}$. If $u \neq 0$, $u = u^p = u \cdot u^{p-1} \neq 0$ and so $u^{p-1} \neq 0$, so that $u^{p-1} = 1$ being an idempotent. If 1, $x \in X^\circ$ are linearly independent and $t \in Z_p$, the equations $(1 + tx)^{p-1} = 1$ give

$$\sum_{j=1}^{p-1} \binom{p-1}{j} t^j x^j = 0$$

for $t = 1, \ldots, p-1$. Since the Vandermonde determinant is non-singular, $\binom{p-1}{j} x^j = 0$ for each j, so $x = 0$, giving a contradiction, and 1 spans X°. *

Note. If $\dim x = 0$, $\beta x = 0$, for $x^p = \mathscr{P}^\circ x = x$ and $\beta x = \beta x^p = p x^{p-1} \cdot \beta x = 0$. This fact will be used without further mention.

While not much explicit use of this will be made, it will be implicit in most of what follows. It is the algebraic analogue of the decomposition of a space into its components.

An n-dimensional Poincaré algebra M is an unstable left algebra over A together with a homomorphism $\phi_M : M^n \to Z_p$ so that

$$M^i \otimes M^{n-i} \to Z_p : x \otimes y \to \phi_M(x \cdot y)$$

defines a non-singular pairing for each i. (*Note*: $M^i = 0$ for $i < 0$ since $\mathscr{P}^\circ x = x$ and $\mathscr{P}^\circ x = 0$ if $x \in M^i$ $i < 0$, and from the pairing $M^i = 0$ for $i > n$. Further, M has a unit for there is an element 1 so that $\phi_M(1 \cdot x) = \phi_M(x)$ for all x in M^n).

An $(n+1)$ dimensional Lefschetz algebra is a 8-tuple $(M, M', M'', i, j, \delta, \phi_M, \phi_{M''})$ in which M, M', M'' are unstable left algebras over A and M'' is an M' module for which

$$\beta(m'm'') = \beta m' \cdot m'' + (-1)^{\deg m'} m' \cdot \beta m''$$
$$\mathscr{P}^i(m'm'') = \sum_{j=0}^{i} (\mathscr{P}^j m') \cdot (\mathscr{P}^{i-j} m'')$$

for $m' \in M'$, $m'' \in M''$. Also $i: M' \to M$, $j: M'' \to M'$, $\delta: M \to M''$ are homomorphisms of left A modules of degrees 0, 0, 1 respectively, such that

$$\begin{array}{ccc} M' & \xrightarrow{i} & M \\ & {}_j\nwarrow \quad \swarrow_\delta & \\ & M'' & \end{array}$$

is exact. In addition, i and j are algebra homomorphisms, $j(m'm'') = m'j(m'')$, $m_0'' \cdot m_1'' = j(m_0'') \cdot m_1''$ (for $m_i'' \in M''$) and $\delta(im' \cdot m) = (-1)^{\deg m'} m' \cdot \delta m$. (Using commutativity rules this could be written $\delta(m \cdot im') = (\delta m) \cdot m'$, except that M' is chosen to multiply on the left of M'' – one may easily introduce right multiplication by using the sign convention). Finally $\phi_M: M^n \to Z_p$ and $\phi_{M''}: M''^{n+1} \to Z_p$ are homomorphisms so that $\phi_M = \phi_{M''}\delta$ with ϕ_M making M an n-dimensional Poincaré algebra and $\phi_{M''}$ defining a dual pairing

$$M'^i \otimes M''^{n+1-i} \to Z_p : x \otimes y \to \phi_{M''}(x \cdot y)$$

for each i.

Notes. 1) Brown and Peterson would not assume M'' to be an algebra, but the multiplication can be defined by $m_0'' m_1'' = j(m_0'') \cdot m_1''$. There is no unit, however, which they assume for algebras.

2) $\phi_{M''}: M''^{n+1} \to Z_p$ gives a unit in M'^0 by $\phi_{M''}(1 \cdot y) = \phi_{M''}(y)$ and $i1 = 1$.

Hopefully, this has described all of the properties needed for the algebraic formalism. There is one property of oriented manifolds or Poincaré duality spaces which is definitely to be avoided: specifically, if M^n is an oriented n-manifold, $\beta: H^{n-1}(M; Z_p) \to H^n(M; Z_p)$ is the zero homomorphism. This follows from the underlying integral structure which will *not* be assumed.

§3. Right Action and the Characteristic Ring

Following J. F. Adams, one now defines a right action of A in an n-dimensional Poincaré algebra M. Given $x \in M^i$ and $\alpha \in A^j$, $x\alpha \in M^{i+j}$ is the unique class so that for all $y \in M^{n-(i+j)}$,

$$\phi_M((x\alpha) \cdot y) = \phi_M(x \cdot (\alpha y)).$$

Following Brown and Peterson, one may define a right action of A on M' for an $(n+1)$-dimensional Lefschetz algebra $(M, M', M'', i, j, \delta, \phi_M, \phi_{M''})$. Given $x \in M'^i$ and $\alpha \in A^j$, $x\alpha \in M'^{i+j}$ is the unique class so that for all $y \in M''^{(n+1)-(i+j)}$,

$\phi_{M''}((x\alpha)\cdot y) = \phi_{M''}(x\cdot(\alpha y))$.

LEMMA 3.1 *$i: M' \to M$ is a homomorphism of right A modules.*

Proof. If $x \in M'^i$, $\alpha \in A^j$, $y \in M^{n-(i+j)}$,

$$\begin{aligned}\phi_M(i(x\alpha)\cdot y) &= \phi_{M''}\delta(i(x\alpha)\cdot y) = \phi_{M''}((-1)^{i+j}(x\alpha)\,\delta y)\\ &= (-1)^{i+j}\,\phi_{M''}(x\cdot(\alpha\delta y)) = (-1)^{i+j}\,\phi_{M''}(x\cdot(-1)^j\,\delta\alpha y)\\ &= (-1)^i\,\phi_{M''}(x\cdot\delta\alpha y) = \phi_{M''}(\delta(ix\cdot\alpha y))\\ &= \phi_M((ix)\cdot\alpha y) = \phi_M(((ix)\,\alpha)\cdot y),\end{aligned}$$

so $i(x\alpha) = (ix)\,\alpha$. *

Returning to Adams, one defines a class of "words" W, by the rules:

1) The letter $\mathscr{E}$ is a word.
2) If w is a word and $\alpha \in A$, then αw and $w\alpha$ are words.
3) If w, w' are words, then ww' is a word.
4) If w, w' are words and $\lambda, \mu \in Z_p$, then $\lambda w + \mu w'$ is a word.

Being given a left algebra H over the Hopf algebra A, with unit 1_H, which is also a right A module, one may define a function $\theta_H: W \to H$ by the obvious rules:

1) $\theta_H(\mathscr{E}) = 1_H$
2) $\theta_H(\alpha w) = \alpha\theta_H(w)$, $\theta_H(w\alpha) = \theta_H(w)\,\alpha$
3) $\theta_H(ww') = \theta_H(w)\cdot\theta_H(w')$
4) $\theta_H(\lambda w + \mu w') = \lambda\theta_H(w) + \mu\theta_H(w')$.

One now divides W into equivalence classes by letting w be equivalent to w' if $\theta_M(w) = \theta_M(w')$ for every Poincaré algebra M. The equivalence classes form the elements of a universal domain U. Denote by $\varrho: W \to U$ the function assigning to w its equivalence class.

It is easy to see that U is a graded algebra over Z_p, with both left and right A actions and is an unstable left algebra over A.

The function $\theta_M: W \to M$, M a Poincaré algebra, clearly defines a homomorphism, preserving all structure, $\theta'_M: U \to M$.

LEMMA 3.2. *If $(M, M', M'', i, j, \delta, \phi_M, \phi_{M''})$ is a Lefschetz algebra, the function $\theta_{M'}: W \to M'$ induces a homomorphism $\theta'_{M'}: U \to M'$ with $\theta'_M = i\theta'_{M'}$.*

Proof. One forms the analogue of the double of a manifold with boundary by letting $L = M' \oplus M''$ with $(m'_0, m''_0).\,(m'_1, m''_1) = (m'_0 m'_1,\ m'_0\cdot m''_1 + m''_0\cdot j(m''_1) + m''_0\cdot m'_0)$ where as before $m''_0\cdot m'_1 = (-1)^{\deg m''_0 \deg m'_1}\, m'_1\cdot m''_0$, with $\alpha(m'_0, m''_0) = (\alpha m'_0, \alpha m''_0)$, $\phi_L(m', m'') = \phi_{M''}(m'')$, making L into a Poincaré algebra. Then $r: L \to M': (m', m'')$

$\to m'$ is a homomorphism, preserving all structure, so that $\theta_{M'}$ factors through θ_L (i.e. $\theta_{M'} = r\theta_L$) and one may let $\theta'_{M'} = r\theta'_L$. *

Notes. 1) M'' actually admits a right A action and in L, $(m', m'')\alpha = (m'\alpha, m''\alpha)$.

2) Thought of as doubling, the double of V is the boundary of $V \times [0, 1]$. The maps $V \times 0 \to \partial(V \times [0, 1]) \to V \times [0, 1]$ induce r and the inclusion of M' in L, while $\partial(V \times [0, 1]) \to (\partial(V \times [0, 1]), V \times 0)$ induces the inclusion of M'' in L.

The algebra U will be called the algebra of characteristic classes. Given a Poincaré algebra M or Lefschetz algebra, the image of θ'_M or $\theta'_{M'}$ will be called the characteristic ring. If M is an n-dimensional Poincaré algebra and $u \in U^n$, $\phi'_M(\theta_M(u))$ will be called the characteristic number of M associated with u.

§4. On Being a Boundary

Let M be an n-dimensional Poincaré algebra. If there is a Lefschetz algebra $(M, M', M'', i, j, \delta, \phi_M, \phi_{M''})$ for which M is the lead term, then M will be said to bound.

LEMMA 4.1 *If M bounds, there is a homogeneous subalgebra $R \subset M$ closed under left and right A action and containing the unit for which R is its own annihilator* $R^\perp = \{m \in R \mid \phi_M(rm) = 0 \forall r \in R\}$.

Proof. Let $R = iM'$. Then R is the direct sum of its subspaces $R^j = R \cap M^j$ (i.e. is homogeneous) and is a subalgebra closed under left and right A actions (since i is a homomorphism preserving the actions) and containing the unit. Further if $r, r' \in R$, $r = im'_0$, $r' = im'_1$, $\phi_M(rr') = \phi_{M''}\delta(im'_0 \cdot im'_1) = \phi_{M''}\delta i(m'_0 m'_1)$ but $\delta i = 0$, so $r' \in R^\perp$; i.e. $R \subset R^\perp$. If $m \in R^\perp$, then for all $m' \in M'$, $\phi_{M''}(m'\delta m) = \phi_{M''}(\pm\delta(im' \cdot m)) = \pm\phi_M(im' \cdot m) = 0$, so $\delta m = 0$ and $m \in R = \text{image } i$. *

LEMMA 4.2. *If M contains a homogeneous subalgebra R closed under left and right A action and containing the unit for which R is its own annihilator, then M bounds.*

Proof. Being given R, let $M' = R$ and $i: M' \to M$ the inclusion. Let $(M'')^i = M^{i-1}/R^{i-1}$ and $\delta: M \to M''$ the map $M^i \to M^i/R^i$ obtained by the quotient map. Let $j: M'' \to M'$ be the zero homomorphism. Then for $r \in R$, $\phi_M(r) = \phi_M(1 \cdot r) = 0$ for $1 \in R$ and $r \in R^\perp$, so ϕ_M induces a homomorphism $\phi_{M''}: (M'')^{n+1} = M^n/R^n \to Z_p$ with $\phi_M = \phi_{M''}\delta$. The required properties are easily verified. *

Notes. 1) In M'' as constructed, the product is trivial; i.e. $x \cdot y = 0$ for all x, y. This is produced by $x \cdot y = j(x) \cdot y$ and $j(x) = 0$. The module structure of M'' as M' module comes from M'' as M module and is non-trivial – M' is *not* an ideal.

