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Stable Vector Bundles over the Projective Orthogonal Groups

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Introduction

Let G be a compact connected Lie group of rank r. If the fundamental group $\pi_1(G)=\pi$ is trivial, then Hodgkin [9] showed that the complex K-theory of G is an exterior algebra (over the integers) generated by r elements arising from the basic irreducible representations of G.

Now suppose that π is a non-trivial, *finite* group. Modulo torsion $K^*(G)$ is again an exterior algebra and therefore

$$K^*(G)\cong\{E_{\mathbf{Z}}(\alpha_1,\ldots,\alpha_r)\otimes T^*(G)\}/S(G),$$

where $\alpha_1,\ldots,\alpha_r\in K^1(G)$ are elements representing generators of the exterior algebra $K^*(G)/\mathrm{Tors}\,K^*(G)$, $T^*(G)=T^0(G)\oplus T^1(G)$ is a certain $\mathbf{Z}_2$-graded subalgebra of $K^*(G)$, generated by 1 and some elements of finite order, and $S(G)$ is the ideal generated by the "relations".

In the case when $\pi\cong\mathbf{Z}_p$, where p is a prime, the authors [8] proved that

$$T^*(G)\cong T^0(G)\cong R(\pi)/(j^*(I_{G_0})),$$

where $R(\pi)$ is the complex representation ring of the covering transformation group π of the universal covering $u:G_0\to G$, $j^*:R(G_0)\to R(\pi)$ the homomorphism induced by the inclusion $j:\pi\hookrightarrow G_0$ and $(j^*(I_{G_0}))$ the ideal generated by j^*-image of the augmentation ideal I_{G_0} of $R(G_0)$. Furthermore $T^0(G)$ coincides with the image of the homomorphism $c^*:K^0(B_\pi)\to K^0(G)$ induced by the map $c:G\to B_\pi$ classifying the universal covering of G. The ideal $S(G)$ in this case is given by

$$S(G)=(\alpha_r\otimes\tilde{T}^0(G)),$$

where $T^0(G)\cong\mathbf{Z}\oplus\tilde{T}^0(G)$.

In this paper we propose to give a complete description of the ringstructure of the unitary K-theory for the family of the *projective orthogonal groups* $PSO(m)$. Note that if m is odd then we have $\mathrm{PSO}(m)=\mathrm{SO}(m)$; the ring $K^*(\mathrm{SO}(m))$ is already known see [7], [8] or [6]. If m is even, say $m=2n$, we shall distinguish between the "*cyclic*" case, i.e. *n odd* and hence $\pi_1(\mathrm{PSO}(2n))\cong Z_4$, and the "*non-cyclic*" case, i.e. *n even* and hence $\pi_1(\mathrm{PSO}(2n))\cong \mathbf{Z}_2\times\mathbf{Z}_2$. In the "cyclic" case it again turns out that $T^1(G)$ is zero and that $T^0(G)$ can be identified with the image$(c^*)\cong R(\pi)/(j^*(I_{G_0}))$, thus in this respect extending the results of [8]. However in the "non-cyclic" case it is no longer true that the ring $K^*(G)$ is generated by the image of the homomorphism c^* and the free generators $\alpha_1,\dots,\alpha_r\in K^1(G)$. The enquiry after the generators of $K^*(\mathrm{PSO}(4t))$ then leads to the definition of a crucial stable vector bundle τ over the suspension of PSO$(4t)$. The element $\tau\in K^1(\mathrm{PSO}(4t))$ will be given in terms of the *transfer maps* associated to the two *semi-spin coverings of PSO*$(4t)$ (see (4.2)). The main result of this paper may then be paraphrased as follows (see (6.2), (7.2)).

Let $G=\mathrm{PSO}(2n)$, n even. Then $T^*(G)=T^0(G)\oplus T^1(G)$ is generated by 1 *and elements ξ_1, $\xi_2\in\mathrm{im}\,c^*\subset K^0(G)$ and $\tau\in K^1(G)$ such that the following relations hold*

(i) *The elements ξ_1, $\xi_1\xi_2$ and $\xi_2\tau$ are of order 2^{k-1} where $k=v_2(n)+2$. The element τ is of order 2^k whereas ξ_2 is of order 2^{n-1}.*

(ii) $\xi_1^2+2\xi_1=0$, $\xi_2^2+2\xi_2=0$, $\tau^2=0$, $\tau\xi_1+2\tau=0$.

The ideal $S(G)\subset E_{\mathbf{Z}}(\alpha_1,\dots,\alpha_r)\otimes T^*(G)$ is generated by the following elements:

$$\alpha_{n-1}\otimes\xi_1,\ \alpha_n\otimes\xi_2,\ \alpha_{n-1}\otimes\tau,\ \alpha_n\otimes\tau,\ 1\otimes2^{k-1}\tau-\alpha_{n-1}\otimes2^{n-2}\xi_2$$

and

$$1\otimes\tau\xi_2+1\otimes2\tau-\alpha_n\otimes\xi_1.$$

(i.e. in $K^*(G)$ one has the relations $\alpha_{n-1}\xi_1=0$, $\alpha_n\xi_2=0$, $\alpha_{n-1}\tau=0$, $\alpha_n\tau=0$, $2^{k-1}\tau=$ $=2^{n-2}\xi_2\alpha_{n-1}$, $\tau\xi_2+2\tau=\alpha_n\xi_1$.)

The proof of this result rests on the relationship between complex *K*-theory and the *complex representation ring* of a Lie group, the *Atiyah-transfer* homomorphism and a very detailed analysis of various *spectral sequences.*

The different geometric and "algebraic topological" features of PSO$(4t+2)$ and PSO$(4t)$ suggest that the two cases be looked at separately. In the layout of this paper the emphasis is put on the "non-cyclic" case (see section 1 to 6), whereas the main steps leading to the result in the "cyclic" case are just summarized; see section 7.

I. THE NON-CYCLIC CASE; $\pi_1(\mathrm{PSO}(2n))\cong\mathbf{Z}_2\times\mathbf{Z}_2$

1. Restricting Representation of Spin $(2n)$ to its Central Subgroups.

(1.1). Throughout Chapter I let $n\geqslant6$ be an *even* integer and $k=v_2(n)+2$, where $v_2(n)$ is the exponent of the highest power of 2 dividing n. The centre of $G_0=\mathrm{Spin}(2n)$ is denoted by π. Hence $\pi\cong\mathbf{Z}_2\times\mathbf{Z}_2$, and in accordance with Tits [11; p. 36] we choose generators z and z' of π. We shall consider the Lie groups of the form G_0/ω where

$\omega \cong \mathbf{Z}_2$ is one of the three possible subgroups of π. If $\omega = \omega_1$ is the subgroup generated by z we get the *semi-spin group* $G_1 = G_0/\omega_1$; if $\omega = \omega_3$ is generated by z' then it is well known that $G_0/\omega_3 = G_3$ is isomorphic to G_1. If $\omega = \omega_2$ is generated by $z \cdot z'$ – (diagonal subgroup of π) – we get the *special orthogonal group* $G_2 = G_0/\omega_2 = \mathrm{SO}(2n)$. The *projective orthogonal group* $\mathrm{PSO}(2n)$ is defined to be $G_0/\pi = G$.

