The Hurwitz Matrix Equations and Multifour Groups.

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§0. Introduction

Let Ω_s be the set of 4^s s-tuples $(\lambda_1, \ldots, \lambda_s)$ where each λ_i stands for e, α, β, or γ, the elements of Klein's four group V_4, which satisfy the following relations:

$$\alpha\circ\alpha=\beta\circ\beta=\gamma\circ\gamma=e, \qquad \alpha\circ\beta=\beta\circ\alpha=\gamma,$$
$$\beta\circ\gamma=\gamma\circ\beta=\alpha, \qquad \gamma\circ\alpha=\alpha\circ\gamma=\beta.$$

If we define multiplication in Ω_s componentwise, i.e.,

$$(\lambda_1, \ldots, \lambda_s)\circ(\mu_1, \ldots, \mu_s)=(\nu_1, \ldots, \nu_s)$$

where ν_i is the group product $\lambda_i\circ\mu_i$ in V_4, then Ω_s becomes a commutative group which we denote by $G(\Omega_s)$. While $G(\Omega_s)$ is simply the direct product of s copies of V_4, the properties of certain of its substructures turn out to be useful in the explicit construction of solutions of systems of matrix equations of the following form:

$$B_h^2=\pm I, \qquad B_hB_k\pm B_kB_h=0, \qquad B_h\pm B_h'=0, \qquad (h, k=1, 2, \ldots; h\neq k) \qquad (*)$$

where the unknown is a set of unspecified number of $n\times n$ matrices $(B_1, \ldots, B_q)$ with entries in a given field F, B_h' is the transpose of B_h, and each of the ambiguity signs can be + or −. When the signs of the last two equations are positive and that of the first equation negative, (∗) reduces to the well known system of Hurwitz matrix equations, first proposed and solved in the complex field by A. Hurwitz in connection with his problem on the composition of quadratic forms [2]. Further investigations were made by Radon [5], Eckmann [1], Lee [3], Wong [7] and others and several far reaching results were obtained. The topic is still of current interest as can be seen in the recent works by Porteous [4] and Semple and Tyrell [6].

Geometrically, the system of Hurwitz matrix equations plays an important role in the study of isoclinic n-planes in Euclidean $2n$-space and the Clifford-parallel $(n-1)$-planes in elliptic $(2n-1)$-space, a work which was initiated by Y. C. Wong [7]. As to other systems of matrix equations of the form (∗) aside from the Hurwitz

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1) The author is indebted to Professor Y. C. Wong for his advice and suggestions during the preparation of this work.

equations, it should be interesting to investigate their corresponding geometric meanings.

In this paper, we shall be concerned mainly with the properties of certain substructures of $G(\Omega_s)$. We shall indicate briefly how the results can be utilized to construct explicitly by means of matrix representations the maximal real solutions of the Hurwitz matrix equations thus verifying the original theorem due to Wong [7]. Our treatment is more elementary compared with the method given by Eckmann [1] and the employment of tools from the representation theory of finite groups is not needed. Using similar technique, the construction of solutions of (*) with signs arbitrarily chosen can also be achieved in many instances. We shall not go into details here.

§1. Definitions and Basic Lemmas

DEFINITION 1.1. The direct product of s copies of Klein's four group V_4, denoted by $G(\Omega_s)$, is called the *multifour group of order* 4^s.

We shall use Roman capital letters to denote the elements in Ω_s, thus we may write L for $(\lambda_1,\dots,\lambda_s)$ and M for $(\mu_1,\dots,\mu_s)$. In particular, we shall denote $(e,\dots,e)$ in Ω_s by I_s. If $L=(\lambda_1,\dots,\lambda_h)$ and $M=(\mu_1,\dots,\mu_k)$ are elements in Ω_h and Ω_k respectively, then $N=(\lambda_1,\dots,\lambda_h,\mu_1,\dots,\mu_k)\in\Omega_{h+k}$ will be denoted by (L, M). If $h=k$, then $(\lambda_1\circ\mu_1,\dots,\lambda_h\circ\mu_k)\in\Omega_h$ will be denoted by $L\circ M$. Thus, $L^2=L\circ L=I_s$ for every $L\in\Omega_s$.

DEFINITION 1.2. Let $L=(\lambda_1,\dots,\lambda_s)$ and $M=(\mu_1,\dots,\mu_s)$ be elements in Ω_s. The unordered pair $[\lambda_i, \mu_i]$ is called the ith *component pair* of L, M.

DEFINITION 1.3. Two elements L and M in Ω_s are said to be *odd-related* (resp. *even-related*) if they have an odd (resp. even) number of component pairs which are of the forms $[\alpha,\beta]$, $[\alpha,\gamma]$ and $[\beta,\gamma]$. We write $L\tau M$ or $L\pi M$ according as they are odd-related or even-related. More generally, two subsets Σ_1, Σ_2 of Ω_s, one of which may be singleton, are said to be odd-related (resp. even-related) if each element of Σ_1 is odd-related (resp. even-related) to each element of Σ_2. Or in symbol, $\Sigma_1\tau\Sigma_2$ (resp. $\Sigma_1\pi\Sigma_2$).

