Structural Theorems for Topological Actions of Z_2-Tori on Real, Complex and Quaternionic Projective Spaces

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§0. Introduction

In this paper, we apply the fundamental fixed point theorem of [2] in cohomology theory of transformation groups to the special cases of Z_2-tori actions on real, complex, and quaternionic projective spaces. The main results of this paper are the following structural theorems which enable us to organize the whole information of cohomological aspect of orbit structure of such actions into a neat, simple, algebraic invariant that we shall call it the system of Z_2-weights. Throughout this paper, all cohomology algebras are over the field Z_2 and will be suppressed from the notation. For example, the notation $X \sim Y$ will mean that X and Y are of the same Z_2-cohomology type. Following [2], we shall consider $H^*(X_G)$ as the *equivariant cohomology* of the G-space X and denote it by $H_G^*(X)$, where X_G is the total space of the universal bundle $X \to X_G \to B_G$ with X as typical fibre. Observe that *the equivariant cohomology* of a G-space X, $H_G^*(X)$, is *an algebra over* $H_G^*(pt) = H^*(B_G)$ which shall be simply *denoted by* R. Again, following [2], we shall call the *connected component of* x *in* $F(G_x, X)$ the *F-variety of* x and denote it by $F(x)$.

THEOREM 1. *Let* G *be a* Z_2*-torus and* $X \sim RP^n$ *with a given* G*-action.* $F(G, X) \neq \emptyset$ *(non-empty). Then, the equivariant cohomology of* X, $H_G^*(X)$, *is isomorphic to* $R[\xi]/\langle f(\xi)\rangle$, $\deg(\xi)=1$, *as an* R*-algebra; where*

$$f(\xi) = (\xi + w_1)^{k_1} \cdots (\xi + w_s)^{k_s}$$

is a splitting polynomial of degree $(n+1)$ *with* $w_j \in H^1(B_G)$ *as distinct roots. And correspondingly, the fixed point set* F *consists of* s *connected components* $\{F^j, 1 \leqslant j \leqslant s\}$ $(s \leqslant 2^{rk(G)})$ *such that*

(i) $F^j \sim RP^{(k_j-1)}$ *and* $\iota_j^*: H_G^*(X) \to H_G^*(q_j)$, $q_j \in F^j$, *maps* ξ *to* w_j,

(ii) *the system of local weights at* F^j, Ω_j, *is given by*

$$\Omega_j = \{(w_i + w_j), \text{ with multi. } k_i \ (i \neq j); 0 \text{ with multi. } (k_j - 1)\}$$

(iii) *for a given point* $x \in X$, *let* $F^{j_1}, \cdots, F^{j_l}$ *be those components of* $F(G, X)$ *contained in the F-variety of* x, $F(x)$, *then* G_x *is given by*

$$w_{j_1} = w_{j_2} = \cdots = w_{j_l} \quad \text{and} \quad F(x) \sim RP^m, \quad m = (k_{j_1} + \cdots + k_{j_l} - 1).$$

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THEOREM 2. *Let G be a $\mathbf{Z}_2$-torus and $X \sim CP^n$ with a given G-action, $F(G, X) \neq \emptyset$ (non-empty). Then, the equivariant cohomology of X, $H_G^*(X)$, is isomorphic to $R[\eta]/\langle f(\eta)\rangle$, $deg(\eta)=2$, as an R-algebra where*

$$f(\eta)=(\eta+\alpha_1)^{k_1}\cdots(\eta+\alpha_s)^{k_s}, \quad (k_1+\cdots+k_s)=(n+1)$$

is a splitting polynomial with $\alpha_j \in H^2(B_G)$ as distinct roots, and η is a suitable lifting of the generator η_0 of $H^*(X)$ such that $Sq^1\eta=\beta\eta$.

Moreover, there are the following two cases according to $\beta=0$ or $\beta\neq 0$:

(a) *In case $\beta=0$. Then all connected components of $F(G, X)$ are of CP-type, i.e., $F^j \sim CP^{(k_j-1)}$ and all roots α_j are perfect squares, namely, $\alpha_j=w_j^2$. And furthermore, the system of local weights at F^j, Ω_j, is given by*

$$\Omega_j=\{(w_i+w_j), \text{ with multi. } 2k_i (i\neq j);\ 0 \text{ with multi. } 2(k_j-1)\}$$

and, for a given point $x\in X$ with $F(x)\cap F(G, X)=F^{j_1}+\cdots+F^{j_l}$, the isotropy group G_x is given by $w_{j_1}=\cdots=w_{j_l}$, $F(x)\sim CP^m$, $m=(k_{j_1}+\cdots+k_{j_l}-1)$.

(b) *In case $\beta\neq 0$. Then all connected components of $F(G, X)$ are of RP-type, i.e., $F^j \sim RP^{(k_j-1)}$ and the roots α_j are of the form $\beta w_j+w_j^2$. And furthermore, the system of local weights at F^j, Ω_j, is given by*

$$\Omega_j=\{(w_i+w_j), (w_i+w_j+\beta), \text{ with multi. } k_i;\ \beta, 0 \text{ with multi. } (k_j-1)\}$$

and $F(x)$ is of CP-type if and only if $\beta \mid G_x=0$.