2) This Lemma is the crux of the Brown-Peterson argument. It is very well hidden.

LEMMA 4.3. *If M bounds, then all characteristic numbers of M are zero. Further* *if $n=\dim M$ is even, there is a subspace $B\subset M^{n/2}$ which is its own annihilator with B containing the characteristic classes of dimension $n/2$.*

Proof. If M is the boundary of $(M, M', M'', i, j, \phi_M, \phi_{M''})$, then every characteristic class lies in $i(M')$, and is killed by ϕ_M. Letting $B=i(M'^{n/2})$ gives the second part. *

LEMMA 4.4. *If M is a Poincaré algebra of positive dimension having all characteristic numbers zero and letting $n=\dim M$, there is a subspace $B\subset M^{n/2}$ which is its own annihilator with B containing the characteristic classes of dimension $n/2$, then M* bounds.

Proof. Let $R\subset M$ be defined as follows. If $i<n/2$, R^i is the i-dimensional part of characteristic subring of M. If $i>n/2$, R^i is the annihilator of R^{n-i}. If $i=n/2$, $R^i=B$.

CLAIM. *R is a homogeneous subalgebra which is its own annihilator containing the unit, and closed under left and right A action. Thus M bounds.*

To see this, let $\chi^i\subset M^i$ be the image $\theta_M(U^i)$; i.e. the characteristic classes of dimension i. Clearly R is homogeneous and its own annihilator. Since $1=\theta'_M(\mathscr{E})$, R contains the unit. Every other step requires a tedious case by case check, and proceeds as follows.

Step 1. $\chi^i\subset R^i$ for all i. If $i\leqslant n/2$, this is by definition. If $i>n/2$, it follows from the fact that χ is self annihilating (i.e. $\chi^i\cdot\chi^{n-i}\subset\chi^n$ and $\phi_M(\chi^n)=0$).

Step 2. $B\subset(\chi^{n/2})^\perp$. Since $\chi^{n/2}\subset B$, taking annihilators gives $B=B^\perp\subset(\chi^{n/2})^\perp$.

Step 3. R is a subalgebra; i.e. $R^j\cdot R^i=R^i\cdot R^j\subset R^{i+j}$ $(i\leqslant j)$. If $i, j<n/2$, $R^i\cdot R^j\subset\chi^{i+j}$ and $\chi^{i+j}\subset R^{i+j}$. If $i=0, j=n/2$ $R^\circ\cdot B=\chi^\circ\cdot B$ and $\chi^\circ\cong Z_p$ with base the unit, so $\chi^\circ B\subset B$. If $0<i<n/2$, $j=n/2$ or $i<n/2$, $j>n/2$, then $R^i\cdot R^j\subset\chi^i\cdot(\chi^j)^\perp$ and $\chi^i\cdot(\chi^\perp)^j\subset(\chi^\perp)^{i+j}\subset R^{i+j}$.

(*Note.* If $x\in\chi^i$, $y\in(\chi^\perp)^j$, then for $z\in\chi^{n-(i+j)}$, $\phi_M((xy)z)=\phi_M((xy)z)=\pm\phi_M(y(xz))=0$ for $xz\in\chi$). If $i, j=n/2$, B is self annihilating so $B\cdot B\subset\{1\}^\perp=(R^\circ)^\perp=R^n$. If $i=n/2, j>n/2$, $M^{i+j}=0$ and similarly, if $i, j>n/2$, $M^{i+j}=0$.

Step 4. R is closed under left action of A; i.e. $A^i\cdot R^j\subset R^{i+j}$. If $j<n/2$, $A^iR^j=A^i\cdot\chi^j\subset\chi^{i+j}\subset R^{i+j}$ and if $i=0$, $j=n/2$, $A^\circ\cong Z_p$ with base $\mathscr{P}^\circ=1$, so $A^\circ\cdot B\subset B$. If $i>0, j=n/2$, then $A^i\cdot R^j\subset A^i(\chi^\perp)^j\subset\chi^{\perp i+j}\subset R^{i+j}$ (if $\alpha\in A^i, x\in(\chi^\perp)^j$, and $y\in\chi^{n-(i+j)}$, $\phi_M((\alpha x)y)=\pm\phi_M(x\cdot y\alpha)$ but $y\alpha\in\chi^{n-j}$ so this is 0).

Step 5. R is closed under right action of A; i.e. $R^j\cdot A^i\subset R^{i+j}$. If $j<n/2$, $R^j\cdot A^i=\chi^jA^i\subset\chi^{i+j}\subset R^{i+j}$ while if $i=0$, $j=n/2$, $A^\circ\cong Z_p$ with base $\mathscr{P}^\circ=1$ so $B\cdot A^\circ\subset B$. If $i>0$ $j=n/2$ or $j>n/2$, $R^j\cdot A^i\subset(\chi^\perp)^j\cdot A^i\subset(\chi^\perp)^{i+j}\subset R^{i+j}$ (if $\alpha\in A^i$, $x\in(\chi^\perp)^j$ and $y\in\chi^{n-(i+j)}$, $\phi_M(x\alpha\cdot y)=\phi_M(x\cdot\alpha y)$ but $\alpha y\in\chi^{n-j}$ so this is 0). *

Note. If $n=0$ there is an analogous statement: namely, M must contain a *subalgebra B* with the unit which is its own annihilator.

To determine when M bounds thus rests on knowing when one can find $B\subset M^{n/2}$

which is its own annihilator and which contains the self annihilating subspace $\chi^{n/2}$.

The easiest case is when $n\equiv 2$ (mod 4). Then $M^{n/2}\otimes M^{n/2}\to Z_p: x\otimes y\to\phi_M(xy)$ is a nondegenerate skew symmetric ($yx=-xy$) bilinear form. Being given $\chi^{n/2}$ which is self annihilating, choose a base $e_1,\ldots,e_r$ for $\chi^{n/2}$ and find $f_1,\ldots,f_r$ in $M^{n/2}$ with $\phi_M(e_ie_j)=0$, $\phi_M(f_if_j)=0$, $\phi_M(e_if_j)=0$ if $i\neq j$ and 1 if $i=j$. (Since $M^{n/2}\to\operatorname{Hom}(\chi^{n/2}; Z_p)$ is epic, one may find f_1. Supposing $f_1,\ldots,f_s$ found, let T be spanned by $e_1,\ldots,e_r,f_1,\ldots,f_s$ and then $M^{n/2}\to\operatorname{Hom}(T;Z_p)$ is epic to find f_{s+1}. Let V be spanned by $e_1,\ldots,e_r,f_1,\ldots,f_r$ and let $V^\perp=\{x\in M^{n/2}\mid\phi_M(xy)=0 \text{ for all } y\in V\}$. Then $V^\perp\otimes V^\perp\to\to Z_p$ is a nondegenerate pairing. If $V^\perp\neq 0$, one may take any $e\neq 0$ and its span is a self annihilating subspace so one may find an f with $\phi_M(e\cdot e)=0$, $\phi_M(f\cdot f)=0$, $\phi_M(e\cdot f)=1$ and take the annihilator of $\{e,f\}$. Proceeding in this way, one finds a base $\{e_i,f_i\}$ $1\leqslant i\leqslant r+s$ for $M^{n/2}$, with $e_1,\ldots,e_r$ spanning $\chi^{n/2}$ and satisfying the symplectic base conditions. B may be taken to be the span of $e_1,\ldots,e_{r+s}$. Thus one has:

Remark. If $n\equiv 2$ (4), B always exists.

The case $n=0$ is next easiest. Given M°, one has a base $e_1,\ldots,e_r$ formed of minimal idempotents, and one has r elements of Z_p given by $a_i=\phi_M(e_i)\neq 0$. If one may reorder the base so that it is $e_1,e_2,\ldots,e_{2s-1},e_{2s}$ with $a_{2j}=-a_{2j-1}$, then taking B to be spanned by the elements $e_{2i}+e_{2i-1}$ gives a subalgebra of the desired form. Conversely, given a subalgebra with unit $B\subset M^\circ$ which is its own annihilator, B has a base consisting of idempotents. Specifically, if $x\in B$, one may write $x=\alpha_1(e_1^1+\cdots+e_{p_1}^1)+\cdots+\alpha_s(e_1^s+\cdots+e_{p_s}^s)$ where the α_i are distinct elements of Z_p and the e_j^i are distinct. Then $x^t=\alpha_1^t(e_1^1+\cdots+e_{p_1}^1)+\cdots+\alpha_s^t(e_1^s+\cdots+e_p^s)$, and $1=x^\circ$ has $\alpha_i^\circ=1$. The $s\times s$ matrix of coefficients of $x^\circ,\ldots,x^{s-1}$ is a Vandermonde determinant – hence invertible, so each $e_1^j+\cdots+e_{p_j}^j$ belongs to the span of $1,x,\ldots,x^{s-1}$ and so to B. Thus B has a base consisting of idempotents which may be written $e_1^j+\cdots+e_{p_j}^j$ (with no common entries). Since B is self annihilating, no p_j can be 1, for $a_i=\phi_M(e_i)\neq 0$. Since B is its own annihilator, $\dim B=(\frac{1}{2})\dim M^\circ$ and each p_j is 2. Reordering, one may suppose $e_1,e_2,\ldots,e_{2s-1},e_{2s}$ is a base of M with $e_{2i}+e_{2i-1}$ forming a base of B. Then $a_{2i}+a_{2i-1}=\phi_M(e_{2i}+e_{2i-1})=0$ since B is self annihilating. This gives

Remark. If $n=0$, B exists if and only if M° has a base of minimal idempotents $e_1,e_2,\ldots,e_{2s}$ with $\phi_M(e_{2j})=-\phi_M(e_{2j-1})$.

Now turning to the case $n>0$ and $n\equiv 0$ (4), $M^{n/2}\otimes M^{n/2}\to Z_p: x\otimes y\to\phi_M(xy)$ is a non-degenerate symmetric ($xy=yx$) bilinear form. Being given such a form $\langle\ ,\ \rangle: V\otimes V\to Z_p$, $\langle x,x\rangle=0$ for all x implies $V=0$ (for $\langle x,y\rangle=\frac{1}{2}\{\langle x+y,x+y\rangle-\langle x-y,x-y\rangle\}$). Thus if $V\neq 0$, there is an x with $\langle x,x\rangle\neq 0$ and let $V'\subset V$ be $\{y\mid\langle x,y\rangle=0\}$. Then $\langle\ ,\ \rangle: V'\otimes V'\to Z_p$ is again such a form. Proceeding in this fashion, one may find a base $v_1,\ldots,v_r$ for V so that $\langle v_i,v_j\rangle=0$ if $i\neq j$ and $\langle v_i,v_i\rangle\neq 0$. Then $\langle av,av\rangle=a^2\langle v,v\rangle$ if $a\in Z_p$ and by taking scalar multiples of the v_i one may change $\langle v_i,v_i\rangle$ by any square factor.