(1.2). The complex representation ring $R(\pi)$ is generated, as a free abelian group, by 1, ϱ_1, ϱ_2 and ϱ_3 where the representations

$$\varrho_i : \pi \to S^1 \quad (i = 1, 2, 3)$$

are defined as follows:

$$\begin{aligned} &\varrho_1(z) = -1 = \varrho_1(z') \\ &\varrho_2(z) = 1, \qquad \varrho_2(z') = -1 \\ &\varrho_3(z) = -1, \qquad \varrho_3(z') = 1 \end{aligned} \tag{1.3}$$

The representations ϱ_i, $(i = 1, 2, 3)$, satisfy

$$\varrho_i^2 = 1, \qquad \varrho_1 \cdot \varrho_2 = \varrho_3 . \tag{1.4}$$

The augmentation ideal I_π of $R(\pi)$ is generated, as a free abelian group, by σ_1, σ_2 and σ_3 where $\sigma_i = \varrho_i - 1$ $(i = 1, 2, 3)$ with relations

$$\sigma_i^2 + 2\sigma_i = 0, \qquad \sigma_1\sigma_2 + \sigma_1 + \sigma_2 = \sigma_3 . \tag{1.5}$$

The representation ring of $\omega_i \cong Z_2$, $(i = 1, 2)$, is given by

$$R(\omega_i) \cong Z[\theta_i]/(\theta_i^2 - 1)$$

where $\theta_i : \omega_i \to S^1$ is the canonical representation. The augmentation ideal I_{ω_i} is generated by $\kappa_i = \theta_i - 1$, with relation $\kappa_i^2 + 2\kappa_i = 0$.

The representation ring of G_0 is a polynomial ring

$$R(G_0) \cong Z[\lambda_1, \lambda_2, \ldots, \lambda_n] \tag{1.6}$$

where the generator λ_s, $(s = 1, 2, \ldots, n-2)$, is the s-th exterior power of the canonical representation $G_0 \xrightarrow{a_2} G_2 \subsetneq U(2n)$ (a_2 being the two-fold covering map of $G_2 = \mathrm{SO}(2n)$), whereas λ_{n-1}, λ_n stand for the spin-representations Δ^+ and Δ^-. Hence the augmentation ideal I_{G_0} is, as a ring, generated by the elements

$$\tilde{\lambda}_s = \lambda_s - \dim \lambda_s \quad (s = 1, 2, \ldots, n) . \tag{1.7}$$

Let $e_i:\omega_i \subsetneq \pi$, $(i=1, 2)$, be the inclusion map. Denoting by $j:\pi \subsetneq G_0$ the inclusion of the centre, we define the map $j_i:\omega_i \subsetneq G_0$ to be $j_i = j \circ e_i$.

Thus the homomorphisms $e_i^*: R(\pi) \to R(\omega_i)$ are given by

$$\begin{aligned} e_1^*(\varrho_1) &= \theta_1 = e_1^*(\varrho_3), \qquad e_1^*(\varrho_2) = 1 \\ e_2^*(\varrho_2) &= \theta_2 = e_2^*(\varrho_3), \qquad e_2^*(\varrho_1) = 1. \end{aligned} \tag{1.8}$$

According to [11; p. 36] the homomorphism $j^*: R(G_0) \to R(\pi)$ is determined by

$$j^*(\lambda_s) = \begin{cases} \binom{2n}{s}\varrho_1, & \text{for } s \text{ odd and } 1 \leqslant s < n-2 \\ \binom{2n}{s}, & \text{for } s \text{ even and } 1 < s \leqslant n-2 \end{cases} \tag{1.9}$$
$$j^*(\lambda_{n-1}) = 2^{n-1}\varrho_2, \qquad j^*(\lambda_n) = 2^{n-1}\varrho_3.$$

The maps $j_i^*: R(G_0) \to R(\mathbf{Z}_2)$, $(i=1, 2)$, are given by (1.8), (1.9) and $j_1^* = e_1^* \circ j^*$, $j_2^* = e_2^* \circ j^*$.

A straight forward calculation using (1.8) and (1.9) establishes the following result.

(1.10) PROPOSITION. (i) *If $J = (j^*(I_{G_0}))$ is the ideal generated by $j^*(I_{G_0})$, then $R(\pi)/J \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{k-1}} \oplus \mathbf{Z}_{2^{n-1}} \oplus \mathbf{Z}_{2^{k-1}}$, where $k = \nu_2(n) + 2$. Generators for the three finite cyclic sumands may be represented by σ_1, σ_2 and $\sigma_1\sigma_2$ respectively.*

(ii) *If $J_1 = (j_1^*(I_{G_0}))$, then $R(\omega_1)/J_1 \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{k-1}}$, with κ_1 representing a generator of $\mathbf{Z}_{2^{k-1}}$.*

(iii) If $J_2 = (j_2^*(I_{G_0}))$, *then $R(\omega_2)/J_2 \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{n-1}}$, with κ_2 representing a generator of $\mathbf{Z}_{2^{n-1}}$.*

(1.11) *Remark.* The canonical ring homomorphisms $h_i: R(\pi)/J \to R(\omega_i)/J_i$, $(i=1, 2)$, are given by $h_1(\sigma_1) = \kappa_1$, $h_1(\sigma_2) = 0$ and $h_2(\sigma_1) = 0$, $h_2(\sigma_2) = \kappa_2$.

2. The Homomorphism in K-theory Induced by the Universal Covering of $G = \mathrm{PSO}(2n)$.

Let us begin with a few observations concerning the universal covering $u: M_0 \to M_0/\omega = M$ of a compact Lie group M of rank r, having finite fundamental group ω. Since $K^*(M_0)$ is torsion free (see [9]) the map $u^*: K^*(M) \to K^*(M_0)$ factors through $K^*(M)/\mathrm{Tors}\,K^*(M)$, thus giving rise to the homomorphism $\bar{u}: K^*(M)/\mathrm{Tors}\,K^*(M) \to K^*(M_0)$. As $\mathbf{Z}_2$-graded Hopf algebras, both $K^*(M)/\mathrm{Tors}\,K^*(M)$ and $K^*(M_0)$ are *exterior algebras* on the group of primitive elements denoted by P and P_0 respectively. The image of u^* is therefore a primitively generated exterior subalgebra of $K^*(M_0)$ and is determined by

$$\bar{u}(P) = (\mathrm{im}\,u^*) \cap P_0.$$

We now aim at giving a description of this latter group. There are elements $v_1, v_2, \ldots, v_r \in K^1(M)$ representing a basis of P and elements $\mu_1, \mu_2, \ldots, \mu_r \in P_0 \subset K^1(M_0)$ forming a basis of P_0 such that

$$u^*(v_s) = m_s \mu_s,\ 0 < m_s \in Z, \qquad (s = 1, 2, \ldots, r). \tag{2.1}$$

(2.2) LEMMA. *The product of the integers $m_1, m_2, \ldots, m_r$ is equal to the order of ω, i.e. $m_1 m_2 \ldots m_r = |\omega|$.*

Proof. In $K^*(M_0)$ we have $u^*(v_1 v_2 \ldots v_r) = m_1 m_2 \ldots m_r \cdot \lambda_1 \lambda_2 \ldots \lambda_r$. We shall prove that $u^*(v_1 v_2 \ldots v_r) = |\omega|\ \lambda_1 \lambda_2 \ldots \lambda_r$. This is seen as follows. For ordinary cohomology with integer coefficients the homomorphism u^* restricted to the top dimensional cohomology class of $H^*(M; \mathbf{Z})$ is multiplication by $|\omega|$. This together with the fact that both M_0 and M are parallelizable compact manifolds and hence stably reducible (see [1]) implies (2.2). (For a different proof of (2.2) see [8; section 2].)

(2.3). From (2.2) we conclude that the subgroup $(\operatorname{im} u^*) \cap P_0$ of P_0 has index $|\omega|$.