LEMMA 1.4. *For any $L, M, N\in\Omega_s$, we have*
(i) $L\tau M \Leftrightarrow L\tau L\circ M$, $\quad L\pi M \Leftrightarrow L\pi L\circ M$;
(ii) $L\tau M, L\tau N \Rightarrow L\pi M\circ N$,
$L\pi M, L\pi N \Rightarrow L\pi M\circ N$,
$L\tau M, L\pi N \Rightarrow L\tau M\circ N$.

The proof of the above lemma is straight forward.

LEMMA 1.5. *Let $L\neq I_s$ be an element in Ω_s. Then L is odd-related to $4^s/2$ elements and even-related to the remaining $4^s/2$ elements in Ω_s.*

Proof. Our lemma can be easily verified for $s=1$ and 2.

Let us assume that our lemma holds for Ω_{s-1}, $s>1$, and write $L=(H, \lambda_s)$ where $H\in\Omega_{s-1}$. Then by induction assumption, there exist $4^{s-1}/2$ elements $H_i\in\Omega_{s-1}$ such that $H_i\tau H$ and $4^{s-1}/2$ elements $K_i\in\Omega_{s-1}$ such that $K_i\pi H$, where $i=1,\dots,4^{s-1}/2$.

Case 1. If $\lambda_s=e$, then for each i, $(H_i, \mu)\tau(H, \lambda_s)$ for any μ and $(K_i, \nu)\pi(H, \lambda_s)$ for any ν. Hence, there are altogether $4\cdot 4^{s-1}/2=4^s/2$ elements which are odd-related to L.

Case 2. If $\lambda_s\neq e$, we may take $\lambda_s=\alpha$ without loss of generality. Then $(H_i, \mu)\tau(H, \lambda_s)$ if and only if $\mu=e$ or α and $(K_i, \nu)\tau(H, \lambda_s)$ if and only if $\nu=\beta$ or γ. Again, there are $4^{s-1}+4^{s-1}=4^s/2$ elements in Ω_s which are odd-related to L.

In either case, the remaining $4^s/2$ elements in Ω_s must be even-related to L.

§2. The Group $\tilde{G}(\Sigma)$

Let Σ be a subset of elements in Ω_s. We shall denote by $\langle\Sigma\rangle$ the subgroup generated by Σ in $G(\Omega_s)$, and by $|\Sigma|$ the number of elements in Σ.

DEFINITION 2.1. A subset Σ of Ω_s is called an *independent set* if for each $N\in\Sigma$, $N\notin\langle\Sigma\setminus N\rangle$. A singleton distinct from I_s is considered as independent.

Let Σ be an independent set in Ω with $|\Sigma|=t$. Then $|\langle\Sigma\rangle|=1+\sum_{r=1}^{t}\binom{t}{r}=2^t$, and it follows that $t\leqq 2s$ since Ω_s has only 2^{2s} elements.

Let us denote by $(\Sigma)_e$ the set of all elements in Ω_s which are even-related to each element of an independent set Σ. By Lemma 1.4 (ii), it is clear that $(\Sigma)_e$ is a subgroup of $G(\Omega_s)$. Since $G(\Omega_s)$ is a commutative group, $(\Sigma)_e$ is normal in $G(\Omega_s)$, and so we may form the quotient group $G(\Omega_s)/(\Sigma)_e$ which we shall denote by $\tilde{G}(\Sigma)$.

LEMMA 2.2. *Let $\Sigma=\{L, M\}$ be an independent set in Ω_s. Then the group $\tilde{G}(\Sigma)$ is isomorphic to V_4.*

Proof. The subgroup $(\Sigma)_e$ of $G(\Omega_s)$ is the set $\{N\colon N\pi L, N\pi M\}$. Let us denote by $P\circ(\Sigma)_e$ the coset of $(\Sigma)_e$ in $G(\Omega_s)$ consisting of all elements $P\circ N$ with $N\in(\Sigma)_e$. We have two distinct cases:

Case 1. $L\tau M$. For any $S\in\Omega_s$, $S\in L\circ(\Sigma)_e$ (resp. $S\in M\circ(\Sigma)_e$) if and only if $S\pi L$ and $S\tau M$ (resp. $S\pi M$, $S\tau L$) by Lemma 1.4. Furthermore, for any $T\in\Omega_s$, $T\in L\circ M\circ(\Sigma)_e$ if and only if $T\tau L$ and $T\tau M$. Since no element in Ω_s can belong to a coset distinct from $(\Sigma)_e$, $L\circ(\Sigma)_e$, $M\circ(\Sigma)_e$ and $L\circ M\circ(\Sigma)_e$, these are just the four cosets of $(\Sigma)_e$ in $G(\Omega_s)$. The representatives $\{I_s, L, M, L\circ M\}$ can be identified with V_4 and it follows that $\tilde{G}(\Sigma)$ is isomorphic to V_4 in this case.

Case 2. $L\pi M$. The proof is similar.

LEMMA 2.3. *Let Σ be an independent set in Ω_s with $|\Sigma|=t\leqq 2s$. For any partition of Σ as the union of Σ_1 and Σ_2, one of which may be empty, there exists some $Q\in\Omega_s$ such that $Q\pi\Sigma_1$ and $Q\tau\Sigma_2$.*

Proof. We prove by induction up to $t=2s$.

For $t=1$ and 2, the lemma follows from Lemma 1.5 and the proof of Lemma 2.2. Let us assume that our lemma holds for all Σ' with $2\leqq|\Sigma|<t\leqq 2s$.