The nonzero elements of Z_p form two disjoint classes: the squares (quadratic residues) and the non-squares (quadratic non-residues). Let P be the number of $\langle v_i, v_i\rangle$ which are squares, and N the number which are not squares. One defines an invariant of the form $\langle\ ,\ \rangle$ on V by

$$I = I(V, \langle\ ,\ \rangle) = \begin{cases} (P, N) \in Z_2 \oplus Z_2 & \text{if } p \equiv 1 \pmod 4 \\ P - N \in Z_4 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

To see that this depends only on the form, not on the base, take any base and look at the determinant $D = \det(\langle x_i, x_j\rangle)$ ($\{x_i\}$ being the base). In any other base, with change of base matrix U, one has $\det(U(\langle x_i, x_j\rangle)\, U^{\mathsf{T}}) = (\det U)^2 D$, where U^{T} is the transpose, and thus D is either a square or nonsquare independently of basis. N is then determined mod 2 by D; i.e. if D is a square N is even and if D is not a square N is odd. Since $P+N=\dim V$, P is then determined mod 2 and also $P-N=(P+N)-2N$ is determined mod 4.

Now let $W \subset V$ be a self annihilating subspace and choose a base $w_1, \ldots, w_r$ of W. One may then find $t_1, \ldots, t_r$ in V so that

$$\begin{aligned} &\langle w_i, w_j\rangle = 0 \quad \text{for all} \quad i, j \\ &\langle w_i, t_i\rangle = 1 \\ &\langle w_i, t_j\rangle = 0 = \langle t_i, t_j\rangle \quad \text{if} \quad i \neq j. \end{aligned}$$

(these will all be linearly independent for t_i cannot lie in the span of the others since it alone does not annihilate w_i). These may be found inductively for $V \to \mathrm{Hom}(T; Z_p)$ is epic if T is spanned by W and $t_1, \ldots, t_i$. Then letting S be the span of W and $t_1, \ldots, t_r$, $\langle\ ,\ \rangle : S \otimes S \to Z_p$ is a dual pairing and $S^\perp$ is a complementary subspace which is also dually paired by $\langle\ ,\ \rangle$.

Now considering the span of w_i and t_i, suppose $\langle t_i, t_i\rangle = a_i$. If $a_i \neq 0$, t_i and $t_i - a_i w_i = s_i$ form a base with $\langle t_i, s_i\rangle = 0$, $\langle t_i, t_i\rangle = a_i$, $\langle s_i, s_i\rangle = -a_i$. If $p \equiv 1 \pmod 4$, -1 is a square so a_i and $-a_i$ are both squares or both nonsquares. If $p \equiv 3 \pmod 4$, -1 is not a square, so one is a square and the other is not.

Thus $I(V, \langle\ ,\ \rangle) = I(S^\perp, \langle\ ,\ \rangle)$. In particular, if V contains a subspace which is its own annihilator, (a W as above with $\dim W = 1/2 \dim V$) then $I(V, \langle\ ,\ \rangle) = 0$ (for $S^\perp$ is zero).

Conversely, if $I(V, \langle\ ,\ \rangle) = 0$ and W is a self annihilating subspace then W lies in a subspace which is its own annihilator. If $\dim W = 1/2 \dim V$, one is done; otherwise $S^\perp \neq 0$ and $I(S^\perp, \langle\ ,\ \rangle) = 0$ and it suffices to show there is an $s \neq 0$ in S with $\langle s, s\rangle = 0$ since adding s to W builds a larger W.

Choose a base v_i of $S^\perp$ with $\langle v_i, v_j\rangle = 0$ $i \neq j$ and (taking scalar multiples) $\langle v_i, v_i\rangle = a^2$ or a' where a' is a non-square.

If $p \equiv 1 \pmod 4$, one may find v, w in the base with $\langle v, v\rangle = \langle w, w\rangle$. If $\beta^2 \equiv -1 \pmod p$ $\langle v + \beta w, v + \beta w\rangle = 0$.

If $p\equiv 3$ (mod4), $a'=-a^2$. If there are v, w in the base with $\langle v, v\rangle = -\langle w, w\rangle$, then $\langle v+w, v+w\rangle = 0$. If not then $\langle v_i, v_i\rangle = b$ for all i and there are at least four v_i. Consider $\alpha v_1+\beta v_2+\gamma v_3 = x$. Then $\langle x, x\rangle = b(\alpha^2+\beta^2+\gamma^2)$ and there is a non-trivial solution of $\alpha^2+\beta^2+\gamma^2=0$. (If not u^2+v^2 is always a square so u^2+1 is always a square, but 1 is a square and inductively everything is a square).

Thus one has:

Remark. If $n>0$, $n\equiv 0$ (mod4), B exists if and only if $I(M^{n/2})=0$, where I is the invariant in $Z_2\oplus Z_2$ ($p\equiv 1$ (4)) or Z_4 ($p\equiv 3$ (4)).

Special note. Nothing about A has been used far except that $A^\circ \cong Z_p$ with base $\mathscr{P}^\circ = 1$. One could consider Poincaré algebras over A together with a homomorphism $f: X \to M$ where X is a fixed unstable left algebra over A with unit. If X is connected ($X^\circ \cong Z_p$), adjoining to A the left multiplications by $f(x)$, $x\in X$, creates a larger operator algebra A' with $(A')^\circ \cong Z_p$, and the arguments apply to Poincaré algebras over A'. If X is not connected, then the minimal idempotents $e_i \in X^\circ$ induce maps $f_i: e_i\cdot X \to f(e_i)\cdot M$ and f bounds if and only if each f_i bounds. This reduces one to the direct sum of "connected" cases. Thus the techniques given can be used to solve the algebraic "bordism" problem with no (or little) additional work.

§5. Algebraic Cobordism Groups

Being given an n-dimensional Poincaré algebra M, with homomorphism $\phi_M: M^n \to Z_p$, one defines $-M$ to be the Poincaré algebra M with a new homomorphism $\phi_{(-M)}: M^n \to Z_p$ given by $\phi_{(-M)}(m) = -\phi_M(m)$.

For M_1 and M_2 two n-dimensional Poincaré algebras, $M = M_1 \oplus M_2$ is the Poincaré algebra obtained from the direct sum with $\phi_M(m_1, m_2) = \phi_{M_1}(m_1) + \phi_{M_2}(m_2)$.

Similarly one may form direct sums and negatives of Lefschetz algebras. If $V = (M, M', M'', i, j, \delta, \phi_M, \phi_{M''})$, denote by ∂V the Poincaré algebra M, with homomorphism ϕ_M.

DEFINITION. Two n-dimensional Poincaré algebras M and M' are *cobordant* if there are $(n+1)$-dimensional Lefschetz algebras V and V' with

$$M\oplus\partial V \cong M'\oplus \partial V'$$

(*Note.* $\cong$ denotes the rather obvious definition of isomorphism).

LEMMA 5.1. *Cobordism is an equivalence relation.*

This is completely trivial. It is very hard to prove transitivity if you use Brown-Peterson's type of definition ($M \sim M'$ if $M\oplus(-M')$ bounds) although it can be done using the results of section 4.

One now defines an operation on the set of cobordism classes of n-dimensional

Poincaré algebras by $[M]+[M']=[M\oplus M']$. With this operation, the set of cobordism classes of n-dimensional Poincaré algebras forms an abelian group, which will be denoted Ω_n^p.

Note. $[-M]=-[M]$, for $M\oplus(-M)\oplus\partial 0=0\oplus\partial V$ where 0 denotes the zero algebras; i.e. $M\oplus(-M)$ bounds. Specifically the diagonal in $M\oplus M$ provides the subalgebra R required in Lemma 4.2.

One now lets Ω_*^p be the direct sum of the groups Ω_n^p.

Being given M and N, Poincaré algebras of dimension m and n, one forms their product $M\times N$ on the algebra $M\otimes N$ with $\phi_{M\times N}(m\otimes n)=\phi_M(m)\cdot\phi_N(n)$. The homomorphism $\theta: M\otimes N\to N\otimes M$ given by $\theta(m\otimes n)=(-1)^{\deg m\cdot\deg n}\, n\otimes m$ gives $M\times N\cong \cong(-1)^{mn}\, N\times M$. Extending this to products $M\times V$ with V a Lefschetz algebra, it is immediate that the product $\times$ makes Ω_*^p into a commutative ring (in the graded sense).

The result of section 4 determine much of the structure of Ω_*^p.

PROPOSITION 5.1. *Ω_0^p is a free abelian group of rank $(p-1)/2$.*

Proof. Define a function

$$c: \Omega_0^p \to \underbrace{Z\oplus\cdots\oplus Z}_{(p-1)/2}$$

by $c(M)=(c_1(M),\ldots,c_{(p-1)/2}(M))$ where $c_i(M)$ is the number of minimal idempotents e in M with $\phi_M(e)\equiv i$ minus the number of minimal idempotents e in M with $\phi_M(e)\equiv -i$. Clearly $c(M_1\oplus M_2)=c(M_1)+c(M_2)$ and $c(M)=0$ if and only if M bounds, so c defines a homomorphism of Ω_0^p into $Z\oplus\cdots\oplus Z$, which is in fact monic. To see that c is epic, one forms the Poincaré algebra $M_a\cong Z_p$ with base the unit 1 and with $\phi_{M_a}(1)=a$ for $a\neq 0$ in Z_p. Then $M_{-a}=-M_a$, and if $1\leqslant a\leqslant (p-1)/2$, $c(M_a)=(0,\ldots,0,1,0,\ldots,0)$ where the 1 occurs in the a-th position. $*$

One may describe the ring structure in Ω_0^p as follows: Letting $x_a=[M_a]$, $a\neq 0$ in Z_p, x_1 is the unit in Ω_*^p, $x_{-a}=-x_a$ and $x_a\cdot x_b=x_{ab}$.

For $n>0$, one defines a homomorphism $\chi:\Omega_n^p\to \mathrm{Hom}(U^n; Z_p)$ where U^n is the n-dimensional part of the algebra of characteristic classes by $\chi(M)\,(u)=\phi_M(\theta'_M(u))$. Since $\theta'_M(u)=(\theta'_{M_1}(u),\theta'_{M_2}(u))$ if $M=M_1\oplus M_2$ this is additive, and if $M=\partial V$, $\phi_M(\theta'_M(u))=\phi_{M''}\delta(i\theta'_{M'}(u))=0$, showing that χ induces a homomorphism on Ω_n^p.

For $n>0$ and $n\equiv 0\,(4)$, one defines a homomorphism

$$I:\Omega_n^p\to\begin{cases} Z_2\oplus Z_2 & p\equiv 1\,(4)\\ Z_4 & p\equiv 3\,(4)\end{cases}$$

by assigning to M the invariant $I(M^{n/2})$. This is easily seen to be additive and sends boundaries to zero, and so induces a homomorphism on Ω^p.

PROPOSITION 5.2. *If $n>0$, the homomorphisms*

$$\chi:\Omega_n^p\to\mathrm{Hom}(U^n;Z_p)\qquad n\not\equiv 0(4)$$

and

$$\chi\oplus I:\Omega_n^p\to\mathrm{Hom}(U^n;Z_p)\oplus\begin{cases}Z_2\oplus Z_2 & p\equiv 1(4),\\ Z_4 & p\equiv 3(4)\end{cases}\qquad n\equiv 0(4)$$

are monic. Thus, for $n\not\equiv 0(4)$, Ω_n^p is a Z_p vector space and for $n\equiv 0(4)$, Ω_n^p is the direct sum of a Z_p vector space and either $Z_2\oplus Z_2$ ($p\equiv 1$) or Z_4 ($p\equiv 3$).

