Spines of Topological Manifolds.

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In this paper we prove that a closed 2-connected topological manifold has a PL-spine, i.e. there is a locally tamely embedded complex such that a regular neighborhood of this complex is the manifold with a disc deleted (dimension is assumed to be at least 6). This "spine method" together with the relative edition of regular neighborhoods of complexes in topological manifolds [5] makes it easy to use general position arguments in topological manifolds. This will be used in a forthcoming paper to extend various embedding theorems to the topological category.

The methods we use are *PL*-approximation theorems due to Cernavskii, Connally, Miller, Rushing... as quoted in [5] theorem 2 and blocktransversality for *PL* complexes and *PL* submanifolds as was first considered by C. Morlet [4] and later extended by D. Stone [6].

DEFINITION 1. A spine of a topological manifold M with $\partial M \neq \emptyset$ is a locally tamely embedded complex $K \subset M$ so that K is a strong deformation retract of M and $K \subset M$ is a simple homotopy equivalence. In case $\partial M = \emptyset$ by the spine of M we mean a spine of M with a disc deleted.

THEOREM 2. *Let* $(M, \partial_- M, \partial_+ M)$ *be a triad of topological manifolds* $\dim(M) = m$, *and assume* $m \geqslant 6$ *and*

$$\Pi_j(M, \partial_+ M) = 0 \quad \textit{for} \quad j < m - r, \qquad r \leqslant m - 3.$$

Further assume there is a PL-complex P locally tamely embedded in the interior of $\partial_+ M$, $\dim(P) = p$ *and* $m - p \geqslant 4$. *Then there is a complex K of dimension* $\max(p+1, r, 2)$, *locally tamely embedded in M such that*

$$K \cap \partial_+ M = P$$
$$K \cap \partial_- M = K'$$

K' a subcomplex of K, K' has a neighborhood of the form $K' \times I$ in K and

$$\partial_- M \cup K \subset M$$

is a strong deformation retract and a simple homotopyequivalence.

Theorem 2 has an immediate corollary:

COROLLARY 3. *Let M be a closed topological manifold,* $\dim(M)\geqslant 6$ *and* $\pi_j(M)=0$ *for* $j\leqslant r$, $r>2$. *Then M has a spine of dimension $m-r$.*

Proof. Let $\bar{M}=M$ – (interior of a disc). Put $\partial_- M=\emptyset$, $\partial_+ M=S^{m-1}$, $P=\emptyset$, and apply Theorem 2.

Proof of Theorem 2. First let us consider the case where $P=\emptyset$. Put $k=\max(r, 2)$. According to Kirby and Siebenmann [3] M has a handlebodydecomposition relative to $\partial_- M$ with no handles of dimension greater than k: Kirby and Siebenmann prove that $(M, \partial_- M)$ has a handlebodydecomposition, and one can then cancel handles to get a minimal handledecomposition. Because of problems with torsion one needs at least 1- and 2-handles.

We filter M by the handlefiltration

$$\partial_- M\times I=M_0\subset M_1\subset\cdots\subset M_s=M$$

where M_{i+1} is obtained from M_i by adjoining a single handle, no handles of dimension greater than $m-3$. The proof will be by downwards induction on the statement:

There is a locally tamely embedded complex

$$K_i\subset\overline{M-M_i}\qquad \dim(K_i)\leqslant k$$

such that

$$K_i'=K_i\cap\partial_+ M_i$$

is contained in the interior of $\partial_+ M_i$, K_i' has a neighborhood in K_i of the form $K_i'\times[0, 1]$ and $M_i\cup K_i$ is a simple strong deformation retract of M.

It is easy to start the induction, we let K_{s-1} be the core of the last handle. Then clearly $M_{s-1}\cup K_{s-1}$ is a simple strong deformation retract of $M=M_s$, so assume the statement for $i+1$. Now

$$M_{i+1}=M_i\cup_{S^{j-1}\times D^{m-j}}D^j\times D^{m-j}$$

for some $j\leqslant k$. Let

$$\bar{E}=D^j\times D^{m-j}\cap\partial M_{i+1}=D^j\times S^{m-j-1}$$

take an outside collar $\partial\bar{E}\times[0, 2]$ of $\partial\bar{E}$ in $\partial_+ M_{i+1}$ and let

$$E_1=\bar{E}\cup\partial\bar{E}\times[0, 1],\quad E_2=\bar{E}\cup\partial\bar{E}\times[0, 2].$$