THEOREM 3. *Let G be a $\mathbf{Z}_2$-torus and $X \sim QP^n$ with a given G-action, $F(G, X)\neq\emptyset$ (non-empty). Then the equivariant cohomology of X, $H_G^*(X)$, is isomorphic to $R[\zeta]/\langle f(\zeta)\rangle$, $\deg(\zeta)=4$, as an R-algebra where*

$$f(\zeta)=(\zeta+\alpha_1)^{k_1}\cdots(\zeta+\alpha_s)^{k_s}, \quad (k_1+\cdots+k_s)=(n+1)$$

is a splitting polynomial with $\alpha_j\in H^4(B_G)$ as distinct roots, and ζ is a suitable lifting of the generator of ζ_0 of $H^*(X)$ such that $Sq^2(\zeta)=\gamma\zeta$.

Moreover, there are the following two cases:

(a) *Case 1. $Sq^2(\zeta)=\gamma\zeta=0$; then all connected components of $F(G, X)$ are of QP-type, i.e., $F^j\sim QP^{(k_j-1)}$ and the roots $\alpha_j=w_j^4$. And furthermore, the system of local weights at F^j, Ω_j, is given by*

$$\Omega_j=\{(w_i+w_j); \text{ with multi. } 4k_i, (i\neq j);\ 0, \text{ with multi. } 4(k_j-1)\}$$

and, for a given point $x\in X$ with $F(x)\cap F(G, X)=F^{j_1}+\cdots+F^{j_l}$, the isotropy group G_x is given by $w_{j_1}=\cdots=w_{j_l}$, $F(x)\sim QP^m$, $m=(k_{j_1}+\cdots+k_{j_l}-1)$.

(b) *Case 2. $Sq^2(\zeta)=\gamma\zeta\neq 0$; then all connected components of $F(G, X)$ are of CP-*

type, i.e., $F^j \sim CP^{(k_j-1)}$ *and the roots* α_j *are of the form* $\alpha_j = v^2 w_j^2 + w_j^2$, *where* $\gamma = v^2$ *and* w_j, $v \in H^1(B_G)$. *And furthermore, the system of local weights at* F^j, Ω_j, *is given by*

$$\Omega_j = \{(w_i + w_j),\ (v + w_i + w_j)\ \textit{multi.}\ 2k_i\ (i \neq j);\ v,\ 0\ \textit{multi.}\ 2(k_j - 1)\}$$

and $F(x)$ *is of CP-type (resp. QP type) if* $v \mid G_x \neq 0$ *(resp.* $v \mid G_x = 0$*).*

In section 1, we shall recall some basic facts in cohomology theory of transformation groups that are needed in the proofs of the above structural theorems. In fact, the above structural theorems can be regarded as some of the simplest applications of the fixed point theorem of [2] and the splitting theorem of [3] to the simple interesting concrete spaces such as real, complex and quaternionic projective spaces. The proofs of theorem 1 and 2 are given in §2 and the proof of theorem 3 are given in §3. In section 4, examples of linear models are analyzed.

Finally, it is interesting to note that a comparison of the results of this paper and the corresponding results of [3] for the cases of odd primes and characteristic zero case will show that the $\mathbf{Z}_2$-case do have some interesting special features.

§1. Some Basic Facts

In this section, we shall recall some basic concepts and fundamental results of [2, 3] that are essential for the proofs of the above structural theorems. Since we shall only need the $\mathbf{Z}_2$-version of such results, it is notation-wise simpler to state them directly in the $\mathbf{Z}_2$-setting rather than the general setting.

(A) *A Fundamental Fixed Point Theorem*

Let G be a $\mathbf{Z}_2$-torus and X be a given G-space with finite dimensional cohomology. Let $H_G^*(X) = H^*(X_G)$ be the equivariant cohomology algebra of X which is clearly an R-algebra, $\mathrm{R} = H^*(B_G)$. The following theorem of [2] correlates the torsion-free part of $H_G^*(X)$ (as an R-module) and the cohomology structure of the fixed point set $F = F(G, X)$.

THEOREM A ($\mathbf{Z}_2$-version) [2]: *Let* G *be a* $\mathbf{Z}_2$*-torus,* R_0 *be the quotient field of* $R = H^*(B_G) = \mathbf{Z}_2[t_1, \cdots, t_l]$, $l = \mathrm{rk}(G)$. Suppose that $H_G^*(X) \otimes_R R_0$ *is given by the following presentation* ϱ *(as an* R_0*-algebra) in terms of generators and relations: Namely,* ϱ *is an epimorphism of the polynomial algebra* $R_0[x_1, \cdots, x_m]$ *onto* $H_G^*(X) \otimes_R R_0$ *with* $I = \mathrm{Ker}(\varrho)$ *as the ideal of defining relations. Then*

(i) *The radical of* I, $\sqrt{I}$, *decomposes into the intersection of* s *maximal ideals* $M_j = M(\alpha_j)$ *whose varieties are respectively the "rational" points* $\alpha_j = (\alpha_{j1}, \cdots, \alpha_{jm}) \in R_0^m$, *i.e.,* $\sqrt{I} = M_1 \cap \cdots \cap M_s$, $V(I) = \{\alpha_1, \cdots, \alpha_s\} \subseteq R_0^m$.

(ii) *There is a natural bijection between the connected components of the fixed point set,* $\{F^j\}$, *and the above points in* R_0^m, $\{\alpha_j\}$, *such that the restriction homomorphism of an arbitrary point* $q_j \in F^j \subseteq X$ *maps the generator* $x_i \in H_G^*(X) \otimes_R R_0$ *to* $\alpha_{ji} \in H_G^*(q_j) \otimes_R R_0 = R_0$.