The universal covering $u: M_0 \to M$ is classified by a map $c: M \to B_\omega$. We view

$$\Lambda = (M_0 \xrightarrow{u} M \xrightarrow{c} B_\omega)$$

– up to homotopy equivalence – as a principal fibre bundle over B_ω, u representing the homotopy class of the fibre inclusion; (see [5]). (The classifying map $B_\omega \to B_{M_0}$ of the M_0-bundle Λ is induced by the inclusion $j: \omega \to M_0$.)

According to [9] the α and β-constructions together with the K-theory exact sequence of the pair (M, M_0) give rise to the following commutative diagram.

$$\begin{array}{ccccccc}
K^1(M) & \xrightarrow{u^*} & K^1(M_0) & \xrightarrow{\delta} & K^0(M, M_0) & \to & K^0(M) \\
 & & \uparrow{\scriptstyle -\beta} & & \uparrow{\scriptstyle \bar{c}^*} & & \uparrow{\scriptstyle c^*} \\
 & & & \overset{\alpha(\Lambda)}{\nearrow} & K^0(B_\omega, pt) & \rightarrowtail & K^0(B_\omega) \\
 & & & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \alpha} \\
 & & I_{M_0} & \xrightarrow{j^*} & I_\omega & \rightarrowtail & R(\omega).
\end{array} \tag{2.4}$$

(For the definition of α see [2]).

(2.5) LEMMA. *The homomorphism $\bar{c}^* \circ \alpha: I_\omega \to K^*(M, M_0)$ factors through $I_\omega / I_\omega \cdot \operatorname{im} j^*$.*

Proof. In $K^0(M, M_0)$ products of the form $\xi \cdot \delta(\eta)$ vanish; [3; p. 87]. The lemma then follows from the commutativity of (2.4), i.e. from $\bar{c}^* \circ \alpha \circ j^* = -\delta \circ \beta$.

Let $F \subset I_{M_0}$ be the free abelian group generated by $\tilde{\lambda}_s = \lambda_s - \dim \lambda_s$, $(s = 1, \ldots, r)$, where $\lambda_1, \ldots, \lambda_r$ are the basic irreducible representations of M_0. By [9] the homomorphism β maps F isomorphically onto the group of primitive elements $P_0 \subset K^1(M_0)$. In the following we shall identify P_0 and F, in particular we shall write $\lambda \in P_0$ for any element $\beta(\lambda)$ with $\lambda \in F$.

With (2.4) and (2.5) we then get the commutative diagram

$$\begin{array}{ccc} P_0 = F & \xrightarrow{\ \delta \mid P_0\ } & K^0(M, M_0) \\ & \searrow^{\varphi} \qquad \nearrow & \\ & I_\omega / I_\omega \cdot \operatorname{im} j^* & \end{array} \tag{2.6}$$

where φ is induced by j^*.
Hence

$$\ker \varphi \subseteq (\ker \delta) \cap P_0 = (\operatorname{im} u^*) \cap P_0 . \tag{2.7}$$

Recalling the notations introduced in section 1, we now revert to the three coverings $u: G_0 = \operatorname{Spin}(2n) \to \operatorname{PSO}(2n) = G$, $a_1: G_0 \to G_0/\omega_1 = G_1$ and $a_2: G_0 \to G_0/\omega_2 = \operatorname{SO}(2n)$. These coverings yield the following commutative diagram

$$\begin{array}{ccc} F & & \\ {\scriptstyle \varphi}\downarrow & \searrow^{\varphi_i} & \\ I_\pi / I_\pi \cdot \operatorname{im} j^* & \to & I_{\omega_i} / I_{\omega_i} \cdot \operatorname{im} j_i^* \end{array} \tag{2.8}$$

where φ, φ_i are induced by j^*, j_i^* respectively; $(i = 1, 2)$.

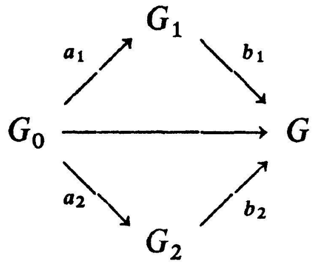
(2.9) PROPOSITION. *There is a basis* $\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n$ *of* $F \subset I_{G_0}$ *such that*
(i) $\beta_1, \ldots, \beta_{n-2}, 2\gamma_{n-1}, 2\gamma_n$ are a basis of $\ker \varphi$
(ii) $\beta_1, \ldots, \beta_{n-2}, 2\gamma_{n-1}, \gamma_n$ are a basis of $\ker \varphi_1$
(iii) $\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, 2\gamma_n$ are a basis of $\ker \varphi_2$.
Moreover, for $\beta_1, \ldots, \beta_{n-3}$ *and* γ_{n-1} *we can choose a linear combination of* $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-2}$ *whereas* $\beta_{n-2} = \Delta^+ - \Delta^-$ *and* $\gamma_n = \lambda_n = \Delta^- - \dim \Delta^-$; *(see* (1.7)*).*

We omit the proof of (2.9) which amounts to a plain computation based on (1.8), (1.9) and the relations (1.5).

It follows from (2.9) that the subgroup $\ker \varphi$ of $F = P_0$ has index 4 and we conclude with (2.3) and (2.7) that

$$\ker \varphi = (\operatorname{im} u^*) \cap P_0, \quad \text{and similarly} \quad \ker \varphi_i = (\operatorname{im} a_i^*) \cap P_0 . \tag{2.10}$$

The following proposition is then a consequence of (2.9), (2.10) and the commutativity of the diagram



(2.11)

where all the maps are canonical covering projections.

(2.12) PROPOSITION. *There are generators* $\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n$ *of the exterior algebra* $K^*(G_0)$ *and elements* $\nu_1, \nu_2, \ldots, \nu_{n-2}, \varepsilon_{n-1}, \varepsilon_n \in K^1(G)$, $\nu_1^{(i)}, \ldots, \nu_{n-2}^{(i)}, \varepsilon_{n-1}^{(i)}, \varepsilon_n^{(i)} \in K^1(G_i)$, $(i=1, 2)$, *such that*

(i) *the elements* $\nu_1, \ldots, \nu_{n-2}, \varepsilon_{n-1}, \varepsilon_n$ *generate an exterior algebra in* $K^*(G)$ *which, under projection, is isomorphic to* $K^*(G)/\mathrm{Tors}\,K^*(G)$. *Furthermore*

$$u^*(\nu_s)=\beta_s, \quad (s=1, \ldots n-2); \qquad u^*(\varepsilon_{n-1})=2\gamma_{n-1}, \qquad u^*(\varepsilon_n)=2\gamma_n.$$

(ii) *the elements* $\nu_1^{(i)}, \ldots, \nu_{n-2}^{(i)}, \varepsilon_{n-1}^{(i)}, \varepsilon_n^{(i)}$ *generate an exterior algebra in* $K^*(G_i)$ *which, under projection, is isomorphic to* $K^*(G_i)/\mathrm{Tors}\,K^*(G_i)$, $(i=1, 2)$. *Furthermore*

$$a_i^*(\nu_s^{(i)})=\beta_s, \quad (s=1, \ldots, n-2), (i=1, 2),$$

and

$$a_1^*(\varepsilon_{n-1}^{(1)})=2\gamma_{n-1}, \qquad a_1^*(\varepsilon_n^{(1)})=\gamma_n, \qquad a_2^*(\varepsilon_{n-1}^{(2)})=\gamma_{n-1}, \qquad a_2^*(\varepsilon_n^{(2)})=2\gamma_n$$

whereas

$$b_i^*(\nu_s)=\nu_s^{(i)}, \qquad (s=1, \ldots, n-2), (i=1, 2)$$

and

$$b_1^*(\varepsilon_{n-1})=\varepsilon_{n-1}^{(1)}, \qquad b_2^*(\varepsilon_n)=\varepsilon_n^{(2)}.$$