For any partition of Σ in Ω_s as the disjoint union of Σ_1 and Σ_2, where we may assume that Σ_1 is non-empty, the partition gives rise to a partition of $\Sigma\setminus S$, where $S\in\Sigma_1$, as the disjoint union of $\Sigma_1\setminus S$ and Σ_2. By induction assumption, there exists $T\in\Omega_s$ such that $T\pi\Sigma_1\setminus S$, $T\tau\Sigma_2$. By Lemma 2.2, there exists $P\in\Omega_s$ such that $P\pi S$ and $P\pi T$. Then $Q=P\circ T\in\Omega_s$ is the element satisfying our lemma.

PROPOSITION 2.4. *Let Σ be an independent set in Ω_s with $|\Sigma|=t\leqq 2s$. For any partition of Σ as the union of Σ_1 and Σ_2, one of which may be empty, there exist 2^{2s-t} elements in Ω_s each of which is even-related to Σ_1 and odd-related to Σ_2.*

Proof. By Lemma 2.3, there exists $Q\in\Omega_s$ such that $Q\pi\Sigma_1$ and $Q\tau\Sigma_2$. Then $Q\circ(\Sigma)_e$ is the coset of $(\Sigma)_e$ in $G(\Omega_s)$ each element of which is related to Σ_1 and Σ_2 in the same manner as Q. Since Σ can be partitioned in 2^t ways, we obtain 2^t cosets of $(\Sigma)_e$ in $G(\Omega_s)$ which are elements of $\tilde{G}(\Sigma)$. Clearly, each element in Ω_s must belong to one of these cosets and it follows that each coset consists of 2^{2s-t} elements.

COROLLARY 2.5. *Let Σ be an independent set in Ω_s with $|\Sigma|=2s$. Then there exists exactly one element $P\in\Omega_s$ such that $P\tau\Sigma$ and there is no element in Ω_s distinct from I_s which is even-related to Σ.*

PROPOSITION 2.6. *Let Σ be an independent set in Ω_s with $|\Sigma|=t\leqq 2s$. Then according as $t=2k$ or $2k+1$, the group $\tilde{G}(\Sigma)$ is isomorphic to $G(\Omega_k)$ or $C_2\times G(\Omega_k)$, where C_2 denotes the cyclic group $\langle\delta\rangle$ of order 2.*

Proof. Case 1. $t=2k$. Let $\Sigma=\{S_1, T_1,\dots, S_k, T_k\}$. A correspondence between the cosets of $(\Sigma)_e$ in $G(\Omega_s)$ (i.e., the elements of $\tilde{G}(\Sigma)$), and the elements of $G(\Omega_k)$ can be set up in the following manner: For any representative Q of a given coset of $(\Sigma)_e$ in $G(\Omega_s)$, we let $Q\circ(\Sigma)_e$ correspond to the element $(v_1,\dots, v_k)$ in $G(\Omega_k)$ where

$$v_i=e \quad\text{if}\quad Q\pi S_i \quad\text{and}\quad Q\pi T_i,$$
$$v_i=\alpha \quad\text{if}\quad Q\pi S_i \quad\text{and}\quad Q\tau T_i,$$
$$v_i=\beta \quad\text{if}\quad Q\tau S_i \quad\text{and}\quad Q\pi T_i,$$

and

$$v_i=\gamma \quad\text{if}\quad Q\tau S_i \quad\text{and}\quad Q\tau T_i.$$

That the above correspondence is in fact a group isomorphism is easily verified.

Case 2. $t=2k+1$. Let $\Sigma=\{R, S_1, T_1,\ldots, S_k, T_k\}$ and let P be a representative of any given coset of $(\Sigma)_e$ in $G(\Omega_s)$. If we let $P\circ(\Sigma)_e$ correspond to the element $(\varrho, \nu_1,\ldots, \nu_k)$ in $C_2\times G(\Omega_k)$ by setting $\varrho=e$ or δ according as $P\pi R$ or $P\tau R$, and $\nu=e, \alpha, \beta$, or γ in the same way as in Case 1, then the correspondence is a group isomorphism between $\tilde{G}(\Sigma)$ and $C_2\times G(\Omega_k)$.

§3. 0-sets and E-sets in Ω_s

DEFINITION 3.1. Let Σ be an independent set in Ω_s and P the product (meaning group product) of all the elements in Σ. Then $P\notin\Sigma$ and we call the set $\bar{\Sigma}=\Sigma\cup\{P\}$ (or simply $\Sigma\cup P$) a *complete set* in Ω_s. (In the sequel, the symbol "$\cup$" will denote disjoint union.)

It follows immediately from definition that if $\bar{\Sigma}$ is a complete set in Ω_s, then the product of all the elements in $\bar{\Sigma}$ is equal to I_s and every element in this set is the product of the remaining elements in the set.

DEFINITION 3.2. An independent or complete set consisting of two or more mutually odd-related (resp. even-related) elements in Ω_s is called an *0-set* (resp. *E-set*) in Ω_s. An 0-set (resp. E-set) in Ω_s is said to be *maximal* if it is not a proper subset of a larger 0-set (resp. E-set) in Ω_s.