(iii) $H^*(F^j) \otimes_{\mathbf{Z}_2} R_0 \cong A/I_j$ *where* $I_j = I_{M_j} \cap A$, I_{M_j} *is the localization of* I *at* M_j *and moreover* $I = I_1 \cap \cdots \cap I_s = I_1 \cdot I_2 \cdots I_s$.

Remark. Obviously, it is always possible to choose $\{x_i\}$ so that they already lie in $H_G^*(X)$. Then, their restrictions α_{ji} lie in $H_G^*(q_j) = R$. Namely, $\{\alpha_j\} \subseteq R^m \subseteq R_0^m$.

As one of the simplest direct consequence of the above theorem, let us mention the following generalization of a result of Bredon [1]. Namely

COROLLARY. *Suppose* $H_G^*(X) \otimes_R R_0$ *is generated (as an* R_0*-algebra) by* $\{x_i \in H_G^{n_i}(X),\ 1 \leqslant i \leqslant m\}$. *Then the number of connected components of the fixed point set* $F(G, X)$ *is bounded by the number of elements in the following* $\mathbf{Z}_2$*-vector space*

$$H^{n_1}(B_G) \times H^{n_2}(B_G) \times \cdots \times H^{n_m}(B_G).$$

Proof. Since F^j are indexed by *different* elements α_j in the above $\mathbf{Z}_2$-vector space. (The special case $m=1$ and $G=\mathbf{Z}_2$ (resp. $\mathbf{Z}_p$) is included in statement of Th. (4.1) of [1]).

(B) *A Special Case*

As one of the simplest application of the above fixed point theorem let us consider the case that $X \sim RP^n$ (resp. CP^n, QP^n) as follows:

Since $H^*(X) \cong \mathbf{Z}_2[\xi_0]/\langle \xi_0^{n+1} \rangle$, $\deg(\xi_0) = 1$ (resp. 2, 4), it is easy to see that the E_2-term of the $\mathbf{Z}_2$-Serre spectral sequence of the fibration $X \to X_G \to B_G$ is equal to $H^*(B_G) \otimes_{\mathbf{Z}_2} H^*(X)$, and the generator ξ_0 is clearly *transgressive.* On the other hand, it follows trivially from the existence of fixed points that $H^*(B_G) \to H^*(X_G)$ is injective. Therefore, the transgression of ξ_0 must be zero and hence $E_2 = E_\infty$, $H^*(X_G) \cong R \otimes_{\mathbf{Z}_2} H^*(X)$ as an R-module; $H^*(X_G) \to H^*(X)$ is surjective. Let ξ be an arbitrary *lifting of* ξ_0 into $H_G^*(X)$. Then, ξ clearly generates the R-algebra $H_G^*(X)$, namely

$$H_G^*(X) \cong R[\xi]/\langle f(\xi) \rangle \quad \text{(as an } R\text{-algebra)}$$

where $f(\xi) = (\xi^{n+1} + c_1 \xi^n + \cdots + c_{n+1}) \in R[\xi]$ is the defining relation. It is then a direct consequence of theorem A that $f(\xi)$ *splits into product of linear factors,* say

$$f(\xi) = (\xi - \alpha_1)^{k_1} \cdots (\xi - \alpha_s)^{k_s}$$

where $\alpha_j \in H^1(B_G)$ (resp. $H^2(B_G)$, $H^4(B_G)$) are the distinct roots. And correspondingly, the fixed point set F consists of s connected components, $\{F^j, 1 \leqslant j \leqslant s\}$, indexed by

$\{\alpha_j\}$ respectively and $H^*(F^j)\cong \mathbf{Z}_2[\eta]/\langle\eta^{k_j}\rangle$, $\deg(\eta)\leqslant\deg(\xi)$. Hence, the beginning parts of the structural theorems 1, 2 and 3 are simply direct consequences of the fixed point theorem of [2]. In fact, it is exactly the above setting of neat correlation between the algebraic structure of $H_G^*(X)$ and the geometric structure of F that makes possible both the formulation and the proof of such structural theorems.

(C) *Structure of F-Varieties and System of Local Weights*

Geometrically, G-spaces are quite analogous to algebraic varieties in the sense that both consist of points of various degree of singularity. An analogy to the concept of Zariski closure of points in an algebraic variety, the following definition F-variety was introduced in [2] and plays a rather useful basic rôle in recent development of cohomology theory of transformation groups.

DEFINITION. The F-variety of x (or rather, spanned by x), denoted by $F(x)$, is the connected component of x in $F(G_x, X)$.

Clearly, for a given G-space X, the isotropy subgroups $\{G_x; x\in X\}$ are those distinguished subgroups and the F-varieties $\{F(x); x\in X\}$ are those distinguished sub-G-spaces. Suppose $Y=F(x)$. Then x is called a *generic point* of Y and G_x is called the *generic isotropy subgroup of* Y, which is obviously the smallest isotropy subgroup among $\{G_y; y\in Y\}$ and shall be denoted by G_Y. The rank (resp. corank) of an F-variety Y is defined to be the rank (resp. corank) of its generic isotropy subgroup G_Y, i.e. $\mathrm{rk}(Y)=\mathrm{rk}(G_Y)$.