(iii) *The above elements can be chosen such that with respect to the various transfer maps (see* [10]) *arising from* (2.11) *one has*

$$(a_1)_*(\gamma_{n-1})\equiv\varepsilon_{n-1}^{(1)} \quad (\mathrm{mod\ torsion}), \qquad (a_2)_*(\gamma_n)\equiv\varepsilon_n^{(2)} \quad (\mathrm{mod\ torsion}),$$
$$\varepsilon_{n-1}=(b_2)_*(\varepsilon_{n-1}^{(2)}), \qquad \varepsilon_n=(b_1)_*(\varepsilon_n^{(1)})$$

and hence

$$b_2^*(\varepsilon_{n-1})=2\varepsilon_{n-1}^{(2)}, \qquad b_1^*(\varepsilon_n)=2\varepsilon_n^{(1)}.$$

(For (iii) see [8; (2.4), (2.7)].)

(2.13) *Remark.* The element $\gamma_n \in K^1(G_0)$ can be represented by the homomorphism $G_0 \xrightarrow{\Delta^-} U(2^{n-1}) \subsetneq U$ which factors through G_3, giving rise to a homomorphism $\Delta_3: G_3 \to U$. The map Δ_3 represents an element in $K^1(G_3)$ which we denote by $\varepsilon_n^{(3)}$. The element $\varepsilon_n^{(1)} \in K^1(G_1)$ can not be represented by a group homomorphism. However, combining the two canonical Hopf multiplications on U, it is possible to write down explicitly a map $\Delta_1: G_1 \to U$ representing $\varepsilon_n^{(1)}$.

3. Generators of Finite Order in $K^0(G)$.

Using the main result of [8] and reverting to (1.10) and (2.12) we first list the following two propositions.

(3.1) *There are elements* $v_1^{(1)}, \dots, v_{n-2}^{(1)}, \varepsilon_{n-1}^{(1)}, \varepsilon_n^{(1)} \in K^1(G_1)$ *and* $\zeta_1 \in \tilde{K}^0(G_1)$ *which generate the ring* $K^*(G_1)$ *and such that*

(i) $K^*(G_1) \cong \{E_{\mathbf{Z}}(v_1^{(1)}, \dots, v_{n-2}^{(1)}, \varepsilon_{n-1}^{(1)}, \varepsilon_n^{(1)}) \otimes T^0(G_1)\}/(\varepsilon_{n-1}^{(1)} \otimes \zeta_1)$ *where* $T^0(G_1)$ *is the subring of* $K^0(G_1)$ *generated by* 1 *and* ζ_1.

(ii) *The element* $1 + \zeta_1$ *is represented by the complex line bundle associated to the twofold covering* $G_0 \xrightarrow{a_1} G_1$; ζ_1 *is subject to the relations*

$$\zeta_1^2 + 2\zeta_1 = 0, \qquad 2^{k-1}\zeta_1 = 0, \qquad (k = v_2(n) + 2).$$

In particular $T^0(G_1) \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{k-1}}$.

(3.2) *There are elements* $v_1^{(2)}, \dots, v_{n-2}^{(2)}, \varepsilon_{n-1}^{(2)}, \varepsilon_n^{(2)} \in K^1(G_2)$ *and* $\zeta_2 \in \tilde{K}^0(G_2)$ *which generate the ring* $K^*(G_2)$ *and such that*

(i) $K^*(G_2) \cong \{E_{\mathbf{Z}}(v_1^{(2)}, \dots, v_{n-2}^{(2)}, \varepsilon_{n-1}^{(2)}, \varepsilon_n^{(2)}) \otimes T^0(G_2)\}/(\varepsilon_n^{(2)} \otimes \zeta_2)$ *where* $T^0(G_2)$ *is the subring of* $K^0(G_2)$ *generated by* 1 *and* ζ_2.

(ii) *The element* $1 + \zeta_2$ *is represented by the complex line bundle associated to the twofold covering* $G_0 \xrightarrow{a_2} G_2$ *and* ζ_2 *is subject to the relations*

$$\zeta_2^2 + 2\zeta_2 = 0, \qquad 2^{n-1}\zeta_2 = 0.$$

In particular $T^0(G_2) \cong \mathbf{Z} \oplus \mathbf{Z}_{2^{n-1}}$.

Remark. The complex K-theory tells the homotopy types of G_1 and G_2 apart, a result which also appears in [4, (9.1)]. In [4] however the Steenrod algebra structure of the ordinary cohomology of G_1 and G_2 is used to distinguish the homotopy types of G_1 and G_2.

We now determine the image of the homomorphism induced by the map $c: G \to B_\pi$ classifying the universal covering of G.

(3.3) PROPOSITION. *Let* $T^0(G) = \mathrm{im}\,[K^0(B_\pi) \xrightarrow{c^*} K^0(G)]$. *Then* $T^0(G)$ *is a direct* *summand of* $K^0(G)$ *and the homomorphism* $c^*\circ\alpha: R(\pi)\to K^0(G)$ *of* (2.4) *induces an isomorphism*

$$T^0(G)\cong R(\pi)/(j^*(I_{G_0}))\cong \mathbf{Z}\oplus\mathbf{Z}_{2^{k-1}}\oplus\mathbf{Z}_{2^{n-1}}\oplus\mathbf{Z}_{2^{k-1}}; \qquad (k=\nu_2(n)+2).$$

Generators of the three finite cyclic summands of $T^0(G)$ *are given by* ξ_1, ξ_2 *and* $\xi_1\cdot\xi_2$, *where the element* $1+\xi_1$ *(respectively* $1+\xi_2$*) is represented by the complex line bundle associated to the twofold covering* $b_2: G_2\to G$ *(respectively* $b_1: G_1\to G$*). The elements* ξ_1 *and* ξ_2 *are subject to the relations* $\xi_1^2+2\xi_1=0$, $\xi^2+2\xi_2^2=0$.

Proof. It follows from [2; (7.2)] that $c^*\circ\alpha$ maps $R(\pi)$ onto $\operatorname{im} c^*=T^0(G)$. Invoking (2.4) we infer that $c^*\circ\alpha$ induces an epimorphism

$$R(\pi)/(j^*(I_{G_0}))\twoheadrightarrow T^0(G).$$

Now consider the composite

$$G_1\times G_2\xrightarrow{b_1\times b_2} G\times G\xrightarrow{m} G\xrightarrow{c} B_\pi$$

where m is the multiplication map on G, and set $t=m_0(b_1\times b_2)$. Applying K^0 we get

$$R(\pi)\overset{\alpha}{\rightarrowtail} K^0(B_\pi)\xrightarrow{c^*} K^0(G)\xrightarrow{t^*} K^0(G_1\times G_2). \tag{3.4}$$

Clearly, the elements $\sigma_i\in R(\pi)$ map onto $\xi_i\in K^0(G)$, $(i=1,2)$. Furthermore, looking at the Chern classes of the line bundles involved, one has $t^*(1+\xi_1)=(1+\zeta_1)\otimes 1$, $t^*(1+\xi_2)=1\otimes(1+\zeta_2)\in K^0(G_1)\otimes K^0(G_2)\subset K^0(G_1\times G_2)$. With (3.1) and (3.2) we then obtain

$$t^*\circ c^*\circ\alpha(\sigma_1)=\zeta_1\otimes 1\in T^0(G_1)\otimes 1$$
$$t^*\circ c^*\circ\alpha(\sigma_2)=1\otimes\zeta_2\in 1\otimes T^0(G_2)$$

which implies that $t^*\circ c^*\circ\alpha$ maps $R(\pi)$ *onto* the direct summand $T^0(G_1)\otimes T^0(G_2)$ of $K^0(G_1\times G_2)$. Hence there is an epimorphism

$$R(\pi)/j^*(I_{G_0}))\twoheadrightarrow T^0(G_1)\otimes T^0(G_2)\cong\mathbf{Z}\oplus\mathbf{Z}_{2^{k-1}}\oplus\mathbf{Z}_{2^{n-1}}\oplus\mathbf{Z}_{2^{k-1}}$$

and the proposition is established.