Proof. That χ and $\chi\oplus I$ are monic follows from section 4 immediately. For the last part, one needs to see that I is epic. Consider $M^4=H^*(CP(2);Z_p)$. Dimensional considerations show that M^4 has trivial A action ($\mathcal{P}^0=1$, $\mathcal{P}^i=0$ $i>1$, $\beta=0$), and $I(M^4)=(1,0)$ if $p\equiv 1(4)$ or $I(M^4)=1$ if $p\equiv 3(4)$. Given two Poincaré algebras M^{4j} and N^{4k}, $(M\times N)^{2(j+k)}$ is the direct sum $\oplus(M^r\oplus M^s)$ for $r+s=2(j+k)$ and the terms with $r<2j$ give a self annihilating subspace W with S formed of the terms with $r\neq 2j$. Thus $I((M\times N)^{2(j+k)})=I(M^{2j}\oplus N^{2k})$. From the tensor product of two "diagonal" bases, this is $I(M^{2j})\cdot I(N^{2k})$.

Note. In $Z_2\oplus Z_2$ $(P_1,N_1)\cdot(P_2,N_2)=(P_1P_2+N_1N_2,P_1N_2+P_2N_1)$. Then, $(M^4)^k=M^4\times\cdots\times M^4$ (k copies) has trivial A action, so $\chi((M^4)^k)=0$ and $I((M^4)^k)=(1,0)$ or 1. If $a\neq 0$ in Z_p, and M_a is the 0-dimensional algebra with $\phi_{M_a}(1)=a$ and M^{4k} is a $4k$-dimensional algebra with diagonal basis v_i, $M_a\times M^{4k}$ has diagonal basis $1\otimes v_i$, with $\langle 1\otimes v_i,1\otimes v_i\rangle=a\langle v_i,v_i\rangle$. Thus if a is a square, $I(M_a\times M^{4k})=I(M^{4k})$ and if a is not a square, $I(M^{4k})=(P,N)$ or $P-N$ gives $I(M_a\times M^{4k})=(N,P)$ or $N-P$. Thus I is epic. *

This permits the description of the multiplicative structure in the 2-primary part of Ω_{4*}^p. One has a 4 dimensional generator y_4 with $2y_4=0$ ($p\equiv 1$) or $4y_4=0$ ($p\equiv 3$) and $x_ay_4=y_4$ if a is a square.

Note. It is no coincidence that $\chi((M^4)^k)=0$. Multiplication by p^2 defines a projection onto the 2 primary part of Ω_n^p, since $p^2\equiv 1\,(4)$, killing the p-torsion, and multiplication by $1-p^2$ defines a projection onto the p primary part of Ω_n^p killing the 2 torsion.

§6. Construction of Poincaré Algebras

In order to complete the calculation of Ω_*^p one must find the image of $\chi:\Omega_n^p\to\to\mathrm{Hom}(U^n;Z_p)$. For this one may show how to construct Poincaré algebras.

Let X be an unstable left algebra over A with unit and let $\phi:X^n\to Z_p$ be a homomorphism. Let $I^j=\{x\in X^j\mid\phi(xy)=0\text{ for all }y\in X^{n-j}\}$, with $I\subset X$ being the sum of the I^j.

If $x, x' \in I^j$, $\lambda, \lambda' \in Z_p$, then $\phi((\lambda x + \lambda' x')\cdot y) = \lambda\phi(xy) + \lambda'\phi(x'y) = 0$ for all $y \in X^{n-j}$, so $\lambda x + \lambda' x' \in I^j$. If $x \in I^j$, $z \in X^k$, then for any $y \in X^{n-(j+k)}$, $\phi((zx)y) = (-1)^{jk}\phi(x(zy)) = 0$, so $zx \in I^{j+k}$. Thus I is an algebra ideal in X, and $M = X/I$ is a commutative Z_p algebra, with $\pi: X \to M$ the quotient homomorphism.

$\phi: X^n \to Z_p$ sends I^n to zero, for if $x \in I^n$, $\phi(x) = \phi(x\cdot 1) = 0$, and so ϕ induces a homomorphism $\phi_M: M^n \to Z_p$.

If $m \in M^i$ and $\phi_M(m\cdot m') = 0$ for all $m' \in M^{n-i}$, then letting $x \in X^i$ represent m, $\pi(x) = m$, $\phi(x\cdot x') = \phi_M(m\cdot\pi(x')) = 0$ for all $x' \in X^{n-i}$ and so $x \in I^i$ or $m = 0$. Thus $M^i \otimes M^{n-i} \to Z_p: m \otimes m' \to \phi_M(mm')$ is a nonsingular pairing for each i.

Clearly, if I is an A ideal, M becomes an unstable left A algebra, with π a homomorphism of left A modules. This being the case, there are elements $b \in M^1$ and $v_i \in M^{2i(p-1)}$ so that

$$\phi_M(\beta m) = \phi_M(b\cdot m) \quad \text{for all} \quad m \in M^{n-1}$$

and

$$\phi_M(\mathcal{P}^i m) = \phi_M(v_i m) \quad \text{for all} \quad m \in M^{n-2i(p-1)}.$$

Hence there are elements $b' \in X^1$ and $v_i' \in X^{2i(p-1)}$ with $\phi(\beta x - b'x) = 0$ for all $x \in X^{n-1}$ and $\phi(\mathcal{P}^i x - v_i' x) = 0$ for all $x \in X^{n-2i(p-1)}$.

Conversely, suppose there are elements $b' \in X^1$, $v_i' \in X^{2i(p-1)}$ with $\phi(\beta x - b'x) = 0$, $x \in X^{n-1}$, and $\phi(\mathcal{P}^i x - v_i' x) = 0$, $x \in X^{n-2i(p-1)}$. Then I is an A-ideal. To see this proceed as follows: Let $x \in I^j$ and $y \in X^{n-(j+1)}$ and $\phi((\beta x)\cdot y) = \phi(\beta(xy) - (-1)^j x\cdot\beta y) = \phi(b'(xy)) + (-1)^{j+1}\phi(x\cdot(\beta y)) = 0 + 0 = 0$ so $\beta x \in I^{j+1}$. If $x \in I^j$, $\mathcal{P}^0 x = x \in I^j$, so suppose inductively that if $i < k$, $x \in I^j$ implies $\mathcal{P}^i x \in I^{j+2i(p-1)}$. Then for $x \in I^j$, $y \in X^{n-(j+2k(p-1))}$, $\phi((\mathcal{P}^k x)\cdot y) = \phi(\mathcal{P}^k(xy) - \sum_{i=0}^{k-1}\mathcal{P}^i x\cdot\mathcal{P}^{k-i}y) = \phi(v_k' xy) - \sum_{i=0}^{k-1}\phi((\mathcal{P}^i x)\cdot(\mathcal{P}^{k-i}y)) = 0$, so $\mathcal{P}^k x \in I^{j+2k(p-1)}$.

DEFINITION. In any unstable left A algebra with unit which is a right A module, let $b = 1\beta$ and $v_i = 1\mathcal{P}^i$.

LEMMA 6.1. a) *If M is a Poincaré algebra and $m \in M$, then*

$$m\beta = (-1)^{\deg m}(bm - \beta m)$$

$$m\mathcal{P}^i = v_i m - \sum_{j=0}^{i-1}(\mathcal{P}^{i-j}m)\,\mathcal{P}^j.$$

b) *If $u \in U$, then*

$$u\beta = (-1)^{\deg u}(bu - \beta u)$$

$$u\mathscr{P}^i = v_i u - \sum_{j=0}^{i-1} (\mathscr{P}^{i-j}u)\,\mathscr{P}^j .$$

Proof. $\phi_M(bmn)=\phi_M(\beta(mn))=\phi_M((\beta m)\,n+(-1)^{\deg m}\,m(\beta n))$ and

$$\phi_M(v_i mn) = \phi_M(\mathscr{P}^i(mn)) = \phi_M\left(\sum_{j=0}^{i-1} (\mathscr{P}^{i-j}m)\,(\mathscr{P}^j n) + m(\mathscr{P}^i n)\right)$$

for all n of appropriate dimension, giving a). Applying θ'_M to the formulae of b) gives an identity in any Poincaré algebra, and by universality the formulae hold in U. *

LEMMA 6.2. *Let $F_n \subset U^n$ be the subspace spanned by the elements $\beta u - bu$, $u \in U^{n-1}$, and $\mathscr{P}^i u - v_i u$, $u \in U^{n-2i(p-1)}$. Then $\phi \in \mathrm{Hom}(U^n; Z_p)$ lies in $\chi(\Omega_n^p)$ if and only if $\phi(F_n)=0$.*

Proof. If M is an n-dimensional Poincaré algebra and $\phi=\chi(M)$, then $\phi(\beta u-bu)$ $=\phi_M(\theta'_M(\beta u-bu))=\phi'_M(\beta\theta_M(u)-\theta'_M(1\beta)\cdot\theta'_M(u))=\phi_M(\beta\theta'_M(u)-(1\cdot\beta)\,\theta'_M(u)=0$ and similarly $\phi(\mathscr{P}^i u-v_i u)=0$. Thus $\phi(F_n)=0$.

If $\phi: U^n \to Z_p$ with $\phi(F_n)=0$, let M be the n-dimensional Poincaré algebra U/I formed from ϕ by the above construction, with $\pi: U\to M$ the quotient map. Since $\phi(F_n)=0$, $\pi(b)=b$ and $\pi(v_i)=v_i$. π is a homomorphism of algebras with unit and left A modules, and by the formulae of Lemma 6.1 is a homomorphism of right A modules (use induction on i to get $\pi(u\mathscr{P}^i)=\pi(u)\,\mathscr{P}^i$). Since $\theta'_M: U\to M$ is characterized as the unique homomorphism of algebras with which is a right and left A module homomorphism, $\pi=\theta'_M$. This being the case, $\chi(M)\;(u)=\phi_M(\theta'_M(u))=\phi_M(\pi(u))$ $=\phi(u)$. *

This is, of course, the analogue of Dold's formulation of relations among characteristic numbers. In an algebraic context, the Brown-Peterson formulation will be more convenient.

Let $A^+ \subset A$ denote the augmentation ideal consisting of elements of positive degree. The quotient homomorphism $U\to U/UA^+$ induces a monomorphism Hom $((U/UA^+)^n; Z_p)\to \mathrm{Hom}(U^n; Z_p)$, and identify $\mathrm{Hom}((U/UA^+)^n; Z_p)$ with the homomorphisms $\phi: U^n\to Z_p$ with $\phi((UA^+)^n)=0$.

PROPOSITION 6.1. $\chi(\Omega_n^p)=\mathrm{Hom}((U/UA^+)^n; Z_p)$.

Proof. The identities

$$\beta u - bu = (-1)^{\deg u+1}\, u\beta$$

$$\mathscr{P}^i u - v_i u = -\sum_{j=1}^{i} (\mathscr{P}^{i-j}u)\,\mathscr{P}^j$$

give $F_n \subset (UA^+)^n$. Using the some identities an induction on i to prove $u\mathscr{P}^i \in F_n$ gives $(UA^+)^n \subset F_n$. *

§7. The Structure of U

In order to calculate $\chi(\Omega_n^p)$ explicitly, it will be necessary to know U precisely. Adams has done most of the work, and one need only repeat his arguments.