In case X is a differentiable G-manifold and x_0 is a fixed point. Then the local structure of F-varieties passing x_0 is given by the "linear model" whose G-action is that of the local representation of G on the tangent space of x_0. Group theoretically, such a representation ϕ_{x_0} of a $\mathbf{Z}_2$-torus G is a homomorphism of G into $0(n)$, which always factors through a maximal $\mathbf{Z}_2$-torus of $0(n)$, i.e., $G\xrightarrow{\phi_{x_0}}0(1)^n\subseteq 0(n)$. Geometrically, the vector space $\mathbf{R}^n$ splits into the direct sum of one-dimensional invariant linear subspaces. Hence, it is easy to see the following relationship between the weight system of ϕ_{x_0}, Ω_{x_0}, and the local structure of F-varieties at x_0: Namely,

multi. of zero weight $=$ dim of F at x_0

multi. of a non-zero weight $w=$ codim of F in $F(w^\perp, X)$ at x_0.

In the more general case that X is a $\mathbf{Z}_2$-cohomology G-manifold, then all F-varieties are again $\mathbf{Z}_2$-cohomology manifolds and, as a consequence of Borel formula, the same dimensional relation still holds. Namely,

$$\dim X-\dim F^0=\sum(\dim Y-\dim F^0)$$

where F^0 is the connected component of x_0 in F and Y runs through all corank one F-varieties passing through x_0.

DEFINITION. Let G be a $\mathbf{Z}_2$-torus, X be a $\mathbf{Z}_2$-cohomology G-manifold and $x_0 \in X$ be a fixed point. Then the *system of local weights at* x_0, again denoted by Ω_{x_0}, consists of the following $\mathbf{Z}_2$-weights with multiplicities:

$$\Omega_{x_0} = \{w, m_w; 0, m_0\}$$

where $m_0 = \dim F(x_0)$, $\{w^{\perp}\}$ *are the generic isotropy subgroups of those corank one F-varieties* $\{Y_w\}$ *passing through* x_0 *and* $m_w = (\dim Y_w - \dim F(x_0))$.

We refer to [4] for detail discussion of system of local weights and its many applications.

(D) *A Splitting Theorem*

We shall need the following $\mathbf{Z}_2$-version of a splitting theorem of [3].

THEOREM B ($\mathbf{Z}_2$-version). *Let G be a* $\mathbf{Z}_2$*-torus and X be a connected* $\mathbf{Z}_2$*-cohomology G-manifold. Suppose that*

$$E_2 = E_\infty = H^*(B_G) \otimes_{\mathbf{Z}_2} H^*(X)$$

in the $\mathbf{Z}_2$*-Serre spectral sequence of the fibration* $X \to X_G \to B_G$. *Then*

(i) *For each element* $f \in H^*(F)$, *the following ideal* $I_X(f)$:

$$I_X(f) = \{a \in R; a \otimes f \in \operatorname{Im}(H_G^*(X) \to H_G^*(F) = \mathrm{R} \otimes H^*(F))\}$$

is a principal ideal with its generator $a(f)$ *splits into product of linear factors, i.e.,*

$$a(f) = w_1^{k_1} \cdots w_l^{k_l}, \qquad (l=0 \text{ if } I(f)=R),\ w_j \in H^1(B_G).$$

(ii) *Correspondingly,* $\{H_j = w_j^{\perp}\}$ *are exactly those maximal* $\mathbf{Z}_2$*-subtori satisfying the condition* $I_Y(f) \neq R$; $Y = F(H, X)$; *and moreover,* $I_{Y_j}(f) = (w_j^{k_j})$ *for* $Y_j = F(H_j, X)$.

(iii) *In particular, if* f_j *is the fundamental cohomology class of* F^j, *then* $I_X(f_j)$ *is generated by the product of non-zero local weights (with multiplicities) at* F^j, *i.e.,* $a(f_j) = w_1^{k_1} \cdots w_l^{k_l}$ *where* $\{w_1, k_1; \cdots; w_l, k_l\}$ *is the system of non-zero local weights at* F^j.

We refer to [3] for a proof of above theorem, and to [5] for a rather far-reaching generalization of the above theorem by T. Chang and T. Skjelbred.

§2. Proof of Theorem 1 and 2

(A) *Proof of Theorem* 1

Since the beginning part of Theorem 1 has already been proved in (B) of §1, we need only to prove the assertions (ii) and (iii) as follows:

Let $\xi_j \in H^1(F^j)$ be the generator of $H^*(F^j)$, i.e., $H^*(F^j) \cong \mathbf{Z}_2[\xi_j]/\langle \xi_j^{k_j} \rangle$. Then

$\iota^*(\xi)=(\xi_1+w_1, \xi_2+w_2, \cdots, \xi_s+w_s)\in H_G^1(F)$, and

$$f_j=(0,\dots, 0, \xi_j^{(k_j-1)}, 0,\dots, 0)$$

is the fundamental cohomology class of F^j. Suppose a is an arbitrary element of $I_X(f_j)$. Then, by definition, there exists an element of $H_G^*(X)$, say $g(\xi)$, such that $\iota^*(g(\xi))=a\cdot f_j$. Hence

$$\iota^*(g(\xi)\cdot(\xi-w_j))=(0,\dots, 0, a\xi_j^{(k_j-1)}, 0,\dots, 0)\cdot(*\cdots*, \xi_j, *,\dots, *)=0.$$

Therefore, it follows from the injectivity of ι^* that $g(\xi)\cdot(\xi-w_j)$ is divisible by $f(\xi)$, or equivalently, $g(\xi)$ is divisible by $f(\xi)/(\xi-w_j)$. Then, it is easy to see that

$$\iota^*\left(\frac{f(\xi)}{(\xi-w_j)}\right)=a(f_j)\cdot f_j$$

where $a(f_j)=\prod_{i\neq j},(w_i+w_j)^{k_i}$ is the generator of $I_X(f_j)$. Hence, (ii) follows from the above computation and (iii) of theorem B.