4. A Basic Generator of Finite Order in $K^1(G)$.

The elements ξ_1, $\xi_2\in K^0(G)$ and $v_1,\dots,v_{n-2}$, ε_{n-1}, $\varepsilon_n\in K^1(G)$ do not yet generate the ring $K^*(G)$. In fact it can be shown, comparing the spectral sequences of the bundles $\Lambda=(G_0\xrightarrow{u} G\xrightarrow{c} B_\pi)$ and $\Gamma_1=(G_0\xrightarrow{a_1} G_1\xrightarrow{c_1} B_{\omega_1})$ that there must exist an element $\tau\in K^1(G)$ with $b_1^*(\tau)=\zeta_1\cdot\varepsilon_n^{(1)}\in K^1(G_1)$. Such an element τ can not be expressed in terms of the elements in $K^*(G)$ described as yet. (Note $b_1^*(\varepsilon_n)=2\varepsilon_n^{(1)}$.)

We are now going to define an element $\tau \in K^1(G)$ of finite order which together with the above elements will generate the ring $K^*(G)$.

To begin with let us consider $\varepsilon_n^{(1)}$, $\varepsilon_n^{(3)}$ and γ_n in $K^1(G_1)$, $K^1(G_3)$ and $K^1(G_0)$ respectively. By (2.12) and (2.13) these elements are related as follows.

$$a_1^*(\varepsilon_n^{(1)}) = \gamma_n = a_3^*(\varepsilon_n^{(3)}). \tag{4.1}$$

We now define

$$\tau = (b_3)_*(\varepsilon_n^{(3)}) - (b_1)_*(\varepsilon_n^{(1)}) \in K^1(G), \tag{4.2}$$

where $(b_i)_*: K^*(G_i) \to K^*(G)$, $(i=1, 3)$, is the Atiyah-transfer map associated to the twofold covering $b_i: G_i \to G$.

(4.3) PROPOSITION. *The element $\tau \in K^1(G)$ has the following properties*
(i) $b_1^*(\tau) = \zeta_1 \varepsilon_n^{(1)} \in K^1(G_1)$
(ii) $b_2^*(\tau) = 0 \in K^1(G_2)$.

Proof. For the basic properties of the transfer map $f_*: K^*(X) \to K^*(Y)$ associated to a finite covering projection $f: X \to Y$ we refer to [2] and [10]. In particular we point out the validity of the "Frobenius reciprocity law", i.e.

$$f_*(f^*(y) \cdot x) = y \cdot f_*(x)$$

where $x \in K^*(X)$, $y \in K^*(Y)$ and $f^*: K^*(Y) \to K^*(X)$ the map induced by f. Consider the following morphisms of coverings

$$\begin{array}{ccc} G_0 & \xrightarrow{a_i} & G_i \\ {\scriptstyle a_j}\downarrow & & \downarrow{\scriptstyle b_i} \\ G_j & \xrightarrow[b_j]{} & G \end{array}$$

where $i \neq j$ and $i, j = 1, 2, 3$.

The transfer is natural with respect to such morphisms and with (4.1) we compute

$$b_2^* \circ (b_i)_*(\varepsilon_n^{(i)}) = (a_2)_* \circ a_i^*(\varepsilon_n^{(i)}) = (a_2)_*(\gamma_n), \qquad (i=1, 3),$$

thus establishing part (ii) of (4.3). On the trivial line bundle $1 \in K^0(G_0)$ the transfer $(a_1)_*$ is given by $(a_1)_*(1) = 2 + \zeta_1$; (see [2; p. 45]). Using the Frobenius law we then calculate

$$b_1^* \circ (b_3)_*(\varepsilon_n^{(3)}) = (a_1)_* \circ a_3^*(\varepsilon_n^{(3)}) = (a_1)_*(\gamma_n) = (a_1)_*(a_1^*(\varepsilon_n^{(1)}) \cdot 1) = \varepsilon_n^{(1)}(2 + \zeta_1).$$

Furthermore $b_1^* \circ (b_1)_*(\varepsilon_n^{(1)}) = 2\varepsilon_n^{(1)}$ and part (i) of (4.3) is verified.

(4.4) COROLLARY. *The following relations hold in* $K^0(G)$.

(i) $\xi_1\tau+2\tau=0$

(ii) $\xi_2\tau+2\tau-\xi_1\varepsilon_n=0$

(iii) $\tau\varepsilon_{n-1}=0$, $\tau\varepsilon_n=0$

(iv) $\tau^2=0$.

Proof. Recall that $\varepsilon_n=(b_1)_*(\varepsilon_n^{(1)})$ and $\varepsilon_{n-1}=(b_2)_*(\varepsilon_{n-1}^{(2)})$. Now observe that $(b_1)_*(1)=2+\xi_2$ and $(b_2)_*(1)=2+\xi_1$; (see definition of ξ_1, ξ_2 in (3.3)). Using (4.3) and the "Frobenius law" we get

$$(2+\xi_1)\,\tau=(b_2)_*(1)\,\tau=(b_2)_*(1\cdot b_2^*(\tau))=0$$

and analogously

$$(2+\xi_2)\,\tau=(b_1)_*(1)\,\tau=(b_1)_*(1\cdot b_1^*(\tau))=(b_1)_*(\zeta_1\cdot\varepsilon_n^{(1)})=\xi_1\cdot\varepsilon_n$$

thus establishing parts (i) and (ii) of (4.4). Next we verify

$$\tau\varepsilon_n=(b_1)_*(b_1^*(\tau)\cdot\varepsilon_n^{(1)})=(b_1)_*(\zeta_1\cdot\varepsilon_n^{(1)}\cdot\varepsilon_n^{(1)})=0$$
$$\tau\varepsilon_{n-1}=(b_2)_*(b_2^*(\tau)\cdot\varepsilon_{n-1}^{(2)})=0.$$

Eventually the fact that G is a finite CW complex and $\tau\in K^1(G)$ implies that $\tau^2=0$. This completes the proof of this corollary.

We now proceed to determine the order of τ.

(4.5) PROPOSITION. *The element* $\tau\in K^1(\mathrm{PSO}(2n))$ *is of order* 2^k *where* $k=$ $=\nu_2(n)+2$.