PROPOSITION 3.3. *An 0-set in Ω_s is maximal if and only if it is complete.*

Proof. Let $\bar{\Phi}=\Phi\cup P$ be a complete 0-set in Ω_s. If $|\Phi|=2s$, then by Corollary 2.5, P is the only element such that $P\tau\Phi$ and so Φ is maximal. If $|\Phi|<2s$, then by Proposition 2.4, there exists some $Q\in\Omega_s$ distinct from P such that $Q\tau\Phi$. Since $P\tau\Phi$, $|\Phi|$ must be even, and this implies that $Q\pi P$ by Lemma 1.4. Therefore, for any such Q, the enlarged set $\Phi\cup P\cup Q$ is not an 0-set showing that $\bar{\Phi}$ is also maximal in this case.

On the other hand, let Φ be any 0-set in Ω_s which is not complete. Since Φ is then an independent set, we have $|\Phi|\leqq 2s$. By Lemma 2.3, there exists some $Q\in\Omega_s$ such that $\Phi\cup Q$ is an 0-set. Hence, Φ is not maximal.

PROPOSITION 3.4. *Let $\bar{\Phi}=\Phi\cup S\cup T$ be an 0-set in Ω_s and $|\Phi\cup S\cup T|=2k+1$ where $k\leqq s$. Then $\bar{\Phi}$ is a complete 0-set if and only if $\tilde{G}(\Phi\cup S)$ and $\tilde{G}(\Phi\cup T)$ are identical.*

Proof. If $\bar{\Phi}$ is complete, then T can be expressed as the product of all the elements in $\Phi\cup S$ which are even in number. Hence, for any $Q\in\Omega_s$, $Q\pi\Phi\cup S$ if and only $Q\pi\Phi\cup T$. This means that $(\Phi\cup S)_e=(\Phi\cup T)_e$ and so the two groups $\tilde{G}(\Phi\cup S)$ and $\tilde{G}(\Phi\cup T)$ are identical.

To prove the converse, we observe that $T\circ(\Phi\cup S)_e\in\tilde{G}(\Phi\cup S)$ is odd-related to $\Phi\cup S$. Since the two given groups are identical, $T\circ(\Phi\cup S)_e$ appears as $T\circ(\Phi\cup T)_e$ in

$\tilde{G}(\Phi \cup T)$, and in $T \circ (\Phi \cup T)_e$, every element is even-related to T. It follows that T is the only element in $T \circ (\Phi \cup S)$ which is odd-related to $\Phi \cup S$, showing that $\bar{\Phi}$ is maximal and hence complete.

PROPOSITION 3.5. *In Ω_s, there exist complete* 0*-sets with* $2k+1$ *elements for* $k=1,\dots,s$.

The proof is straight forward.

PROPOSITION 3.6. *Let $\bar{\Phi}$ be a complete* 0*-set in Ω_s with* $2s+1$ *elements. For any $P \in \bar{\Phi}$, $\bar{\Phi} \setminus P$ is a set of generators of $G(\Omega_s)$.*

Proof. Since $\langle \bar{\Phi} \setminus P \rangle$, the group generated by $\bar{\Phi} \setminus P$, is of order 4^s, it must coincide with $G(\Omega_s)$.

Similar results concerning complete E-sets can be easily derived. We state without proof three propositions as follows.

PROPOSITION 3.7. *An E-set in Ω_s is maximal if and only if it is complete.*

PROPOSITION 3.8. *In Ω_s, there exist complete E-set with k elements for* $k=3, 4,\dots, s+1$.

PROPOSITION 3.9. *Let $\bar{\Psi}$ be a complete E-set in Ω_s with $s+1$ elements. Then any s elements in $\bar{\Psi}$ constitute a set of generators of the identity element $(\Psi)_e$ in $\tilde{G}(\bar{\Psi} \setminus Q)$ where Q is the element deleted from $\bar{\Psi}$.*

§4. Complete 0-sets which are Mutually Even-related and Complete E-sets which are Mutually Odd-related

PROPOSITION 4.1. *For any positive integers j, k such that $1 \leqq j, k < s$ and $j+k \leqq s$, there exist in Ω_s complete* 0*-sets $\bar{\Phi}_1$ and $\bar{\Phi}_2$ with* $2j+1$ *and* $2k+1$ *elements such that $\bar{\Phi}_1 \pi \bar{\Phi}_2$.*

Proof. Since $j+k \leqslant s$, there exist 0-sets Φ_1, Φ_2 with $2j$ and $2k$ elements such that $\Phi_1 \cup \Phi_2$ is an independent 0-set. Let P (resp. Q) be the product of all the elements in Φ_1 (resp. Φ_2). Then $\bar{\Phi}_1 = \Phi_1 \cup P$ and $\bar{\Phi}_2 = P \circ \Phi_2 \cup Q$ are two complete 0-sets which are even-related.

PROPOSITION 4.2. *For any positive integer t such that $1 < t \leqq s$, there exist in Ω_s* 0*-sets $\bar{\Phi}_1,\dots, \bar{\Phi}_t$ which are complete, disjoint, mutually even-related, and such that $\sum_{i=1}^{t} |\bar{\Phi}_i| \leqq 2s+t$.*

Proof. Let $\bar{\Phi}_1 = \Phi_1 \cup P_1$ where P_1 is any element in $\bar{\Phi}_1$, and let $|\Phi_1| = 2j$. By Proposition 4.1, there exists an independent 0-set Φ with $2(s-j)$ elements such that $\Phi \pi \Phi_1$.

Since $\langle\Phi\rangle \subset (\Phi_1)_e$, the identity element of $\tilde{G}(\Phi_1)$, we have $\langle\Phi\rangle = (\Phi_1)_e$ since they have the same number of elements.