Let $Y=F(x)$ be the F-variety of x and $Y\cap F=F^{j_1}+\ \ +F^{j_l}$, and $\bar{\xi}$ be the restriction of ξ to $H_G^*(Y)$. Then, it is clear that $Y\sim RP^m$, $m=(k_{j_1}+\cdots+k_{j_l}-1)$ and $H_G^*(Y)\cong R[\bar{\xi}]/\langle\bar{f}(\bar{\xi})\rangle$, where $\bar{f}(\bar{\xi})=(\bar{\xi}-w_{j1})^{k_{j1}}\dots(\bar{\xi}-w_{jl})^{k_{jl}}$. On the other hand, it follows from (ii) that the generic isotropy subgroup of X, $G_X=\mathrm{Ker}(X)$ is given by putting all local weights in Ω_1 equal to zero, i.e.

$$w_1+w_2=w_1+w_3=\cdots=w_1+w_s=0,$$

or equivalently,

$$w_1=w_2=\cdots=w_s.$$

Finally, applying the above result to Y instead of X, we see that G_Y is given by $w_{j1}=w_{j2}=\cdots=w_{jl}$. q.e.d.

(B) *Proof of Theorem* 2

We choose the lifting η of the generator η_0 of $H_G^*(X)$ into $H\ \ (X)$ so that $\alpha_1=$ the restriction of η to $H_G^*(q_1)=0$. Then, it is easy to see that $Sq^1(\eta)=\beta\cdot\eta$, and we shall divide the proof into two cases according to $\beta=0$ or $\beta\neq 0$:

(1) *The case* $\beta=0$, *(i.e.,* $Sq^1(\eta)=0$*)*. We claim that there are no components of RP-type. Suppose the contrary, say F^1 is of RP-type. Let ξ_1 be the generator of $H^*(F^1)$, $\deg(\xi_1)=1$. Then $Sq^1(\eta)=0$ implies that

$$Sq^1(\iota_1^*\eta)=Sq^1(*+\beta_1\xi_1+\alpha_1)=\beta_1\xi_1^2+\beta_1^2\xi_1+Sq^1\alpha_1=0.$$

Hence $\beta_1=0$, which implies that $H^*(F^1)$ is generated by ξ_1^2, a contradiction. On the other hand, $Sq^1(\eta)=0$ implies $Sq^1(\alpha_j)=0$, (notice that α_j is the restriction of η to $H_G^*(q_j)$). Hence α_j must be a perfect square, i.e., $\alpha_j=w_j^2$, $w_j\in H^1(B_G)$. Therefore

$$\iota^*(\eta)=(\eta_1+w_1^2, \eta_2+w_2^2, \ldots, \eta_s+w_s^2)\in H_G^2(F)$$

and the same computation as in the proof of Theorem 1 will show that

$$I_X(f_j)=\prod_{i\neq j}(w_i^2+w_j^2)^{k_i}=\prod_{i\neq j}(w_i+w_j)^{2k_i}.$$

The rest of the proof for this case is the same as that of Theorem 1.

(2) *The case $\beta\neq 0$.* In this case, we claim that there are no components of CP-type Suppose the contrary that $F^1\sim CP^{(k_1-1)}$, $(k_1-1)>0$, and η_1 be the generator of $H^*(F^1)$. Then $\iota_1^*(\eta)=\eta_1+\alpha_1$ and

$$\beta\eta_1+\beta\alpha_1=\iota_1^*(\beta\cdot\eta)=\iota_1^*Sq^1\eta=Sq^1\iota_1^*\eta=Sq^1(\eta_1+\alpha_1)=Sq^1\alpha_1$$

which is clearly a contradiction. Hence, all components are of RP-type. Let $\iota_j^*(\eta)=\gamma_j\xi_j^2+\beta_j\xi_j+\alpha_j$, $\deg\xi_j=1$. Then

$$\beta\cdot(\gamma_j\xi_j^2+\beta_j\xi_j+\alpha_j)=\iota_j^*(Sq^1(\eta))=Sq^1(\gamma_j\xi_j^2+\beta_j\xi_j+\alpha_j)=\beta_j\xi_j^2+\beta_j^2\xi_j+Sq^1\alpha_j$$

implies that $\beta=\beta_j$, $\gamma_j=1$ and $Sq^1\alpha_j=\beta\cdot\alpha_j$. Observe that $\deg\beta=1$ and hence can be considered as a variable in $R=\mathbf{Z}_2[t]$. Write $\alpha_j=\beta\cdot w_j+\alpha_j'$. Then $Sq^1(\beta\cdot w_j+\alpha_j')=$ $=\beta^2w_j+\beta w_j^2+Sq^1\alpha_j'=\beta^2w_j+\beta\alpha_j'$ implies that $\alpha_j'=w_j^2$, that is, $\alpha_j=\beta\cdot w_j+w_j^2$. Therefore

$$\iota^*(\eta)=(\xi_1^2+\beta\xi_1+\beta w_1+w_1^2, \ldots, \xi_s^2+\beta\xi_s+\beta w_s+w_s^2)\in H_G^2(F)$$

and the rest of the proof of this case follows readily from the same type of computation as that of Theorem 1. q.e.d.

§3. Proof of Theorem 3

(A) We choose the lifting ζ of the degree 4 generator ζ_0 of $H^*(X)$ into $H_G^*(X)$ so that $\alpha_1=$the restriction of ζ to $H_G^*(q_1)=0$. Then, it is easy to see that $Sq^1(\zeta)=\beta\cdot\zeta$ and $Sq^2=\gamma\cdot\zeta$; $\beta\in H^1(B_G)$, $\gamma\in H^2(B_G)$.