Proof. The fact that $2^{k-1}\xi_1=0$, (see (3.3)), together with the relation $2\tau=-\xi_1\tau$, (see (4.4)), implies that $2^k\tau=0$. It remains to show that $2^{k-1}\tau\neq 0$. This is done in the following way. The commutative square

$$\begin{array}{ccc} G_0 & \xrightarrow{a_2} & G_2 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle b_2} \\ G_1 & \xrightarrow{b_1} & G \end{array}$$

gives rise to a map of pairs $j:(G_1,G_0)\to(G,G_2)$. (Replace the spaces in the bottom row by the mapping cylinders of a_1 and b_2 respectively.) We thus obtain a morphism of exact sequences

$$\begin{array}{ccccccccccc} \cdots\longrightarrow & K^0(G_2) & \xrightarrow{\delta(2)} & K^1(G,G_2) & \xrightarrow{i^*_2} & K^1(G) & \xrightarrow{b^*_2} & K^1(G_2) & \longrightarrow\cdots \\ & \downarrow{\scriptstyle a^*_2} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle b^*_1} & & \downarrow{\scriptstyle a^*_2} & \\ \cdots\longrightarrow & K^0(G_0) & \xrightarrow{\delta(1)} & K^1(G_1,G_0) & \xrightarrow{i^*_1} & K^1(G_1) & \xrightarrow{a^*_1} & K^1(G_0) & \longrightarrow\cdots. \end{array}$$

Since $b_2^*(\tau)=0$ there is an element $\omega\in K^1(G, G_2)$ such that $i_2^*(\omega)=\tau$. With $b_1^*(\tau)=$ $=\zeta_1\varepsilon_n^{(1)}$ we infer $j^*(\omega)\equiv\zeta_1\cdot\varepsilon_n^{(1)}$ (mod im $\delta^{(1)}$), where in the latter expression the dot denotes the action of $K^*(G_1)$ on $K^*(G_1, G_0)$. Referring to (2.4), (2.9) (ii) and (2.12) we observe that $\delta^{(1)}(\gamma_{n-1})=2^{k-1}\zeta_1\neq 0$ and thus $\delta^{(1)}(\gamma_{n-1}\gamma_n)=2^{k-1}\zeta_1\cdot\varepsilon_n^{(1)}\neq 0$. Hence

$$j^*(2^{k-1}\omega)=2^{k-1}\zeta_1\cdot\varepsilon_n^{(1)}=\delta^{(1)}(\gamma_{n-1}\gamma_n)\neq 0. \tag{4.6}$$

(Note, $2\cdot\operatorname{im}\delta^{(1)}=0$).

We show that $2^{k-1}\tau=0$ leads to a contradiction. The assumption $2^{k-1}\tau=0$ implies $i_2^*(2^{k-1}\omega)=0$; hence there is an element in $K^0(G_2)$, say η, with $\delta^{(2)}(\eta)=2^{k-1}\omega$. By (4.6) we then get

$$\delta^{(1)}a_2^*(\eta)=2^{k-1}\zeta_1\cdot\varepsilon_n^{(1)}=\delta^{(1)}(\gamma_{n-1}\gamma_n).$$

According to (2.12) we have $a_2^*(K^*(G_2))=E_{\mathbf{Z}}(\beta_1,\ldots,\beta_{n-2},\gamma_{n-1},2\gamma_n)\subset K^*(G_0)$ and $\ker\delta^{(1)}=a_1^*(K^*(G_1))=E_{\mathbf{Z}}(\beta_1,\ldots,\beta_{n-2},2\gamma_{n-1},\gamma_n)$. One now checks readily that

$$a_2^*(\eta)\not\equiv\gamma_{n-1}\gamma_n \qquad (\text{modulo } \ker\delta^{(1)})$$

and the contradiction becomes evident. Hence the order of τ is indeed 2^k.

5. The Spectral Sequences.

In this section we compute all the differentials in the spectral sequence $(E_r(G), d_r^{\Lambda})$ of the fibre bundle

$$\Lambda=(G_0\underset{u}{\rightarrow}G\underset{c}{\rightarrow}B_\pi). \tag{5.1}$$

This will enable us to fully determine the target term $E_\infty(\Lambda)$. The additional information on $K^*(G)$ we get from $E_\infty(\Lambda)$ will then be sufficient to complete the description of the ring $K^*(G)$.

Basically we shall compare the spectral sequence of Λ with the "known" (see [8]) spectral sequences $(E_r(\Gamma_i), d_r^{\Gamma_i})$, where Γ_i is the fibre bundle

$$\Gamma_i=(G_0\underset{a_i}{\rightarrow}G_i\underset{c_i}{\rightarrow}B_{\omega_i}), \quad (i=1, 2). \tag{5.2}$$

For the E_2-term of the spectral sequence of Γ_i we have

$$E_2(\Gamma_i)\cong H^*(B\omega_i; \mathbf{Z})\otimes K^*(G_0),$$

where $H^*(B_{\omega_i}; \mathbf{Z})\cong\mathbf{Z}[w_i]/(2w_i)$, $w_i\in H^2(B_{\omega_i}; \mathbf{Z})$ and $K^*(G_0)=E_{\mathbf{Z}}(\beta_1,\ldots,\beta_{n-2},$ $\gamma_{n-1},\gamma_n)$, see (2.12). With (1.10) and [8] we obtain

(5.3) PROPOSITION. (i) *All differentials* $d_r^{\Gamma_1}$ *are trivial except for the differential* $d_{2k}^{\Gamma_1}$, $(k=v_2(n)+2)$, *which, evaluated on the element* $1\otimes\gamma_{n-1}$, *is given by*

$$d_{2k}^{\Gamma_1}(1\otimes\gamma_{n-1})=w_1^k\otimes 1\,.$$

The reduced E_∞*-term,* $\tilde{E}_\infty(\Gamma_1)=\bigoplus_{m>0}E_\infty^{m,*}(\Gamma_1)$, *is given by*

$$\begin{aligned}\tilde{E}_\infty(\Gamma_1)&\cong\{\tilde{H}^*(B_{\omega_1};\mathbf{Z})/(w_1^k)\}\otimes E_{\mathbf{Z}}(\beta_1,\dots,\beta_{n-2},\gamma_n)=\\&=\{(w_1)/(w_1^k)\}\otimes E_{\mathbf{Z}}(\beta_1,\dots,\beta_{n-2},\gamma_n)\,.\end{aligned}$$

(ii) *All differentials* $d_r^{\Gamma_2}$ *are trivial except for the differential* $d_{2n}^{\Gamma_2}$ *which, evaluated on the element* $1\otimes\gamma_n$, *is given by*

$$d_{2n}^{\Gamma_2}(1\otimes\gamma_n)=w_2^n\otimes 1\,.$$

The reduced $E_\infty(\Gamma_2)$*-term is given by* $\tilde{E}_\infty(\Gamma_2)\cong\{(w_2)/(w_2^n)\}\otimes E_{\mathbf{Z}}(\beta_1,\dots,\beta_{n-2},\gamma_{n-1})$.
We now focus on the following commutative diagram.

$$\begin{array}{ccccc} G_0 & \xleftarrow{q_i} & G_0\times G_0 & \xrightarrow{m_0} & G_0\\ \downarrow{\scriptstyle a_i} & & \downarrow{\scriptstyle a_1\times a_2} & & \downarrow{\scriptstyle u}\\ G_i & \xleftarrow{pr.} & G_1\times G_2 & \xrightarrow{t} & G\\ \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_1\times c_2} & & \downarrow{\scriptstyle c}\\ B_{\omega_i} & \xleftarrow{p_i} & B_{\omega_1}\times B_{\omega_2} & \xrightarrow[=]{h} & B_\pi \end{array}\qquad (i=1,2). \tag{5.4}$$

In (5.4) m_0 stands for the multiplication map, t is as in (3.4), p_i, q_i and $pr.$ are the canonical projections and h is the identification map induced by $\omega_1\times\omega_2=\pi$, (see 1). We denote the bundle in the middle of (5.4) by $\Gamma_1\times\Gamma_2$ and the corresponding bundle homomorphisms by

$$\Gamma_i\xleftarrow{P_i}\Gamma_1\times\Gamma_2\xrightarrow{M}\Lambda\,. \tag{5.5}$$

For the E_2-terms of the spectral sequences of $\Gamma_1\times\Gamma_2$ and Λ we have

$$\begin{aligned}E_2(\Gamma_1\times\Gamma_2)&\cong H^*(B_\pi;Z)\otimes K^*(G_0\times G_0)\\ E_2(\Lambda)&\cong H^*(B_\pi;Z)\otimes K^*(G_0)\,.\end{aligned}$$

We write $(E_r(B_\pi), d_r^{B_\pi})$ for the spectral sequence of the CW-complex $B_\pi=B_{\omega_1}\times B_{\omega_2}$ and make two basic observations.