The group $G(\Omega_{s-j})$ is generated by an independent 0-set Σ_1 with $2(s-j)$ elements by Proposition 3.6. A correspondence between $G(\Omega_{s-j})$ and $(\Phi_1)_e$ can be set up by associating the elements in Σ_1, the generators of $G(\Omega_{s-j})$, with those in Φ, the generators of $(\Phi_1)_e$, in one-to-one manner. Clearly, this correspondence is a group isomorphism which preserves the odd and even relations.

Now let Σ_2 be an independent set with $2k$ elements in Ω_{s-j} where $1 \leqq k < s-j$. Then there exists an independent 0-set Σ_3 in Ω_{s-j} with $|\Sigma_3| \leqq 2(s-j-k)$ such that $\Sigma_2 \pi \Sigma_3$. By the isomorphism given in the above paragraph, there exist 0-sets Φ_2 and Φ_3 in $(\Phi_1)_e$ corresponding to Σ_2 and Σ_3 respectively such that $\Phi_2 \pi \Phi_3$. Since Φ_2 and Φ_3 are in $(\Phi_1)_e$, it follows that Φ_1, Φ_2 and Φ_3 are mutually even-related. Then $\bar{\Phi}_1$, $\bar{\Phi}_2$ and $\bar{\Phi}_3$ are complete 0-sets which are mutually even-related and $\Sigma\, |\Phi_i| \leqq 2s+3$.

Our proposition then follows by repeating the above arguments.

We state without proof the following propositions concerning complete E-sets in Ω_s.

PROPOSITION 4.3. *Let* $\bar{\Psi}_1 = \{L_1, L_2, P\}$, *and* $\bar{\Psi}_2 = \Psi \cup P$ *be two complete E-sets in* Ω_s. *Then* $\{L_1, L_2\} \tau \Psi$.

PROPOSITION 4.4. *Let* Ψ_1 *be an independent E-set in* Ω_s *and* $|\Psi_1| < s-1$. *There exist E-sets* Ψ_2 *and* Ψ_3 *such that* $\Psi_1 \cup \Psi_2 \cup \Psi_3$ *is an independent set and the three sets are mutually odd-related.*

§5. Construction of Complete 0-Sets and Complete E-Sets in Ω_s from those in Ω_{s-1}

Let $\Sigma = \{L_1, \ldots, L_t\}$ be an arbitrary set of elements in Ω_{s-1}, where $s \geqslant 2$. For simplicity, we shall use the notation $\{\Sigma, \lambda\}$ to denote the set of elements $\{(L_1, \lambda), \ldots, (L_t, \lambda)\}$ in Ω_s.

By virtue of the propositions given in §3 and §4, the following constructions can be achieved. The proofs are omitted.

PROPOSITION 5.1. *The following are complete* 0*-sets in* Ω:

(i) $\{(I_{s-1}, \alpha), (I_{s-1}, \beta), (I_{s-1}, \gamma)\}$,

(ii) $\{(I_{s-1}, \alpha), (L, \beta), (L, \gamma)\}$ for any $L \in \Omega_{s-1}$.

PROPOSITION 5.2. *Let* $\bar{\Phi}$ *be a complete* 0*-set in* Ω_{s-1}. *Then the following are complete* 0*-sets in* Ω_s:

(i) $\{\bar{\Phi}, e\}$,

(ii) $\{(I_{s-1}, \alpha), (I_{s-1}, \beta), \{\bar{\Phi}, \gamma\}\}$.

PROPOSITION 5.3. *Let $\bar{\Phi}_1, \bar{\Phi}_2, \bar{\Phi}_3$ be complete* 0-*sets in Ω_{s-1} which are mutually even-related. Then the following are complete* 0-*sets in Ω_s:*

(i) $\{(I_{s-1}, \alpha), \{\bar{\Phi}_1, \beta\}, \{\bar{\Phi}_2, \gamma\}\}$,

(ii) $\{\{\bar{\Phi}_1, \alpha\}, \{\bar{\Phi}_2, \beta\}, \{\bar{\Phi}_3, \gamma\}\}$.

PROPOSITION 5.4. *Let $\Phi \cup \Phi'$ be a complete* 0-*set in Ω_{s-1} such that $|\Phi|$ is even. Then the following are complete* 0-*sets in Ω_s:*

(i) $\{\{\Phi, \alpha\}, \{\Phi', e\}\}$

(ii) $\{\{\Phi, e\}, (P, \alpha), (P, \beta), \{\Phi', \gamma\}\}$

(iii) $\{\{\Phi, e\}, (P, \alpha), (P, \beta), (P, \gamma)\}$,

where P is the product of all the elements in Φ'.