First, recall that A is a Hopf algebra with a diagonal map $\Delta: A \to A \otimes A$. As convention one writes

$$\Delta(a) = \sum_r a'_r \otimes a''_r$$

and if $x, y \in X$, X an unstable left A algebra,

$$a(xy) = \sum_r (-1)^{\deg x \cdot \deg a''_r} (a'_r x)(a''_r y).$$

The canonical antiautomorphism $\chi: A \to A$ is defined inductively on degree by

$$\chi(1) = 1, \sum_r \chi(a'_r) a''_r = 0 \quad (\dim a > 0).$$

One now defines classes in U by $b = 1\beta$ and $\bar{q}_i = 1\chi(\mathscr{P}^i)$. Using Adam's methods, it will be shown that

PROPOSITION 7.1. *U is multiplicatively generated by the classes b, $\bar{q}_i$, and $\beta\bar{q}_i$.*

The proof proceeds by a sequence of lemmas.

LEMMA 7.2. a) *If* $\dim a > 0$ *and* $u \in U$, *then*

$$(1a) \cdot u = au + \sum (-1)^{\deg u \cdot \deg a''_r} (a'_r u) a''_r + (-1)^{\deg u \cdot \deg a} ua,$$

the sum being for those terms other than $a \otimes 1$ *and* $1 \otimes a$ in Δa.

b) If $\dim a > 0$, then

$$(1a) \cdot (1b) = a(1b) + \sum (-1)^{\deg b \cdot \deg a''_r} (a'_r(1b)) a''_r + (-1)^{\deg b \cdot \deg a} (1b) a.$$

Proof. Just as in Lemma 6.1 or Adams' Lemma 8. *

LEMMA 7.3. *U is multiplicatively generated by the elements $1a$, $a \in A$.*

Proof. Adams' proof of Lemma 9 goes through verbatim, except for the signs in his formula for Wa, which are immaterial. *

Let $D(U)$ denote the decomposables in U.

LEMMA 7.4. *If* $u \in D(U)$, $a \in A$, *then* $ua \in D(U)$.
Proof. Adams' Lemma 10.

LEMMA 7.5. *If* $\dim b > 0$, *then*

$$a(1b) = (-)1^{\deg a \cdot \deg b}\, 1b\chi(a) \quad \bmod D(U).$$

Proof. Adams' Lemma 11 with the appropriate sign convention. *

LEMMA 7.6. $U/D(U)$ *is spanned by the elements* $1, b, \bar{q}_i, \beta\bar{q}_i$.
Proof. From Lemma 7.3, $U/D(U)$ is spanned by the elements $1a$, $a \in A$, and it clearly suffices to consider only a set of elements $a \in A$ which span A.

Let St^i, $i \equiv 0, 1 \bmod 2(p-1)$ be the elements $St^{2k(p-1)} = \mathscr{P}^k$, $St^{2k(p-1)+1} = \beta\mathscr{P}^k$. If $I = (i_1, \ldots, i_r)$, $i_\alpha \equiv 0, 1$ $(2(p-1))$, $St^I = St^{i_1} \ldots St^{i_r}$. The set S of elements St^I with $i_1 \geqslant pi_2, i_2 \geqslant pi_3, \ldots, i_{r-1} \geqslant pi_r$ span A, and in fact form a base of A, by the Adem relations.

The set S contains the elements St^i, and every other element of S is of the form $St^i c$ with $0 < \dim c < i/(p-1)$ $(i_1 \geqslant (p-1)(i_2 + \cdots + i_r) + i_r)$. The set $\chi(S)$ also spans A. $\chi(S)$ contains the elements $\chi(St^i)$ and every other element of $\chi(S)$ is of the form $d\chi(St^i)$ with $0 < \dim d < i/(p-1)$. By Lemma 7.5,

$$1d\chi(St^i) = \pm St^i(1d) \quad \bmod D(U),$$

but $St^i(1d) = 0$ since $\dim(1d) < i/(p-1)$, so $1d\chi(St^i)$ is decomposable.

Thus $U/D(U)$ is spanned by the elements $1\chi(St^i)$. Since $1\chi(\mathscr{P}^i) = \bar{q}_i$ (or 1 if $i = 0$) and $1\chi(\beta) = -b$, and $1\chi(\beta\mathscr{P}^i) = 1\chi(\mathscr{P}^i)\,\chi(\beta) = \bar{q}_i\chi(\beta) \equiv \beta\bar{q}_i$ $(\bmod D(U))$, one has the result. *

This completes the proof of Proposition 7.1.

Let U' denote the free associative, commutative algebra generated by elements b, $\bar{q}_i$, and $\beta\bar{q}_i$; i.e. the tensor product of polynomial algebras $Z_p[\bar{q}_i]$ and exterior algebras $E[b]$ and $E[\beta\bar{q}_i]$. One then has defined an epimorphism $\sigma: U' \to U$.

One now wishes to show that σ is an isomorphism.

For this, form an unstable left A algebra $B = E[b] \cong H^*(S^1; Z_p)$ with trivial A action. Form a Thom space $\tilde{B}$ for B with $\tilde{B}^2 \cong Z_p$ with base u_1 and $\tilde{B}^3 \cong Z_p$ with base u_2 and with A action $\mathscr{P}^0 u_1 = u_1$, $\mathscr{P}^0 u_2 = u_2$, $\beta u_1 = -u_2$ (making $\tilde{B}$ a left A module) with $\Phi_B: B \to \tilde{B}$ a Thom isomorphism given by $\Phi_B(1) = u_1$, $\Phi_B(b) = u_2 = bu_1$.

Following Peterson and Toda [6], let $BSF = BSG$ be the classifying space for oriented spherical fiber spaces, and MSF the associated Thom spectrum, with $\phi: H^*(BSF; Z_p) \to H^*(MSF; Z_p)$ the Thom isomorphism.

Then $B \otimes H^*(BSF; Z_p)$ is an unstable left A algebra and $\Phi_B \otimes \phi: B \otimes H^*(BSF; Z_p) \to \tilde{B} \otimes H^*(MSF; Z_p)$ is taken as Thom isomorphism. $\tilde{B} \otimes H^*(MSF; Z_p)$ is a left

A-module, and one defines a right A module structure on $B \otimes H^*(BSF; Z_p)$ by $ua = (-1)^{\deg a \cdot \dim u} (\Phi_B \otimes \phi)^{-1} \chi(a) (\Phi_B \otimes \phi)(u)$.

Note. A sign has been added to this equation since the order of a and u is reversed.

Thus, there is the canonical function

$$\theta = \theta_{B \otimes H^*(BSF)}: W \to B \otimes H^*(BSF; Z_p).$$

Now let $q' \in H^*(BSF; Z_p)$ be the class defined by $q'_i = \phi^{-1} \mathscr{P}^i \phi(1)$. Then Peterson-Toda show that $H^*(BSF; Z_p)$ contains the free associative, commutative algebra on the classes q'_i and $\beta q'_i$, and it is implicit in their paper that this subalgebra is closed under left and right A action. Further, $\beta\phi(1) = 0$.

Now consider the subalgebra $B \otimes Z_p[q'_i] \otimes E[\beta q'_i]$ contained in $B \otimes H^*(BSF; Z_p)$. This clearly contains 1 and is closed under left A-action. Further, it contains $b = 1\beta = b \otimes 1$ and $\bar{q}_i = 1 \otimes q'_i$, so $\beta \bar{q}_i = 1 \otimes \beta q'_i$.

Now make use of the fact that $\Phi_B \otimes \phi$ is a Thom homomorphism; i.e. $\tilde{B} \otimes H^*(MSF; Z_p)$ is a $B \otimes H^*(BSF; Z_p)$ module. One then has

$$\begin{aligned} u\beta &= (-1)^{\dim u} (\Phi_B \otimes \phi)^{-1} \chi(\beta)(u \cdot u_1 \otimes \phi(1)) \\ &= (-1)^{\dim u} (\Phi_B \otimes \phi)^{-1} \{-\beta u \cdot u_1 \otimes \phi(1) + (-1)^{\dim u} u \cdot (-\beta(u_1 \otimes \phi(1)))\} \\ &= (-1)^{\dim u} (-\beta u + (-1)^{\dim u} ub) = (-1)^{\dim u} (bu - \beta u). \end{aligned}$$

and

$$\begin{aligned} u\chi(\mathscr{P}^i) &= (\Phi_B \otimes \phi)^{-1} \mathscr{P}^i (u \cdot u_1 \otimes \phi(1)) \\ &= (\Phi_B \otimes \phi)^{-1} \sum_{j=0}^{i} \mathscr{P}^{i-j} u \cdot (1 \otimes q'_j) \cdot u_1 \otimes \phi(1)) \\ &= \sum_{j=0}^{i} \mathscr{P}^{i-j} u \cdot \bar{q}_j . \end{aligned}$$

Thus $B \otimes Z_p[q'_i] \otimes E[\beta q'_i]$ is closed under right A action. Since $B \otimes Z_p[q'_i] \otimes E[\beta q'_i]$ is the free associative, commutative algebra on b, $\bar{q}_i$, and $\beta \bar{q}_i$, $\theta(W) = B \otimes Z_p[q'_i] \otimes \otimes E[\beta q'_i]$.

Thus, identifying U' with $\theta(W)$, U' becomes an unstable left A algebra with unit and a right A module.

In addition, U' is a connected coalgebra in which the diagonal map $\Delta: U' \to U' \otimes U'$ is given by $\Delta(b) = b \otimes 1 + 1 \otimes b$, $\Delta(\bar{q}_i) = \sum \bar{q}_{i-j} \otimes \bar{q}_j$ and $\Delta(\beta \bar{q}_i) = \sum \beta \bar{q}_{i-j} \otimes \bar{q}_j + \sum \bar{q}_{i-j} \otimes \otimes \beta \bar{q}_j$. Under the Thom homomorphism, this corresponds to the usual coalgebra structure in $H^*(MSF; Z_p)$ induced by the Whitney sum of oriented spherical fibrations, and the coalgebra structure on $\tilde{B}$ given by $\Delta(u_1) = u_1 \otimes u_1$, $\Delta(u_2) = u_2 \otimes u_1 + u_1 \otimes u_2$. This coalgebra structure on $\tilde{B} \otimes H^*(MSF; Z_p)$ forms a coalgebra over the Hopf algebra A.

If one now considers the action $v: A \to \tilde{B} \otimes H^*(MSF; Z_p)$ given by $v(a) = a(\Phi_B \otimes \phi)(1)$, v is monic. To see this, Peterson-Toda have shown that $\tilde{v}: A \to H^*(MSF; Z_p): a \to a\phi(1)$ has kernel precisely $A\beta$. Now $v: A\beta \to \tilde{B} \otimes H^*(MSF; Z_p)$ sends $a\beta$ to $a(-bu_1 \otimes \phi(1)) = (-1)^{\deg a+1}(bu_1 \otimes a\phi(1))$ and is monic, for if $v(a\beta) = 0$, $a \in A\beta$ and $a\beta = 0$. Then v induces a homomorphism

$$\bar{v}: A/A\beta \to \tilde{B} \otimes H^*(MSF; Z_p)/\tilde{B}^3 \otimes H^*(MSF; Z_p)$$

which may be identified with $\tilde{v}$, so is monic. Thus v is monic.

By Theorem 4.4 of Milnor and Moore [5], $\tilde{B} \otimes H^*(MSF; Z_p)$ is a free A module. Applying the inverse of the Thom isomorphism, or working directly with $\hat{v}: A \to U': a \to 1\chi(a)$, one sees that U' is a free right A module.