(5.6) *Let* $r\geqslant 2$. *We have* $E_{r+1}(\Gamma_1\times\Gamma_2)\cong E_{r+1}(B_\pi)\otimes K^*(G_0\times G_0)$ *if, and only if,* $E_r(\Gamma_1\times\Gamma_2)\cong E_r(B_\pi)\otimes K^*(G_0\times G_0)$ *and* $d_r(1\otimes K^*(G_0\times G_0))=0$. *A similar remark can be made about the spectral sequence of* Λ.

This fact is easy to verify. Note, $E_r(B_\pi)$ is a differential subring of $E_r(B_\pi)\otimes \otimes K^*(G_0\times G_0)$ with $K^*(G_0\times G_0)$ torsion free, and similarly for $E(\Lambda)$.

(5.7). *If* $E_r(\Gamma_1\times\Gamma_2)\cong E_r(B_\pi)\otimes K^*(G_0\times G_0)$ *for some* $r\geqslant 2$, *then* $E_r(\Lambda)\cong E^r(B_\pi)\otimes K^*(G_0)$.

This is true for $r=2$ and it follows for $r>2$ by induction from (5.6) and the fact that the bundle map $M:\Gamma_1\times\Gamma_2\to\Lambda$ induces the monomorphism

$$E_{r-1}(B_\pi)\otimes K^*(G_0)\xrightarrow{\text{id.}\otimes m^*_0} E_{r-1}(B_\pi)\otimes K^*(G_0\times G_0).$$

We then derive from that

(5.8) LEMMA. *For the bundles* $\Gamma_1\times\Gamma_2$ *and* Λ *one has*

$$E_{2k}(\Gamma_1\times\Gamma_2)\cong E_{2k}(B_\pi)\otimes K^*(G_0\times G_0)$$
$$E_{2k}(\Lambda)\cong E_{2k}(B_\pi)\otimes K^*(G_0),\qquad (k=\nu_2(n)+2).$$

Proof. Referring to (5.6) and (5.7) we have to show that

$$d_s^{\Gamma_1\times\Gamma_2}(1\otimes K^*(G_0\times G_0))=0,\qquad (s=2,3,\ldots,2k-1),\tag{5.9}$$

By (5.3) the differentials $d_s^{\Gamma_i}$, $(s=2,3,\ldots,2k-1$ and $i=1,2)$, are trivial (note that $k=\nu(n)+2<n$) and since $E_s^{0,*}(\Gamma_1\times\Gamma_2)\cong 1\otimes K^*(G_0\times G_0)\cong 1\otimes K^*(G_0)\otimes K^*(G_0)$ is generated by the images of the spectral sequence maps $E_s(P_i)$, $(i=1,2)$, statement (5.9) follows.

We now list the relevant facts about the spectral sequence of $B_\pi=B_{\omega_1}\times B_{\omega_2}$. This spectral sequence is not trivial. However a computation of C. T. C. Wall (see [2; p. 61]) shows that

$$E_4(B_\pi)\cong E_\infty(B_\pi)\cong \text{Gr.}\,R(\pi)\cong \mathbf{Z}[x,y]/(2x,2y,x^2y-xy^2)\tag{5.10}$$

with

$$\text{Gr.}_{2s}R(\pi)=I_\pi^s/I_\pi^{s+1},\qquad \text{Gr.}_{\text{odd}}R(\pi)=0$$

where $x, y\in\text{Gr.}_2R(\pi)=I_\pi/I_\pi^2$ are represented by σ_1, σ_2 respectively. We introduce the following notation

$$R_s=\text{Gr.}_{2s}R(\pi),\qquad R=\bigoplus_{s=0}^{\infty}R_s=\text{Gr.}\,R(\pi),\qquad \tilde{R}=\bigoplus_{s=1}^{\infty}R_s=\text{Gr.}\,I_\pi.\tag{5.11}$$

We then have $R_1\cong Z_2\oplus Z_2$, where x and y generate the two cyclic summands. For $s\geqslant 2$ the cyclic summands of $R_s\cong\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$ are generated by x^s, y^s and xy^{s-1} respectively.

For later use it is convenient to set

$$z_s = y^s + xy^{s-1} \in R_s, \qquad (s=2,3,\ldots)$$

and hence we have

$$x^r z_s = 0, \qquad y^r z_s = z_{r+s} = z_r z_s, \qquad x^r y^s = z_{r+s} - y^{r+s}. \tag{5.12}$$

We are now ready to give an explicit description of the $2k$-level of the spectral sequence of the bundle Λ.

(5.13) LEMMA. (i) $E_{2k}(\Lambda) = R \otimes K^*(G_0) \cong \{\mathbf{Z}[x, y]/(2x, 2y, x^2y - xy^2)\} \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n)$

(ii) $d^\Lambda_{2k}(R \otimes 1) = 0, \quad d^\Lambda_{2k}(1 \otimes \beta_s) = 0, \quad (s=1,2,\ldots,n-2),$
$d^\Lambda_{2k}(1 \otimes \gamma_n) = 0, \quad d^\Lambda_{2k}(1 \otimes \gamma_{n-1}) = x^k \otimes 1.$

Proof. Part (i) is a consequence of (5.8) and (5.10), since $2k > 4$. Also from (5.10) we infer that $d^\Lambda_{2k}(R \otimes 1) = 0$. Now the bundle maps of (5.4) induce homomorphisms of the corresponding spectral sequences, which on the $2k$-level are given as follows

$$\begin{array}{ccccc} H^*(B_{\omega_i}: \mathbf{Z}) \otimes K^*(G_0) & \xrightarrow[p^*_i \otimes q^*_i]{} & R \otimes K^*(G_0 \times G_0) & \xleftarrow[\mathrm{id.} \otimes m^*_0]{} & R \otimes K^*(G_0) \\ \wr\| & & \wr\| & & \wr\| \\ E_{2k}(\Gamma_i) & \longrightarrow & E_{2k}(\Gamma_1 \times \Gamma_2) & \longleftarrow & E_{2k}(\Lambda). \end{array}$$

Using (5.3), the fact that $p^*_1(w^k_1) = x^k \otimes 1$ and the primitivity of the elements $\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n$ with respect to m^*_0 we immediately complete the proof of this lemma. (Again note that $k < n$.)

A short computation involving (5.12) and (5.13) shows that

$$\left.\begin{array}{l} E^{0,*}_{2k+1}(\Lambda) \cong \mathbf{Z} \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, 2\gamma_{n-1}, \gamma_n) \\ \text{and} \\ \tilde{E}_{2k+1}(\Lambda) \cong \tilde{R}/(x^k) \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_n) \\ \qquad \oplus (z_2) \otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_n) \cdot \gamma_{n-1}. \end{array}\right\} \tag{5.14}$$

(Here (v) stands for the ideal generated by $v \in R$).