PROPOSITION 5.5. *Let $\bar{\Psi}$ be a complete E-set in Ω_{s-1}. For any partition of $\bar{\Psi}$ as the union of Ψ_1 and Ψ_2, where $|\Psi_1|$ is even and Ψ_2 may be singelton or empty, the following is a complete E-set in Ω_s:*

$$\{\{\Psi_1, \alpha\}, \{\Psi_2, e\}\}.$$

PROPOSITION 5.6. *Let $\Psi_1 \cup P$ and $\Psi_2 \cup P$ be complete E-sets in Ω_{s-1} such that $|\Psi_1|$ and $|\Psi_2|$ are both even and $\Psi_1 \tau \Psi_2$. Then the following is a complete E-set in Ω_s:*

$$\{\{\Psi_1, \alpha\}, \{\Psi_2, \beta\}\}.$$

In particular, if $\Psi_1 = \{L, M\}$, the following E-set in Ω_s is complete:

$$\{(L, \alpha), (M, \alpha), \{\Psi_2, \beta\}\}.$$

PROPOSITION 5.7. *Let Ψ_1, Ψ_2, Ψ_3 be E-sets in Ω_{s-1} which are mutually odd-related and such that their union is an independent set. Let P be the product of all the elements in the three sets. If the number of elements in Ψ_i are all odd or all even, then the following is a complete E-set in Ω_s:*

$$\{(P, e), \{\Psi_1, \alpha\}, \{\Psi_2, \beta\}, \{\Psi_3, \gamma\}\}.$$

Remark. In the propositions of this section, the last components α, β and γ may be interchanged. Thus, by Proposition 5.3 (i), $\{(I_{s-1}, \beta), \{\bar{\Phi}_1, \alpha\}, \{\bar{\Phi}_2, \gamma\}\}$ and $\{(I_{s-1}, \gamma), \{\bar{\Phi}_1, \alpha\}, \{\bar{\Phi}_2, \beta\}\}$ are also complete 0-sets in Ω_s.

There are other possible constructions but they are not useful in connection with the solution of matrix equations.

§6. Maximal 0-sets in $\Omega_s^*(\beta)$

Let $\Omega_s^*(\beta)$ be the subset of Ω_s defined by the following condition: $L = (\lambda_1, \ldots, \lambda_s)$

$\in\Omega_s^*(\beta)$ if and only if among the components λ_i of L, the element β in V_4 appears an odd number of times. Then $\Omega_s\backslash\Omega_s^*(\beta)$ is the subset of Ω_s consisting of those elements, among the components of each of which, β appears an even number of times or does not appear at all. The determination of maximal 0-sets and maximal E-sets with elements lying entirely in $\Omega_s^*(\beta)$ or entirely in $\Omega_s\backslash\Omega_s^*(\beta)$ are especially useful in applications. We shall confine our discussion to maximal 0-sets in $\Omega_s^*(\beta)$.

LEMMA 6.1. *If L, M are both in $\Omega_s^*(\beta)$ or both in $\Omega_s\backslash\Omega_s^*(\beta)$ and $L\tau M$, then $L\circ M\in\Omega_s^*(\beta)$.*

Proof. Since L, M are both in $\Omega_s^*(\beta)$ or both in $\Omega_s\backslash\Omega_s^*(\beta)$, the total number of their component pairs of the forms $[e, \beta]$, $[\alpha, \beta]$ and $[\gamma, \beta]$ is even. On the other hand, $L\tau M$ implies that the total number of their component pairs of the forms $[\alpha, \beta]$, $[\gamma, \beta]$ and $[\alpha, \gamma]$ is odd. It follows that the total number of component pairs of the forms $[e, \beta]$ and $[\alpha, \gamma]$ is odd and consequently $L\circ M$ has an odd number of components equal to β. Our lemma is proved.

LEMMA 6.2. *If Φ_1 is an independent 0-set in $\Omega_s^*(\beta)$ with $2k$ elements, where $k<s$, then an element $N\in\Omega_s^*(\beta)$ can be chosen so that $N\pi\Phi_1$.*

Proof. It suffices to consider the case when $k=s-1$. The subgroup $(\Phi_1)_e$ in $G(\Omega_s)$ is isomorphic to V_4 according to the proof given in Proposition 4.2. Let $(\Phi_1)_e=\{I_s, L, M, N\}$. Then if $L, M\notin\Omega_s^*(\beta)$, N must be in $\Omega_s^*(\beta)$ by Lemma 6.1.

LEMMA 6.3. *If there exists a maximal 0-set Φ in $\Omega_s^*(\beta)$ with $2k+1$ elements, where $k<s$, then there exists a maximal 0-set in $\Omega_{s+2}^*(\beta)$ with $2k+5$ elements.*

Proof. If $k<s$, then there exists $N\in\Omega_s^*(\beta)$ such that $N\pi\Phi$, by Lemmas 6.1 and 6.2. Then $\{(I_s, \beta), (N, \alpha), (N, \gamma)\}$ is a maximal 0-set in $\Omega_{s+1}^*(\beta)$ which is even-related to $\{\Phi, e\}$ in $\Omega_{s+1}^*(\beta)$. By Proposition 5.3 (i), the following is a maximal 0-set in $\Omega_{s+2}^*(\beta)$ with $2k+5$ elements:

$$\{(I_{s+1}, \beta), ((I_s, \beta), \alpha), ((N, \alpha), \alpha), ((N, \gamma), \alpha), \{\{\Phi, e\}, \gamma\}\}.$$

PROPOSITION 6.4. *In $\Omega_s^*(\beta)$, $s\geqslant 2$, there exist maximal 0-sets with k elements, where $k=3, 7, \ldots, 4[s/2]-1$, and $[s/2]$ denotes the greatest integer not exceeding $s/2$.*

Proof. There exist maximal 0-sets in $\Omega_2^*(\beta)$ and $\Omega_3^*(\beta)$ with 3 elements, for instance, $\{(e, \beta), (\beta, \alpha), (\beta, \gamma)\}$ and $\{(e, e, \beta), (a, \beta, \alpha), (a, \beta, \gamma)\}$. The proposition then follows from Proposition 5.2 (i) and Lemma 6.3.