Let us summarize these facts.

LEMMA 7.7. *If U' is the free associative, commutative Z_p algebra on b, $\bar{q}_i$, $\beta\bar{q}_i$, U' can be made into an unstable left algebra over A and a right A module with $b = 1\beta$, $\bar{q}_i = 1\chi(\mathscr{P}^i)$. Further, as a right A module U' is free, and the right A module structure satisfies the identities*

$$u\beta = (-1)^{\dim u}(bu - \beta u), \quad u\chi(\mathscr{P}^i) = \sum_{j=0}^{i} (\mathscr{P}^{i-j}u)\cdot\bar{q}_j.$$

In order to compare U' with U, one must relate U' with Poincaré algebras. Unfortunately, U' is expressed in terms of $\bar{q}_i$ with all formulae involving $\chi(\mathscr{P}^i)$ while Poincaré algebras are expressed in v_i with formulae involving $\mathscr{P}^i$. First these must be reconciled.

Denote by $\bar{q}$ the class $1 + \bar{q}_1 + \bar{q}_2 + \cdots$, and similarly, let $v = 1 + v_1 + v_2 + \cdots$, $\mathscr{P} = 1 + \mathscr{P}^1 + \mathscr{P}^2 + \cdots$. Then $\chi(\mathscr{P}) = \mathscr{P}^{-1}$, i.e. $\chi(\mathscr{P})\cdot\mathscr{P} = \mathscr{P}\cdot\chi(\mathscr{P}) = 1$.

LEMMA 7.8. *In U' and in any Poincaré algebra the classes $\bar{q} = 1\chi(\mathscr{P})$ and $v = 1\mathscr{P}$ are related by*

$$\bar{q}\cdot(\mathscr{P}v) = 1.$$

Proof. In a Poincaré algebra M, $\phi_M(\bar{q}\cdot(\mathscr{P}v)\cdot x) = \phi_M(\mathscr{P}^{-1}(\mathscr{P}v\cdot x)) = \phi_M(\mathscr{P}^{-1}\mathscr{P}v\cdot\mathscr{P}^{-1}x) = \phi_M(v\cdot\mathscr{P}^{-1}x) = \phi_M(\mathscr{P}\mathscr{P}^{-1}x) = \phi_M(1\cdot x)$ for all x, so $\bar{q}\cdot(\mathscr{P}v) = 1$. In U', $(\Phi_B \otimes \Phi)(\bar{q}\cdot(\mathscr{P}v)) = (\mathscr{P}v)\cdot\bar{q}(\Phi_B \otimes \phi)(1) = (\mathscr{P}v)\cdot\mathscr{P}(\Phi_B \otimes \phi(1)) = \mathscr{P}(v\cdot\Phi_B \otimes \phi(1)) = \mathscr{P}(\mathscr{P}^{-1}(\Phi_B \otimes \phi(1))) = \Phi_B \otimes \phi(1)$, so $\bar{q}\cdot(\mathscr{P}v) = 1$. *

LEMMA 7.9. *In any Poincaré algebra*

$$u\chi(\mathscr{P}^i) = \sum_{j=0}^{i} (\mathscr{P}^{i-j}u)\cdot\bar{q}_j.$$

Proof. For any x,

$$\phi_M\left(\sum_{j=0}^{i}(\mathscr{P}^{i-j}u)\cdot\bar{q}_j\cdot x\right)=\phi_M\left(\sum_{j=0}^{i}\chi(\mathscr{P}^j)\{(\mathscr{P}^{i-j}u)\cdot x\}\right)$$
$$=\phi_M\left(\sum_{j=0}^{i}\sum_{k=0}^{j}\{\chi(\mathscr{P}^{j-k})\,\mathscr{P}^{i-j}u\}\cdot\chi(\mathscr{P}^k)\,x\right)$$
$$=\phi_M\left(\sum_{k=0}^{i}\sum_{j=k}^{i}\{\chi(\mathscr{P}^{j-k})\,\mathscr{P}^{i-j}u\}\cdot\chi(\mathscr{P}^k)\,x\right)$$

but $\sum_{j=k}^{i}\chi(\mathscr{P}^{j-k})\,\mathscr{P}^{i-k}$ is 0 if $i-k>0$, 1 if $i-k=0$, so this is $\phi_M(u\cdot\chi(\mathscr{P}^i)\,x)$. *

Now consider pairs (M,f) where M is a Poincaré algebra and $f: U'\to M$ is a homomorphism of algebras with unit and A modules with $f(b)=b$.

CLAIM. *The following are equivalent:*
a) $f(\bar{q}_i)=\bar{q}_i$ *for all* i.
b) $f(v_i)=v_i$ *for all* i.
c) f *is a homomorphism of right* A *modules.*

Proof. b) implies a) for $f(v)=v$ gives $f(\mathscr{P}v)=\mathscr{P}v$ and $f(\bar{q})=f(1/\mathscr{P}v)=1/f(\mathscr{P}v)=1/\mathscr{P}v=\bar{q}$. a) implies c) for

$$f(u\chi(\mathscr{P}^i))=f\left(\sum_{j=0}^{i}(\mathscr{P}^{i-j}u)\cdot\bar{q}_j\right)=\sum_{j=0}^{i}\mathscr{P}^{i-j}f(u)\cdot\bar{q}_j=f(u)\,\chi(\mathscr{P}^i)$$

and

$$f(u\beta)=f((-1)^{\dim u}(bu-\beta u))=(-1)^{\dim f(u)}(bf(u)-\beta f(u))=f(u)\,\beta.$$

Finally c) implies b) for $f(v_i)=f(1\mathscr{P}^i)=f(1)\,\mathscr{P}^i=1\mathscr{P}^i=v_i$. *

Now following Brown and Peterson [3] (*Note*: Part I of this series is restricted to Z_2 but has the results needed, Part II covers Z_p but does not apply here), one lets $I_n(U')^i$ be the set of classes in $(U')^i$ sent to zero by every homomorphism $f: U'\to M$ of left A algebras with unit and right A modules where M is an n-dimensional Poincaré algebra.

Let $K(Z_p, n-i)$ denote the Eilenberg-MacLane space and let $i\in H^{n-i}(K(Z_p, n-i); Z_p)$ denote the fundamental class. Being given any unstable left A algebra with unit X and class $x\in X^{n-i}$, there is a unique homomorphism $\varrho(x): H^*(K(Z_p, n-i); Z_p)\to X$ of left A algebras with unit for which $\varrho(x)\,(i)=x$.

Let $F_i^n\subset\{U'\otimes H^*(K(Z_p, n-i); Z_p)\}^n$ be the subspace spanned by all elements of the forms $\beta x-(b\otimes 1)\cdot x$, $\dim x=n-1$, and $\mathscr{P}^j x-(v_j\otimes 1)\cdot x$, $\dim x=n-2j(p-1)$.

LEMMA 7.10. $I_n(U')^i$ *is the set of classes* $u\in(U')^i$ *for which* $u\otimes i\in F_i^n$.

Proof. If $u\otimes i\in F_i^n$ and $f: U'\to M$ is any appropriate homomorphism, then for any $x\in M^{n-i}$, one has a homomorphism of left A algebras with unit $g=f\cdot\varrho(x): U'\otimes H^*(K(Z_p, n-i); Z_p)\to M: u'\otimes z\to f(u')\cdot\varrho(x)(z)$. $g(b\otimes 1)=f(b)=b$ and $g(v_j\otimes 1)=f(v_j)=v_j$, so $g(F_i^n)=0$ and thus $f(u)\cdot x=g(u\otimes i)=0$. Thus $\phi_M(f(u)\cdot x)=0$ for all $x\in M^{n-i}$ and $f(u)=0$. Thus $u\in I^n(U')^i$.

If $u\otimes i\notin F_i^n$, there is a homomorphism $\phi:(U'\otimes H^*(K(Z_p, n-i); Z_p)^n\to Z_p$ with $\phi(F_i^n)=0$ and $\phi(u\otimes i)\neq 0$. As in section 6, there is an n-dimensional Poincaré algebra M with an epimorphism $\pi: U'\otimes H^*(K(Z_p, n-i); Z_p)\to M$ which is a homomorphism of left A algebras so that $\phi_M(\pi(x))=\phi(x)$, and with $\pi(b\otimes 1)=b$, $\pi(v_j\otimes 1)=v_j$. Let $f: U'\to M$ by $f(u')=\pi(u'\otimes 1)$. Then f is a homomorphism of left A algebras with unit and $f(b)=b$, $f(v_j)=v_j$ so a homomorphism of right A modules. Since $\phi_M(f(u)\cdot\pi(1\otimes i))=\phi_M(\pi(u\otimes 1)\cdot\pi(1\otimes i))=\phi_M(\pi(u\otimes i))=\phi(u\otimes i)\neq 0$, $f(u)\neq 0$. Thus $u\notin I_n(U')^i$. *

Since U' is a right A module and $H^*(K(Z_p, n-i); Z_p)$ is a left A module, one may form their tensor product over A, $U'\otimes_A H^*(K(Z_p, n-i); Z_p)$, which is obtained from the tensor product over Z_p by dividing out the subspace spanned by all $ua\otimes v-u\otimes av$, $a\in A$.

LEMMA 7.11. *F_i^n is the kernel of the quotient homomorphism* $q:(U'\otimes H^*(K(Z_p, n-i); Z_p))^n\to(U'\otimes_A H^*(K(Z_p, n-i); Z_p))^n$.

Proof. To see that $\ker q\subset F_i^n$, let $\phi:(U'\otimes H^*(K(Z_p, n-i); Z_p))^n\to Z_p$ be any homomorphism with $\phi(F_i^n)=0$ and form the associated Poincaré algebra M as in section 6. Using the notation of the last lemma,

$$\begin{aligned}\phi(ua\otimes v)&=\phi_M(\pi(ua\otimes v))=\phi_M(\pi(ua\otimes 1)\cdot\pi(1\otimes v))\\&=\phi_M(f(ua)\cdot\pi(1\otimes v))=\phi_M(f(u)\,a\cdot\pi(1\otimes v))=\phi_M(f(u)\cdot\pi(a(1\otimes v)))\\&=\phi_M(\pi(u\otimes 1)\cdot\pi(1\otimes av))=\phi_M\,\pi(u\otimes av)=\phi(u\otimes av).\end{aligned}$$

Thus $ua\otimes v-u\otimes av\in\ker\phi$. Since this holds for all ϕ, $ua\otimes v-u\otimes av\in F_i^n$.