LEMMA 1. $Sq^1(\zeta)=\beta\cdot\zeta=0.$

Proof. Suppose the contrary that $Sq^1(\zeta)=\beta\cdot\zeta\neq 0$, i.e. $\beta\neq 0\in H^1(B_G)$. Then there exists a rank one subtorus $K\subseteq G$ such that $\beta\mid K\neq 0$. Let t be the generator of $H^*(B_K)$ and $\bar{\zeta}$ be the image of ζ in $H_K^*(X)$. By the naturality of Sq^1, we have $Sq^1(\bar{\zeta})=t\cdot\bar{\zeta}\neq 0$ We claim that $F(K, X)$ must be connected. For otherwise, $H_K^*(X)\cong\mathbf{Z}_2[t][\bar{\zeta}]/\langle\bar{f}(\bar{\zeta})\rangle$ and the defining equation $\bar{f}(\bar{\zeta})=0$ must have *more than one distinct roots* $\bar{\alpha}_j\in H^4(B_K)$.

However, $H^4(B_K)$ consists of only two elements, namely, 0 and t^4. Hence $\bar{\alpha}_1=0$ and $\bar{\alpha}_2=t^4$, and

$$0=Sq^1(\bar{\alpha}_2)=Sq^1(\zeta \mid H_K^*(\bar{q}_2))=Sq^1(\zeta)\mid H_K^*(\bar{q}_2)=t\zeta \mid H_K^*(\bar{q}_2)=t^5$$

which is clearly a contradiction. Hence $F(K, X)$ is connected and we may assume that $\bar{\alpha}_1=0$. Then there are the following three possibilities, namely, $F(K, X)\sim QP^n$, or CP^n, or RP^n. In case $F(K, X)\sim QP^n$ or CP^n, it is easy to show that $\iota^*Sq^1(\zeta) = Sq^1(\iota^*(\zeta))=0$ and hence, by the injectivity of $\iota^*: H_K^*(X)\to H_K^*(F(K, X))$, $Sq^1(\zeta)=0$ which contradicts to the assumption. In case $F(K, X)\sim RP^n$, we have

$$\iota^*(\zeta)=t^3\zeta_1+a_2\zeta_1^2+a_3\zeta_1^3+a_4\zeta_1^4$$

where ζ_1 is the generator of $H^*(F(K, X))$. Then

$$t\iota^*(\zeta)=t^4\zeta_1+ta_2\xi_1^2+ta_3\xi_1^3+ta_4\xi_1^4$$
$$Sq^1(\iota^*(\zeta))=t^4\xi_1+(t^3+Sq^1a_2)\xi_1^2+\cdots$$

and the fact $Sq^1(a_2)=0$ ($a_2=0$, or t^2) imply that $a_2=t^2$. Notice that $Sq^2(\zeta)=0$ or $t^2\zeta$, which implies that $Sq^2(\iota^*(\zeta))=0$ or $Sq^2(\iota^*(\zeta))=t^2(\iota^*(\zeta))$. However

$$Sq^2(\iota^*(\zeta))=Sq^2(t^3\xi_1+t^2\xi_1^2+\cdots)$$
$$=t^5\xi_1+0\xi_1^2+\cdots$$

which is neither 0 nor $t^2(\iota^*(\zeta))$, hence, a contradiction. All the above contradictions prove that the assumption $Sq^1(\zeta)=\beta\cdot\zeta\neq 0$ is impossible. Hence $Sq^1(\zeta)=0$. q.e.d.

Now, we shall divide the proof of Theorem 3 into two cases according to $Sq^2(\zeta)=\gamma\cdot\zeta$ is zero or non-zero.

(B) *The case* $Sq^2(\xi)=\gamma\cdot\zeta=0$

We claim that there are no components of either *RP*-type or *CP*-type. In fact, it is easy to show that the existence of a component of *RP*-type implies $Sq^1(\zeta)\neq 0$ and the existence of a component of *CP*-type implies $Sq^2(\zeta)\neq 0$. Moreover $Sq^1(\zeta)=0$, $Sq^2(\zeta)=0$ imply that

$$Sq^1(\alpha_j)=0 \quad \text{and} \quad Sq^2(\alpha_j)=0 \quad \text{for all} \quad 1\leqslant j\leqslant s,$$

and it is not difficult to show that such degree 4 elements must be of the form $\alpha_j=w_j^4$, $w_j\in H^1(B_G)$. Therefore

$$\iota^*(\zeta)=(\zeta_1+w_1^4, \zeta_2+w_2^4, \ldots, \zeta_s+w_s^4)\in H_G^4(F),$$

and the same computation as in the proof of Theorem 1 will show that

$$I_X(f_j)=\prod_{i\neq j}(w_i^4+w_j^4)^{k_i}=\prod_{i\neq j}(w_i+w_j)^{4k_i}.$$

The rest of the proof of this case is again the same as that of Theorem 1.