PROPOSITION 6.5. *In $\Omega_s^*(\beta)$, where $s=4k+3$, there exists a maximal 0-set with $2s+1$ elements.*

Proof. In $\Omega_2^*(\beta)$, $\Phi_1=\{(e, \beta), (\beta, \alpha), (\beta, \gamma)\}$ and $\Phi_2=\{(\beta, e), (\alpha, \beta), (\gamma, \beta)\}$ are the only two maximal 0-sets which are even-related. Hence, $\Phi_3=\{(I_2, \beta), \{\Phi_1, \alpha\}, \{\Phi_2, \gamma\}\}$ is a maximal 0-set in $\Omega_3^*(\beta)$ with 7 elements. Now $\{I_3, \Phi_3\}\pi\{\Phi_3, I_3\}$ in $\Omega_6^*(\beta)$, it follows that the following is a maximal 0-set in $\Omega_7^*(\beta)$ with 15 elements:

$$\Phi_4=\{(I_6, \beta), \{\{I_3, \Phi_3\}, \alpha\}, \{\{\Phi_3, I_3\}, \gamma\}\}.$$

Our proposition is thus true for $k=0, 1$.

Assume that our proposition holds for all $\Omega_t^*(\beta)$ with $t=4h+3$, $0\leqslant h<k$. We write $k=h_1+h_2+1$, $h_1, h_2\geqslant 0$. By induction assumption, there exist maximal 0-sets $\Phi'_1\in\Omega_{t_1}^*(\beta)$, $\Phi'_2\in\Omega_{t_2}^*(\beta)$, where $t_i=4h_i+3$, such that $|\Phi'_i|=2t_i+1$. Then $\{I_{t_1}, \Phi'_2\}$ $\pi\{\Phi'_1, I_{t_2}\}$ in $\Omega_{t_1+t_2}^*(\beta)$, and so we can construct the following maximal 0-set in $\Omega_s^*(\beta)$

$$\{(I_{s-1}, \beta), \{\{I_{t_1}, \Phi'_2\}, \alpha\}, \{\{\Phi'_1, I_{t_2}\}, \gamma\}\}.$$

which has $2(t_1+t_2)+3=2s+1$ elements.

PROPOSITION 6.6. *In $\Omega_s^*(\beta)$ where $s=4k$, there exists a maximal 0-set with $2s$ elements which is not complete.*

Proof. By Proposition 6.5, there exists a maximal 0-set Φ in $\Omega_{s-1}^*(\beta)$ with $2s-1$ elements. By Proposition 5.2 (ii), $\bar{\Phi}=\{(I_{s-1}, \alpha), (I_{s-1}, \beta), \{\Phi, \gamma\}\}$ is a complete 0-set in Ω_s with $2s+1$ elements, which after deleting the element $(I_{s-1}, \alpha)\notin\Omega_s^*(\beta)$, gives rise to a maximal 0-set in $\Omega_s^*(\beta)$ with $2s$ elements which is not complete.

Remark. If $\Phi\in\Omega_{s-1}^*(\beta)$ has less than $2s-1$ elements, then $\{(I_{s-1}, \beta), \{\Phi, \gamma\}\}$ is not maximal in $\Omega_s^*(\beta)$ because it is contained in the maximal 0-set $\{(I_{s-1}, \beta), \{\Phi, \gamma\}, \{\Phi', \alpha\}\}$ where Φ' is another maximal 0-set in $\Omega_{s-1}^*(\beta)$ such that $\Phi\pi\Phi'$.

PROPOSITION 6.7. *In $\Omega_s^*(\beta)$, where $s=4k+1$, there exists a maximal 0-set with $2s-1$ elements which is not complete.*

Proof. By Proposition 6.6, $\{(I_{s-2}, \beta), \{\Phi, \gamma\}\}$ is a maximal 0-set in $\Omega_{s-1}^*(\beta)$ with $2s-2$ elements, where Φ is a maximal 0-set in $\Omega_{s-2}^*(\beta)$ with $2s-3$ elements. Then the following 0-set in $\Omega_s^*(\beta)$ has $2s-1$ elements:

$$\{(I_{s-1}, \beta), ((I_{s-2}, \beta), \alpha), \{\{\Phi, \gamma\}, \alpha\}\}.$$

Clearly, this 0-set is maximal but not complete.

§7. The Associative Algebra $A(\Omega_s)$ and its Matrix Representation

We may consider the 4^s s-tuples $(\lambda_1,\dots, \lambda_s)$ in Ω_s as the base elements of a vector space over a field F. If we define the product of two base elements L and M, denoted by $L\cdot M$, to be their group product $L\circ M$ in $G(\Omega_s)$ multiplied by a structure constant $C_{L,M} \in F$, where $C_{L,M}$ need not equal to $C_{M,L}$, this vector space becomes an algebra over F. When F is the complex field, it is convenient to choose the structure constants as follows: For each pair of elements L and M not equal to I_s, we set $C_{L,M}=(\sqrt{-1})^{j+k}$ (or $-(\sqrt{-1})^{j+k}$), where j is the number of component pairs of L, M which are of the forms $[e, \beta]$ and $[\alpha, \gamma]$, and $k=0$ if exactly one of L, M lies in $\Omega_s^*(\beta)$ and $k=1$ if otherwise. Once $C_{L,M}$ is fixed, we set $C_{M,L}=C_{L,M}$ or $-C_{L,M}$ according as $L\pi M$ or $L\tau M$. Also, we set $C_{I_s,I_s}=C_{I_s,L}=C_{L,I_s}=1$ and $C_{L,L}=-1$, where $L\neq I_s$. With proper choice of sign for $C_{L,M}$ for each pair L, M, we obtain an associative algebra over the complex field. The algebra so defined will be denoted by $A(\Omega_s)$ and it can be represented by matrices with entries in the complex field in the following manner.