To see that $F_i^n\subset\ker q$, it suffices to show that $a(u\otimes v)-(1a\otimes 1)\ (u\otimes v)\in\ker q$ where $a=\beta$ or $\mathscr{P}^j$, for the $u\otimes v$ span $U'\otimes H^*(K(Z_2, n-i); Z_p)$. Now

$$\begin{aligned}a(u\otimes v)-(1a\otimes 1)(uv)&=\sum a_r'u\otimes(-1)^{\dim u\,\dim a_r''}\,a_r''v-(1a\cdot u\otimes v)\\&=\sum(-1)^{\dim u\cdot\dim a''_r}\,(a_r'u)\,a_r''\otimes v-(1a\cdot u)\otimes v\qquad(\operatorname{mod}\ker q)\\&=\{\sum(-1)^{\dim u\,\dim a''_r}\,(a_r'u)\,a_r''-1a\cdot u\}\otimes v\qquad(\operatorname{mod}\ker q)\end{aligned}$$

When $a=\beta$, this is

$$\{(-1)^0\,(\beta u)+(-1)^{\dim u}\,u\beta-b\cdot u\}\otimes v$$

which is zero. When $a=\mathscr{P}^j$, it is

$$\left\{\sum_{k=0}^{j} (\mathscr{P}^{j-k}u)\,\mathscr{P}^k - 1\mathscr{P}^j\cdot u\right\}\otimes v.$$

Now in U' one has

$$(\Phi_B\otimes\phi)\left(\sum(\mathscr{P}^{j-k}u)\,\mathscr{P}^k\right)=\sum\chi(\mathscr{P}^k)\,\{\mathscr{P}^{j-k}u\cdot(\Phi_B\otimes\Phi(1))\}$$
$$=\sum_{k=0}^{j}\sum_{s=0}^{k}\chi(\mathscr{P}^{k-s})\,\mathscr{P}^{j-k}u\cdot\chi(\mathscr{P}^s)\,(\Phi_B\otimes\Phi)\,(1)$$

and as in Lemma 7.9, this is

$$u\cdot\chi(\mathscr{P}^j)\,(\Phi_B\otimes\phi)\,(1)=u\cdot 1\mathscr{P}^j\cdot(\Phi_B\otimes\phi)\,(1)=(\Phi_B\otimes\phi)\,(1\mathscr{P}^j\cdot u).$$

Thus $F_i^n\subset\ker q$. *

LEMMA 7.12. *If $2i<n$, then $I_n(U')^i=0$.*

Proof. Consider the composite

$$U'\xrightarrow{r}U'\otimes H^*(K(Z_p,n-i);Z_p)\xrightarrow{q}U'\otimes_A H^*(K(Z_p,n-i);Z_p)$$

where $r(u)=u\otimes i$.

In dimensions less than $2(n-i)$, $Z_p\oplus A\to H^*(K(Z_p,n-i);Z_p)$ sending $(1,0)$ to 1 and $(0,a)$ to ai is an isomorphism of left A modules, where A acts trivially on Z_p. Thus $U'\otimes_A H^*(K(Z_p,n-i);Z_p)\cong U'/U'A^+\oplus U'$ in dimensions less than $2(n-i)$, and under this identification, $qr(u)=(0,u)$. In particular, qr is monic on $(U')^j$ if $j+(n-i)<2(n-i)$. Since $2i<n$, $i+(n-i)<2(n-i)$ and qr is monic on $(U')^i$, but $I_n(U')^i$ is the kernel of qr. *

PROPOSITION 7.2. *U is the free associative, commutative algebra over Z_p on the classes b, $\bar{q}_i$, and $\beta\bar{q}_i$. Further, U is a free right A module and is a coalgebra with $\Delta(b)=b\otimes 1+1\otimes b$, $\Delta(\bar{q}_i)=\sum\bar{q}_{i-j}\otimes\bar{q}_j$. With this coalgebra structure and the right A action, U is a coalgebra over the Hopf algebra A.*

Proof. It suffices to show that $\sigma\colon U'\to U$ is an isomorphism and a homomorphism of left and right A modules, since U' has the properties described.

Let $f\colon U'\to M$ be a homomorphism of left A algebras with unit and right A modules with M a Poincaré algebra. Then $f(b)=b=1\beta=\theta'_M(\sigma(b))$, $f(\bar{q}_i)=\bar{q}_i=1\chi(\mathscr{P}^i)=\theta'_M(\sigma(\bar{q}_i))$ and $f(\beta\bar{q}_i)=\beta\bar{q}_i=\theta'_M(\sigma(\beta\bar{q}_i))$. Since f and $\theta'_M\sigma$ are algebra homomorphisms agreeing on generators, $f=\theta'_M\sigma$. Thus, if $\sigma(u')=0$, $f(u')=0$ for all such (M,f). Since $I_n(U')^i=0$ if $n>2i$, $f(u')=0$ for all such (M,f) implies $u'=0$, and thus σ is monic. Since σ was epic by Proposition 7.1, σ is an isomorphism.

Now let $x \in U'$, $a \in A$, and suppose $a\sigma(x) = \sigma(y)$, $y \in U'$. Then for all (M, f), $f(y - ax) = f(y) - af(x) = \theta'_M(\sigma(y)) - a\theta'_M(\sigma(x)) = \theta'_M(\sigma(y) - a\sigma(x)) = \theta'_M(0) = 0$, and so $y = ax$. Thus $a\sigma(x) = \sigma(ax)$.

If $x \in U'$, $a \in A$ and $\sigma(x)\,a = \sigma(y)$, $y \in U'$, then for all (M, f), $f(y - xa) = f(y) - f(x)\,a = \theta'_M(\sigma(y)) - \theta'_M(\sigma(x))\,a = \theta'_M(\sigma(y) - \sigma(x)\,a) = \theta'_M(0) = 0$ and so $y = xa$. Thus $\sigma(x)\,a = \sigma(xa)$. *

§8. Structure of Ω^p_*

In section 7, it was shown that U is a coalgebra. Since one wants to use this structure heavily, let us review it briefly.

LEMMA 8.1. *For each u in U there is a unique element* $\Delta u = \sum u'_s \otimes u''_s$ *in* $U \otimes U$ *such that*

$$\theta'_{M' \times M''}(u) = \sum \theta'_{M'}(u'_s) \otimes \theta'_{M''}(u''_s)$$

for all pairs M', M'' *of Poincaré algebras. The map* $\Delta : U \to U \otimes U$ *makes U into a Hopf algebra, and for* $a \in A$

$$\Delta(au) = \sum (-1)^{\dim u'_s \cdot \dim a''_r}\, a'_r u'_s \otimes a''_r u''_s$$

$$\Delta(ua) = \sum (-1)^{\dim u''_s \cdot \dim a'_r}\, u'_s a'_r \otimes u''_s a''_r .$$

Proof. (This is Adams' Lemma 14). Make $U \otimes U$ into a left A algebra and right A module by using the diagonal in A. One then has a function from W into $U \otimes U$.

No nonzero class in U^i goes to zero in all n dimensional Poincaré algebras if $n > 2i$, so there is an n-dimensional Poincaré algebra M' with $\theta'_{M'}$ monic on U^i (To see this let M_1 be any n dimensional Poincaré algebra. If $\theta'_{M_1}(u) = 0$ choose M_2 with $\theta'_{M_2}(u) \neq 0$, then $\ker \theta'_{M_1 \oplus M_2} \subset \ker \theta'_{M_1}$ properly. Since U^i is finite dimensional, this process is finite, giving M'), and then $\theta'_{M'}$ is monic in dimensions less than or equal to i. $\theta'_{M'} \otimes \theta'_{M'} : U \otimes U \to M' \otimes M'$ is then monic in dimensions less than i.

If $w \in W^i$ with $\varrho(w) = 0$ in U, then $\theta'_{M'} \otimes \theta'_{M'}(\theta_{U \otimes U}(w)) = \theta_{M' \times M'}(w) = \theta'_{M' \times M'}(\varrho(w)) = 0$ and so $\theta_{U \otimes U}(w) = 0$. Thus $\theta_{U \otimes U}$ induces a homomorphism $\Delta : U \to U \otimes U$ of left A algebras and right A modules. *

Note. This is the same coproduct as previously discussed for the rule for $\Delta(ua)$ gives the previous formulae.

The coproduct in U makes $\mathrm{Hom}(U; Z_p)$ into an algebra and $\Delta(UA^+) \subset UA^+ \otimes U + U \otimes UA^+$ so U/UA^+ becomes a coalgebra with $\mathrm{Hom}(U/UA^+; Z_p)$ a subalgebra of $\mathrm{Hom}(U; Z_p)$. The rule for $\theta'_{M' \times M''}$ shows that $\chi : \Omega^*_p \to \mathrm{Hom}(U/UA^+; Z_p)$ is a ring homomorphism.

For any $\omega=(i_1,\ldots,i_r)$, let S_ω be the polynomial expressing the symmetric function $\sum t_1^{i_1}\ldots t_r^{i_r}$ in terms of the elementary symmetric functions σ_i of the t's: i.e. $\sum t_1^{i_1}\ldots t_r^{i_r}=S_\omega(\sigma_1,\ldots,\sigma_n,\ldots)$. Let $s_\omega\in U$ be the class $s_\omega=S_\omega(\bar q_1,\ldots,\bar q_n,\ldots)$. Then $\Delta s_\omega=\sum s_{\omega'}\otimes s_{\omega''}$, the sum being over pairs ω', ω'' with $\omega'\cup\omega''=\omega$. The classes s_ω form a base for $Z_p[\bar q_i]$.

The classes $s_i\in U^{2i(p-1)}$ (corresponding to $\omega=(i)$) are primitive, and satisfy Newton's formula

$$s_i-\bar q_1 s_{i-1}+\cdots+(-1)^r\bar q_r s_{i-r}+\cdots+(-1)^i i\bar q_i=0.$$

Thus, if $i\not\equiv 0 \bmod p$, βs_i is also a nonzero primitive, being congruent to $-(-1)^i i\beta\bar q_i$ modulo decomposables. (If $i=pj$, $s_i=(s_j)^p$ and $\beta s_i=p(s_j)^{p-1}\beta s_j=0$). This gives nonzero primitives in each dimension $2i(p-1)$ and also $2i(p-1)+1$ if $i\not\equiv 0(p)$.

If $i=0$, $b\in U^1$ is a nonzero primitive of dimension $2i(p-1)+1$. If $i=p_j^k$, $j\not\equiv 0(p)$,

$$\mathscr{P}^{jp^{k-1}(p-1)}\ldots\mathscr{P}^{jp(p-1)}\mathscr{P}^{j(p-1)}\beta s_j$$

is a primitive of dimension $2i(p-1)+1$. To see that it is nonzero, consider U as U' and map into $H^*(BSF;Z_p)$ by killing b. According to Peterson and Toda (proof of Proposition 3.1) $\mathscr{P}^t\beta q'_s=(-1)^t\binom{s(p-1)}{t}\beta q'_{s+t}$+decomposables. Thus, modulo decomposables, this class is $(-1)^{jp^{k-1}(p-1)+\cdots+j(p-1)}\cdot(-1)^{j+1}j\beta\bar q_i$ which is nonzero.

In the following, let $\mathscr{P}^I\beta S_j$ denote the nonzero primitive element of dimension $2i(p-1)+1$, understanding for $i=0$ that this is the class b.

Since $\mathscr{P}^I\beta s_j$ is always indecomposable, the elements $s_\omega\cdot(\mathscr{P}^{I_1}\beta s_{j_1})\ldots(\mathscr{P}^{I_r}\beta s_{j_r})$ $(i_1<\cdots<i_r)$ form a base for U. In $\operatorname{Hom}(U;Z_p)$ let $x_{2i(p-1)}$ and $y_{2i(p-1)+1}$ ($i>0$ and $i\geqslant 0$ respectively) be the elements of the dual base with $s_i(x_{2i(p-1)})=1$ and $\mathscr{P}^I\beta s_j(y_{2i(p-1)+1})=1$. Then for $\omega=(k_1,\ldots,k_s)$,

$$x_{2k_1(p-1)}\cdots x_{2k_s(p-1)}y_{2i_1(p-1)+1}\cdots y_{2i_r(p-1)+1}$$

is the base element of $\operatorname{Hom}(U;Z_p)$ dual to $s_\omega(\mathscr{P}^{I_1}\beta s_{j_1})\ldots(\mathscr{P}^{I_r}\beta s_{j_r})$.