(C) *The case* $Sq^2(\zeta)=\gamma\cdot\zeta\neq 0$

In this case, it is easy to show that all connected components are of CP-type. Let η_j be the generator of $H^*(F^j)$, $\imath_j^*$ be the restriction homomorphism of $H_G^*(X)$ to $H_G^*(F^j)$, and $\imath_j^*(\zeta)=(a_j\eta_j^2+b_j\eta_j+\alpha_j)$. Then

$$Sq^2(\zeta)=\gamma\cdot\zeta \quad \text{implies} \quad a_j=1, b_j=\gamma \quad \text{and} \quad Sq^2\alpha_j=\gamma\cdot\alpha_j$$
$$Sq^1(\zeta)=0 \quad \text{implies} \quad Sq^1(b_j)=Sq^1(\gamma)=0.$$

Hence $\gamma=v^2$ is a perfect square. Now, let us choose basis in $H^1(B_G)$ so that v is the first base, and express α_j as polynomial in v with coefficients in terms of the other "variables". Then it is not difficult to show that $Sq^1(\alpha_j)=0$ and $Sq^2(\alpha_j)=v^2\cdot\alpha_j$ imply that $\alpha_j=v^2\cdot w_j^2+w_j^4$ for a suitable $w_j\in H^1(B_G)$. Therefore

$$\imath^*(\zeta)=(\eta_1^2+v^2\eta_1+v^2w_1^2+w_1^4,\ldots,\eta_s^2+v^2\eta_s+v^2w_s^2+w_s^4)\in H_G^4(F)$$

and the rest of the proof of this case follows easily from similar computation as that of Theorem 1.2. q.e.d.

§4. Examples and Concluding Remarks

(A) *Examples of Linear Models*

EXAMPLE 1. Let G be a $\mathbf{Z}_2$-torus acting on $\mathbf{R}^{(n+1)}$ via a linear representation $\phi:G\to 0(n+1)$. Suppose the system of $\mathbf{Z}_2$-weights of ϕ, $\Omega(\phi)=\{w_j, k_j;\ 1\leqslant j\leqslant s\}$ and $X=RP^n$ with the induced G-action of the above linear action on $\mathbf{R}^{(n+1)}$. Then, it is not difficult to see that

(i) $H_G^*(X)=R[\xi]/\langle f(\xi)\rangle$ where $f(\xi)=(\xi+w_1)^{k_1}\ldots(\xi+w_s)^{k_s}$ and ξ is the transgression of the fibration $\mathbf{Z}_2\to S_G^n\to RP_G^n$,

(ii) $F(G,X)=RP^{(k_1-1)}+\cdots+RP^{(k_s-1)}$ where $RP^{(k_j-1)}$ is the projective space of $F(w_j^\perp,\mathbf{R}^{n+1})=\mathbf{R}^{k_j}$,

(iii) the weight system of local representation of G at $F^j=RP^{(k_j-1)}$ is given by $\Omega_j=\{(w_i+w_j), k_i\ (i\neq j);\ 0, (k_j-1)\}$.

EXAMPLE 2a. Let G be a $\mathbf{Z}_2$-torus acting on $\mathbf{C}^{(n+1)}$ via a complex linear representation $\phi:G\to U(n+1)$. Suppose the system of $\mathbf{Z}_2$-weights of ϕ (considered as a real representation $G\to U(n+1)\subseteq 0(2n+2)$), $\Omega(\phi)=\{w_j, 2k_j; 1\leqslant j\leqslant s\}$ and $X=CP^n$ with the induced G-action of the above linear action on $\mathbf{C}^{(n+1)}$. Then it is not difficult to see that

(i) $H_G^*(X)=R[\eta]/\langle f(\eta)\rangle$ where $f(\eta)=(\eta+w_1^2)^{k_1}\ldots(\eta+w_s^2)^{k_s}$, and η is the transgression of the fibration $S^1\to_G^{(2n+1)}\to CP_G^n$,

(ii) $F(G, X)=CP^{(k_1-1)}+\cdots+CP^{(k_s-1)}$ where $CP^{(k_j-1)}$ is the complex projective space of $F(w_j^\perp, \mathbf{C}^{(n+1)})=\mathbf{C}^{k_j}$,

(iii) the weight system of local representation of G at $F^j=CP^{(k_j-1)}$ is given by $\Omega_j=\{(w_i+w_j), 2k_i\ (i\neq j); 0, 2(k_j-1)\}$.

EXAMPLE 2b. Let $G=\mathbf{Z}_2\times G'$ be a $\mathbf{Z}_2$-torus acting on $\mathbf{C}^{(n+1)}=\mathbf{C}\otimes\mathbf{R}^{(n+1)}$ via a real linear representation $\phi=\beta\otimes\phi'$, where β is the conjugation of $\mathbf{C}$ and ϕ' is a real linear representation of $G'\to 0(n+1)$ with $\Omega(\phi')=\{w_j, k_j; 1\leqslant j\leqslant s\}$. Then, there is an induced G-action on $X=CP^n$, and it is not difficult to see that

(i) $H_G^*(X)\cong R[\eta]/\langle f(\eta)\rangle$ where $f(\eta)=(\eta+\beta w_1+w_1^2)^{k_1}\ldots(\eta+\beta w_s+w_s^2)^{k_s}$ and $Sq^1(\eta)=\beta\cdot\eta$, η is the transgression of $S^1\to S_G^{2n+1}\to CP_G^n$,

(ii) $F(G, X)=RP^{(k_1-1)}+\cdots+RP^{(k_s-1)}$,

(iii) the weight system of local representation of G at $F^j=RP^{(k_j-1)}$ is given by $\Omega_j=\{(w_i+w_j), (\beta+w_i+w_j), k_i\ (i\neq j); \beta, 0, (k_j-1)\}$.