For $s\geq h\geq 1$, we represent the elements I_h, (I_{h-1}, α), (I_{h-1}, β) and (I_{h-1}, γ) in Ω_h by

$$\begin{pmatrix} J & \\ & J \end{pmatrix}, \quad \sqrt{-1}\begin{pmatrix} J & \\ & -J \end{pmatrix}, \quad \begin{pmatrix} & J \\ -J & \end{pmatrix}, \quad \text{and} \quad \sqrt{-1}\begin{pmatrix} & J \\ J & \end{pmatrix}$$

respectively, where J stands for the identity matrix of order $2^{h-1}m$ (m odd). Note that when $h=1$, (I_{h-1}, α), etc. mean simply α, etc. If $L(\neq I_h)\in\Omega_h$ is represented by a matrix A of order t, we represent the elements (L, e), (L, α), (L, β) and (L, γ) in Ω_{h+1} respectively the following matrices of order $2t$:

$$\begin{pmatrix} A & \\ & A \end{pmatrix}. \quad \begin{pmatrix} A & \\ & -A \end{pmatrix}, \quad \sqrt{-1}\begin{pmatrix} & A \\ -A & \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} & A \\ A & \end{pmatrix}.$$

In this way, when any element $(\lambda_1,\dots,\lambda_s)\in\Omega_s$ is given, we can start from its first component and construct step by step a matrix of order $2^s m$ which is its representation. Since the elements of Ω_s are the base elements of $A(\Omega_s)$, a faithful representation of this algebra is achieved.

DEFINITION 7.1. The 4^s matrices of order $n=2^s m$ (m odd) representing the elements of Ω_s obtained in the manner described above are called the *basic matrices of order n.*

§8. Maximal Real Solutions of Hurwitz Matrix Equations

As an illustration of the application of our results, we proceed to show how the maximal real solutions of the following system of Hurwitz matrix equations can be constructed explicitly:

$$\begin{aligned} &B_h^2=-I, \quad B_hB_k+B_kB_h=0, \quad B_h+B_h'=0, \quad (h, k=1, 2,\dots; h\neq k), \\ &\text{order of } B_h=2^s m \quad (m \text{ odd}). \end{aligned} \tag{**}$$

Let B_h, B_k be basic matrices which are representations of two distinct elements L_h, L_k in Ω_s. It is clear that $L_h \tau L_k$ if and only if $B_h B_k = -B_k B_h$. If $L_h \neq I_s$, then the relation $B_h^2 = -I$ is always satisfied. Furthermore, B_h is real if and only if L_h lies in $\Omega_s^*(\beta)$, in which case, we also have $B_h = -B_h'$. From these observations, we conclude that if $\Sigma = \{B_1, ..., B_q\}$ is a set of basic matrices representing a maximal 0-set in $\Omega_s^*(\beta)$, then Σ is a maximal set of real solutions of (**), and conversely. Since the constructions as given in Propositions 6.4, 6.5, 6.6 and 6.7 exhaust all possible types of maximal 0-sets in $\Omega_s^*(\beta)$, we are led to the following proposition discovered originally by Wong [7].

PROPOSITION 8.1. *There exist q-dimensional maximal real solutions of Hurwitz matrix equations of order* $2^s m$ (m odd) *for the following values of q and s:*

$$
\begin{array}{llll}
s \equiv 1 \pmod 4: & q = 4k+3, & k = 0, 1, ..., (s-3)/2; & q = 2s-1. \\
s \equiv 2 \pmod 4: & q = 4k+3, & k = 0, 1, ..., (s-2)/2. & \\
s \equiv 3 \pmod 4: & q = 4k+3, & k = 0, 1, ..., (s-1)/2. & \\
s \equiv 0 \pmod 4: & q = 4k+3, & k = 0, 1, ..., (s-2)/2; & q = 2s.
\end{array}
$$

It was proved by Wong [7] that every set of maximal real solutions of (**) is unitary similar to what he called a maximal set of quasisolutions. Basing on this fact, we have the following result.

PROPOSITION 8.2. *Every maximal real solution of Hurwitz matrix equations of order* $2^s m$ (m odd) *is orthogonally similar to a maximal solution consisting of real basic matrices which are representations of a maximal* 0*-set in* $\Omega_s^*(\beta)$.

We may consider maximal E-sets instead of maximal 0-sets, or we may restrict such sets to other specified subsets of Ω_s such as $\Omega_s \setminus \Omega_s^*(\beta)$ instead of $\Omega_s^*(\beta)$, in order to construct real or complex solutions of (*) given in the Introduction with signs differ from those appear in (**). In some cases, the use of other types of matrix representations may be necessary. We leave the details to interested readers for their own investigation.

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