PROPOSITION 8.1. $\operatorname{Hom}(U;Z_p)$ *is the free associative, commutative algebra over* Z_p *on the classes* $x_{2i(p-1)}$ $(i>0)$ *and* $y_{2i(p-1)+1}$ $(i\geqslant 0)$.

Proof. The given free associative commutative algebra maps into $\operatorname{Hom}(U;Z_p)$ and the monomials forming its base are sent to a base of $\operatorname{Hom}(U;Z_p)$. *

Now one follows Liulevicius [4] to determine the structure of $\operatorname{Hom}(U/UA^+;Z_p)$. Denote by U^* the algebra $\operatorname{Hom}(U;Z_p)$ and by A^* the dual of the Steenrod algebra.

In addition to its algebra structure, U^* has the structure of an A^* comodule. Dualizing the right A action $U\otimes A\to U$ gives this coaction $U^*\to U^*\otimes A^*$.

Let H be the free associative commutative algebra over Z_p on generators $x'_{2i(p-1)}$

and $y'_{2i(p-1)+1}$ with i not of the form $(p^r-1)/(p-1)$. Let $f: U^* \to H$ be the algebra homomorphism given by

$$f(x_{2i(p-1)}) = \begin{cases} x'_{2i(p-1)} & i \neq (p^r-1)/(p-1) \\ 0 & i = (p^r-1)/(p-1) \end{cases}$$

$$f(y_{2i(p-1)+1}) = \begin{cases} y'_{2i(p-1)+1} & i \neq (p^r-1)/(p-1, \\ 0 & i = (p^r-1)/(p-1). \end{cases}$$

Let $g: U^* \to H \otimes A^*$ be defined by $(f \otimes 1) \circ \mu$.

Giving $H \otimes A^*$ the obvious algebra structure and the "free" A^* comodule structure, g is a homomorphism of algebras and A^* comodules according to Proposition 2 of Liulevicius.

As algebra, A^* is the free associative commutative algebra on generators $x'_{2i(p-1)}$ $(i>0)$ and $y'_{2i(p-1)+1}$ $(i \geqslant 0)$ where i is of the form $(p^r-1)/(p-1)$ dual to primitive elements of dimensions $2(p^r-1)$ and $2(p^r-1)+1$ in A.

To prove that g is an isomorphism, it suffices to prove g is epic (since everything is of finite type and has the same dimension as Z_p vector space in each dimension). For this, it suffices to show that $g(x_{2i(p-1)})$ and $g(y_{2i(p-1)+1})$ are indecomposable. Since all generators have different dimensions, it is sufficient to show that

$$\phi_1: U^* \xrightarrow{g} H \otimes A^* \to H \otimes A^*/A^*_+ \cong H \otimes Z_p \cong H$$

and

$$\phi_2: U^* \xrightarrow{g} H \otimes A^* \to H/H_+ \otimes A^* \cong Z_p \otimes A^* \cong A^*$$

are epic.

Now ϕ_1 is just the composite

$$U^* \xrightarrow{\mu} U^* \otimes A^* \xrightarrow{\tau} U^* \otimes Z_p \cong U^* \xrightarrow{f} H$$

and $\tau \circ \mu$ is dual to $U^* \to U^*: u \to 1(u)$, $1 \in A$ being the unit, so $\tau \circ \mu = 1$ and $\phi_1 = f$ is epic. The map ϕ_2 is just the composite

$$U^* \xrightarrow{\mu} U^* \otimes A^* \to Z_p \otimes A^* \cong A^*$$

and is dual to $v: A \to U: a \to a(1)$ which is monic, so $\phi_2 = v^*$ is epic.

Thus $g: U^* \to H \otimes A^*$ is an isomorphism of algebras over A^*. Dually $U \cong H^* \otimes A$ is an isomorphism of coalgebras over A, and U/UA^+ is isomorphic to H^* as coalgebra, or $\mathrm{Hom}(U/UA^+; Z_p) \cong H$ as algebra. This gives:

PROPOSITION 8.2. $\mathrm{Hom}(U/UA^+; Z_p)$ *is the free associative, commutative algebra over* Z_p *on generators* $x'_{2i(p-1)}$ *and* $y'_{2i(p-1)+1}$ *with* $i \neq (p^r-1)/(p-1)$.

PROPOSITION 8.3. *Ω^p_* is the associative commutative algebra over Z generated by classes*

x_a, $a \neq 0$ in Z_p, of dimension zero, y_4 of dimension 4,

$x'_{2i(p-1)}$ and $y'_{2i(p-1)+1}$ for $i \neq (p^r-1)/(p-1)$ of dimension $2i(p-1)$ and $2i(p-1)+1$ respectively.

The relations among these classes are:

$x_1=1$ is the unit. $x_{-a}=-x_a$. $x_a \cdot x_b = x_{ab}$.

$2y_4=0$ if $p \equiv 1 \pmod 4$ or $4y_4=0$, if $p \equiv 3 \pmod 4$.

$x_a y_4 = y_4$ if a is a square. $px'_{2i(p-1)}=0$. $py'_{2i(p-1)+1}=0$. $x_a x'_{2i(p-1)} = a x'_{2i(p-1)}$. $x_a y'_{2i(p-1)+1} = a y'_{2i(p-1)+1}$.

Proof. This has all been proved except for the rules $x_a x'_{2i(p-1)} = a x'_{2i(p-1)}$, and similarly for y'. These follow at once from the fact that the characteristic numbers of $M_a \times M$ are precisely a times those of M. *

Note. The epimorphism $U^* \xrightarrow{f} H$ gives a monomorphism $H^* \to U$. The image of H^* has as basis the classes

$$s_\omega(\mathscr{P}^{I_r}\beta s_{j_1}) \dots (\mathscr{P}^{I_r}\beta s_{j_r})$$

with $\omega=(k_1, \dots, k_s)$ where k_α and i_α are not of the form $(p^r-1)/(p-1)$. (Recall $\mathscr{P}^I\beta s_j$ has dimension $2i_\alpha(p-1)+1$. Thus, these characteristic numbers suffice to detect the p-torsion in Ω^p_*.

§9. Image of Oriented Bordism

Being given a closed oriented n-dimensional differentiable manifold M^n, one may assign to M its mod$p$ cohomology, $H^*(M; Z_p)$, giving an n-dimensional Poincaré algebra. This assignment induces a forgetful homomorphism

$$F: \Omega^{so}_* \to \Omega^p_*$$

where Ω^{so}_* denotes the oriented bordism ring studied by Wall [7].

PROPOSITION 9.1. *The image of $F: \Omega^{so}_* \to \Omega^p_*$ is the (polynomial) subalgebra generated by the classes y_4 and $x'_{2i(p-1)}$.*

Proof. According to Wall, the torsion subgroup of Ω^{so}_* is a Z_2 vector space and (in his notation) is generated as Ω^{so}_* module by the classes $\partial(x_{2k_1} \dots x_{2k_r})$, $k_i \neq 2^s$, which are odd dimensional. Since Ω^p_* has no odd dimensional 2-torsion, F annihilates the torsion of Ω^{so}_*. Thus F induces $F': \Omega^{so}_*/\text{Torsion} \to \Omega^p_*$. Now $\Omega^{so}_*/\text{Torsion}$ is the polynomial ring over the integers on generators z_{4j}.

Considering first just the 2 primary part $(\Omega_*^{so}/\text{Torsion})\otimes Z_2$ is the Z_2 polynomial ring on the complex projective spaces $CP(2j)$, and hence the image of F in the 2 primary part of Ω_*^p is generated by the $CP(2j)$. Since the middle dimensional pairing in $CP(2j)$ has $P=1$, $N=0$, $CP(2j)$ hits $(y_4)^j$ in the 2 primary part of Ω_*^p.

Now looking at the p-primary part, let M be a closed oriented manifold and let $\mathscr{P}_i\in H^{4i}(M;Z_p)$ denote its i-th *normal* Pontrjagin class reduced modulo p. Let $s_\omega(\mathscr{P})$ denote the class $S_\omega(\mathscr{P}_1,\dots,\mathscr{P}_n,\dots)$.

If the total Pontrjagin class $\mathscr{P}=1+\mathscr{P}_1+\cdots$ is written formally as $\Pi(1+x_i^2)$ $(\dim x_i=2)$, then $\bar{q}=1+\bar{q}_1+\cdots$ is given by $\Pi(1+x_i^{p+1})$. Thus $\theta'_M:U\to H^*(M;Z_p)$ sends s_ω into $s_{\omega'}(\mathscr{P})$ where for $\omega=(i_1,\dots,i_r)$, $\omega'=(i_1(p-1)/2,\dots,i_r(p-1)/2)$.

Since β into the top dimensional cohomology of M is zero, $\theta'_M(b)=0$, and since $\mathscr{P}_i$ is the reduction of an integral class, θ'_M kills all Bocksteins.

Thus $\chi\circ F'(\Omega_*^{so})$ is detected by the characteristic numbers s_ω where $\omega=(i_1,\dots,i_r)$ and no i_α is of the form $(p^r-1)/(p-1)$.

Since there are oriented manifolds M^{4n} with $s_n(\mathscr{P})\,[M^{4n}]\not\equiv 0\ (\text{mod}\,p)$ except when $2n+1=p^s$, there are manifolds $M^{2i(p-1)}$ with $\phi_M(\theta'_M(s_i))\neq 0$ except when $i(p-1)+1=p^s$. One may then take some multiple (divisible by 4) of $M^{2i(p-1)}$ as a generator $x'_{2i(p-1)}$. These classes generate $\chi\circ F'(\Omega_*^{so})$, which is then the subalgebra generated by the $x'_{2i(p-1)}$. *

Note. The invariant $IF(M)$ in $Z_2\oplus Z_2$ or Z_4 is easily seen to coincide with the Hirzebruch index of M reduced mod 2 or 4. Thus, the Hirzebruch index modulo 4 can be computed from the Z_p cohomology if $p\equiv 3$ mod 4. Modulo 2, the Hirzebruch index is just the Euler characteristic and may be computed with any coefficients.

Letting $A'\subset A$ denote the algebra of reduced powers, one may consider Poincaré algebras over A', as studied by Adams, with corresponding bordism ring $\Omega_*'^p$. Considering an A' algebra as an A algebra with $\beta=0$ (i.e. A' is isomorphic to A mod the two sided ideal generated by β) and considering an A algebra as an A' algebra by restriction gives algebra homomorphisms $\sigma:\Omega_*'^p\to\Omega_*^p$ and $\varrho:\Omega_*^p\to\Omega_*'^p$ with $\varrho\sigma=1$. It is easy to compute $\Omega_*'^p$ using Adams' results and one has

$$\Omega_*'^p\cong\frac{Z[y_4,x'_{2i(p-1)}\mid i\neq(p^r-1)/(p-1)]}{\{px'_{2i(p-1)}=0,\,2y_4=0\ (p\equiv 1)\quad\text{or}\quad 4y_4=0\ (p\equiv 3)\}},$$

It is immediate that $F:\Omega_*^{so}\to\Omega_*^p$ has the same image as σ, and that ϱF is epic.

It should be noted that there are classes in Ω_*^p which cannot be represented as $H^*(M;Z_p)$ for any oriented Poincaré duality space. For an oriented Poincaré duality space M^n of dimension n, $\beta:H^{n-1}(M;Z_p)\to H^n(M;Z_p)$ is zero. Thus, if $n=2i(p-1)+1$ with $i\not\equiv 0\ (\text{mod}\,p)$, the characteristic number βs_i must vanish, and $H^*(M^n;Z_p)$ is decomposable in Ω_*^p.

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