$\bar{E}$ has a PL structure being a codimension 0 submanifold of the boundary of $D^j\times D^{m-j}$, and we can extend this PL structure to E_1 and E_2 using the collar. K_{i+1}' is of codimension more than 3 in $\partial_+ M_{i+1}$, so by [1], see e.g. [5] Theorem 2, since $\dim \partial_+ M_{i+1} \geqslant 5$, there is an ambient ε-isotopy of E_2 fixing ∂E_2 that moves K'_{i+1} to be PL embedded in E_2 except in a neighborhood of ∂E_2 which can be assumed small. So we may assume, since this can be taken to be the restriction of an ambient isotopy of M, that $K'_{i+1} \cap E_1 \subset E_1$ is PL. Using [4] we can isotop K'_{i+1} further by a small ambient isotopy so that K'_{i+1} intersects $\partial \bar{E} = S^{j-1} \times S^{m-j-1}$ blocktransversally. Assume this done, and denote

$$Z = K'_{i+1} \cap \partial \bar{E}.$$

Since the normal blockbundle of $\partial \bar{E}$ in E_1 is a trivial one dimensional bundle we obtain that $\partial \bar{E}$ has a neighborhood in E of the form $\partial \bar{E} \times (-1, 1)$ and

$$\partial \bar{E} \times (-1, 1) \cap K'_{i+1} = Z \times (-1, 1)$$

since it is the restriction of the trivial blockbundle to Z, by blocktransversality. Z is a PL subcomplex of $S^{j-1} \times S^{m-j-1}$, which is the boundary of $S^{j-1} \times D^{m-j}$, of dimension $m-1-r$, so of codimension at least 3. By [2] Theorem 5.2 there is a subcomplex Z' of $S^{j-1} \times D^{m-j}$ of dimension $\min(\dim(Z)+1, j)$ so that

$$Z' \cap S^{j-1} \times S^{m-j-1} = Z$$

and $S^{j-1} \times D^{m-j}$ simplicially collapses to Z' ($S^{j-1} \times D^{m-j}$ is the mapping cylinder of the projection $S^{j-1} \times S^{m-j-1} \to S^{j-1}$, so take Z' to be the mapping cylinder of the restriction to Z). Using [2] lemma 2.20 this implies that

$$S^{j-1} \times D^{m-j} \times I$$

simplicially collapses to

$$S^{j-1} \times D^{m-j} \times 0 \cup Z' \times I \cup S^{j-1} \times D^{m-j} \times 1$$

so taking $S^{j-1} \times D^{m-j} \times I$ to be a collar of $S^{j-1} \times D^{m-j}$ in $D^j \times D^{m-j}$ we see that if we define D to be

$$D = \overline{D^j \times D^{m-j} - S^{j-1} \times D^{m-j} \times I}$$

there is a simple strong deformation retract of $M_{i+1} \cup K_{i+1}$ to $M_i \cup Z' \times I \cup K_{i+1} \cup D$. However D is a disc, and $Z' \times 1 \cup K_{i+1} \cap \partial D$ is of codimension bigger than 3 in ∂D, so we may as before assume it is PL-embedded and D now simplicially collapses to the cone of $Z' \times 1 \cup K_{i+1} \cap \partial D$ thus finishing the induction step. It is clear by construction that K'_i has a product neighborhood in K_i.

In case $P \neq \emptyset$ the proof is the same except we have to go through the motions of the induction step in the initial step of the induction too.

REFERENCES

[1] Connelly, R., *Unknotting close embeddings of polyhedra in codimensions other than two*, to be published, see also Proceedings of The University of Georgia Topology of Manifolds Institute 1969, 384–389.
[2] Hudson, J. F. P., *Piecewise Linear Topology* (Math. Lecture Note Ser.) Benjamin, New York, 1969.
[3] Kirby, R. C. and Siebenman, L. C., *Some theorems on Topological Manifolds.* 'Manifolds', editor N. Kuiper, Springer Lecture Notes, 1971.
[4] Morlet, C., *Topologie des variétés semi-linéaires*, Ann. Scient. Ec. Norm., 4'ème serie, t. *1*, 1968, 313–394.
[5] Pedersen, E. K., *Topological Concordances* (to appear).
[6] Stone, D. A., *Stratified Polyhedra*, Lecture Notes in Mathematics vol. 252, Springer Verlag 1972.

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