EXAMPLE 3a. Let G be a $\mathbf{Z}_2$-torus acting on $\mathbf{Q}^{(n+1)}$ via a quaternionic representation $\phi: G\to Sp(n+1)$. Suppose the system of $\mathbf{Z}_2$-weights of ϕ (considered as a real representation $G\to Sp(n+1)\subseteq 0(4n+4)$), $\Omega(\phi)=\{w_j, 4k_j; 1\leqslant j\leqslant s\}$ and $X=QP^n$ with the induced G-action of the above linear action on $\mathbf{Q}^{(n+1)}$. Then, again, it is not difficult to see that

(i) $H_G^*(X)=R[\zeta]/\langle f(\zeta)\rangle$ where $f(\zeta)=(\zeta+w_1^4)^{k_1}\ldots(\zeta+w_s^4)^{k_s}$ and ζ is the transgression of the fibration $S^3\to S_G^{4n+3}\to QP_G^n$.

(ii) $F(G, X)=QP^{(k_1-1)}+\cdots+QP^{(k_s-1)}$ where $QP^{(k_j-1)}$ is the projective space of $F(w_j^\perp, \mathbf{Q}^{(n+1)})=\mathbf{Q}^{k_j}$,

(iii) the weight system of local representation of G at $F^j=QP^{(k_j-1)}$ is given by $\Omega_j=\{(w_i+w_j), 4k_i\ (i\neq j); 0, 4(k_j-1)\}$.

EXAMPLE 3b. Let $G=\mathbf{Z}_2\times G'$ be a $\mathbf{Z}_2$-torus acting on $\mathbf{Q}^{(n+1)}=\mathbf{Q}\otimes\mathbf{R}^{(n+1)}$ via a real linear representation $\phi=v\otimes\phi'$, where v is the $\mathbf{Z}_2$-automorphism of $\mathbf{Q}$ which changes the sign of j and k; and ϕ' is a real representation of $G'\to 0(n+1)$ with $\Omega(\phi')=\{w_j, k_j; 1\leqslant j\leqslant s\}$. Then, there is an induced G-action on $X=QP^n$ and it is not difficult to see that

(i) $H_G^*(X)\cong R[\zeta]/\langle f(\zeta)\rangle$, $f(\zeta)=(\zeta+v^2w_1^2+w_1^4)^{k_1}\ldots(\zeta+v^2w_s^2+w_s^4)^{k_s}$, $Sq^2\zeta=v^2\zeta$, and ζ is the transgression of the fibration $S^3\to S_G^{4n+3}\to QP_G^n$,

(ii) $F(G, X)=CP^{(k_1-1)}+\cdots+CP^{(k_s-1)}$,

(iii) the weight system of local representation of G at $F^j = QP^{(k_j-1)}$ is given by $\Omega_j = \{(w_i + w_j), (v + w_i + w_j), 2k_i\ (i \neq j);\ v, 0, 2(k_j - 1)\}$.

The above "linear models" not only demonstrate that all the possibilities in the statements of the structural Theorem 1, 2, 3 are respectively geometrically realizable but also show the *cohomological aspect of orbit structures of general G-actions on cohomology projective spaces are actually identical to that of their respective linear models.*

(B) *Concluding Remarks*

(i) Suppose G is a given compact Lie group and X is a cohomology projective space. Then the family of maximal $\mathbf{Z}_2$-tori of G consists of finite many conjugacy classes, say $\{(H_i), 1 \leqslant i \leqslant h\}$, and $\{H_i, 1 \leqslant i \leqslant h\}$ is a complete representative. Applying the structure theorem to the restricted H_i-action on X, we obtain a system of $\mathbf{Z}_2$-weights $\Omega(X \mid H_i)$ for each H_i. *The totality of the h weight systems* $\{\Omega(X \mid H_i), 1 \leqslant i \leqslant h\}$ *is called the* $\mathbf{Z}_2$*-weight system of the G-space X*. Geometrically, it is interesting to investigate the relationship the orbit structure of the G-action on X and the orbit structures of the respective restricted H_i-action on X, which can be read off from their respective weight system $\Omega(X \mid H_1)$ as far as the cohomological aspect is concerned. We shall investigate this direction of applications of the structural theorem in a later paper.

(ii) Suppose a maximal $\mathbf{Z}_2$-torus, H, of G happens to be a normal subgroup of G, and $\bar{H}$ is a maximal $\mathbf{Z}_2$-torus of the quotient group $\bar{G} = G/H$. Then, for a given G-space X of cohomology type of projective spaces, $F(H, X)$ is a sum of cohomology projective spaces with an induced $\bar{G}$-action. Hence, we can again apply the structural theorems to the restricted $\bar{H}$-action on $F(H, X)$ to obtain some weight systems. Such weight systems is called the secondary $\mathbf{Z}_2$-weight system of the G-space X with respect to the pair $(H, \bar{H})$, or rather the 2-primary group K with $H \subseteq K \subseteq G$ and $K/H = \bar{H}$. For example, $G = \mathbf{Z}_4^l$, $H = \mathbf{Z}_2^l$, $\bar{H} = G/H = \mathbf{Z}_2^l$.

Of course, one may similarly define higher order 2-weight systems to deal with higher order 2-primary elements of G.

(iii) Similar computation can also be applied to analyze actions of $\mathbf{Z}_2$-tori on spaces of the $\mathbf{Z}_2$-cohomology type of Cayley projective plane.

(iv) In view of the interesting rôle of Steenrod square operations in the above structural theorems, it is interesting to see whether there are similar rôle of higher order cohomology operations in detecting more delicate geometric behavior of topological actions of primary groups.

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