To get a hold on the differentials d^Λ_r, for $r > 2k$, we consider the bundle maps

$$F_i: \Gamma_i \to \Lambda, \qquad (i=1,2) \tag{5.15}$$

which are given by the commutative diagrams

$$\begin{array}{ccccc} G_0 & \longrightarrow & G_i & \xrightarrow{c_i} & B_{\omega_i} \\ \downarrow 1 & & \downarrow b_i & & \downarrow s_i \\ G_0 & \longrightarrow & G & \xrightarrow{c} & B_\pi \end{array} \qquad (i=1,2).$$

(5.16) LEMMA. (i) *The homomorphism*

$$E_{2k+1}(F_2): E_{2k+1}^{0,*}(\Lambda) \cong E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, 2\gamma_{n-1}, \gamma_n)$$
$$\to E_{2k+1}^{0,*}(\Gamma_2) \cong E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n)$$

is the canonical inclusion.

(ii) $E_{2k+1}(F_2)$ *maps* $(z_2)\otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_n)\cdot\gamma_{n-1} \subset E_{2k+1}(\Lambda)$ *isomorphically onto* $(w^2)\otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_n)\cdot\gamma_{n-1} \subset E_{2k+1}(\Gamma_2)$.

(iii) $E_{2k+1}(F_2): E_{2k+1}^{2p,*}(\Lambda) \to E_{2k+1}^{2p,*}(\Gamma_2)$ *is an isomorphism for* $2p \geqslant 2k+2$. (*Note,* $E_{2k+1}^{\mathrm{odd},*}(\Lambda) = 0 = E_{2k+1}^{\mathrm{odd},*}(\Gamma_2)$.)

Proof. Part (i) is clear. For parts (ii) and (iii) we observe that

$$E_{2k}(F_2): R\otimes K^*(G_0) \to H^*(B_{\omega_2}; \mathbf{Z})\otimes K^*(G_0)$$

is given by $E_{2k}(F_2)(x\otimes 1) = 0$, $E_{2k}(F_2)(y\otimes 1) = w_2\otimes 1$, hence $E_{2k}(F_2)(z_s\otimes 1) = w_2^s\otimes 1$. To complete the proof look at the induced map on the $(2k+1)$-level.

It follows from (5.16) that d_r^Λ, $(r \geqslant 2k+1)$, is trivial as long as $d_r^{\Gamma_2} = 0$, and with (5.3) (ii) we get immediately

(5.17) LEMMA. (i) $d_r^\Lambda = 0$ *for* $r = 2k+1, \ldots, 2n-1$, *i.e.* $E_{2k+1}(\Lambda) \cong E_{2n}(\Lambda)$

(ii) $d_{2n}^\Lambda(1\otimes\gamma_n) = \bar{y}^n\otimes 1$; *(where* $\bar{y}\in\tilde{R}/(x^k)$ *is the element represented by* $y\in\tilde{R}$). d_{2n}^Λ *is zero on the elements* $1\otimes\beta_1, \ldots, 1\otimes\beta_{n-2}, 1\otimes 2\gamma_{n-1}, \bar{x}\otimes 1, \bar{y}\otimes 1, z_2\otimes\gamma_{n-1}$; *(where* $\bar{x}$ *is the element represented by* x*). In particular,* $d_{2n}^\Lambda(z_2\otimes\gamma_{n-1}\gamma_n) = z_{n+2}\otimes\gamma_{n-1}$.

An explicit calculation resting on (5.12), (5.14) and (5.17) then gives

(5.18) $E_{2n+1}^{0,*}(\Lambda) = E_{2n+2}^{0,*}(\Lambda) = 1\otimes A$, where A is the subalgebra of $E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2}, \gamma_{n-1}, \gamma_n)$ generated by $\beta_1, \ldots, \beta_{n-2}, 2\gamma_{n-1}, 2\gamma_n$ and $2\gamma_{n-1}\gamma_n$. Moreover we have

$$\tilde{E}_{2n+1}(\Lambda) \cong \tilde{E}_{2n+2}(\Lambda) \cong \{\tilde{R}/(x^k, y^n)\}\otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2})$$
$$\oplus\{(x)/(x^k)\}\otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2})\,\gamma_n$$
$$\oplus\{(z_2)/(z_{n+2})\}\otimes E_{\mathbf{Z}}(\beta_1, \ldots, \beta_{n-2})\,\gamma_{n-1}\,.$$

Since $E_{2n+2}^{p,*}(\Lambda) = 0$ for $p > 2n+3$, we conclude that $d_r = 0$ for $r \geqslant 2n+3$ and $d_{2n+2}^\Lambda(E_{2n+2}^{q,*}(\Lambda)) = 0$ for $q > 0$. On the other hand elements of the form $2\gamma_{n-1}\gamma_n\alpha \in K^*(G_0)$, where $\alpha = \beta_{i_1}\beta_{i_2}\cdots\beta_{i_s}$ are not in the image of $u^*: K^*(G) \to K^*(G_0)$, (see (2.12)), i.e. these elements can not "survive" in the spectral sequence of Λ. Hence for $1\otimes 2\gamma_{n-1}\gamma_n\,\alpha \in E_{2n+2}^{0,*}(\Lambda)$ we must have

$$d_{2n+2}^\Lambda(1\otimes 2\gamma_{n-1}\gamma_n\alpha) = \bar{z}_{n+1}\otimes\gamma_{n-1}\alpha$$

and thus we get

$$\begin{aligned} E_\infty^{0,*}(\Lambda) &\cong \mathbf{Z}\otimes E_{\mathbf{Z}}(\beta_1,\dots,\beta_{n-2},2\gamma_{n-1},2\gamma_n)\\ \tilde{E}_\infty(\Lambda) &\cong \tilde{R}/(x^k,y^n)\otimes E_{\mathbf{Z}}(\beta_1,\dots,\beta_{n-2})\\ &\quad\oplus (x)/(x^k)\otimes E_{\mathbf{Z}}(\beta_1,\dots,\beta_{n-2})\,\gamma_n\\ &\quad\oplus (z_2)/(z_{n+1})\otimes E_{\mathbf{Z}}(\beta_1,\dots,\beta_{n-2})\,\gamma_{n-1}\,. \end{aligned} \tag{5.19}$$

In particular $E_\infty^{\mathrm{odd},*}(\Lambda)=0$, $E_\infty^{p,*}(\Lambda)=0$ for $p\geqslant 2n+2$.

The ringstructure on the right hand side of (5.19) is the one inherited from $R\otimes E_{\mathbf{Z}}(\beta_1,\dots,\beta_{n-2},\gamma_{n-1},\gamma_n)$.

Note that – as abelian groups – the "quotients" in $\tilde{E}_\infty(\Lambda)$ can be exhibited as follows (the elements under the $\mathbf{Z}_2$-summands indicate the respective generators):

$$\begin{aligned} \tilde{R}/(x^k,y^n) &\cong (\underset{\bar{x}}{\mathbf{Z}_2}\oplus\underset{\bar{y}}{\mathbf{Z}_2})\oplus(\underset{\bar{x}^2}{\mathbf{Z}_2}\oplus\underset{\bar{y}^2}{\mathbf{Z}_2}\oplus\underset{\bar{x}\bar{y}}{\mathbf{Z}_2})\oplus\underset{\dots}{\dots}\oplus(\underset{\bar{x}^{k-1}}{\mathbf{Z}_2}\oplus\underset{\bar{y}^{k-1}}{\mathbf{Z}_2}\oplus\underset{\bar{x}\bar{y}^{k-2}}{\mathbf{Z}_2})\oplus(\underset{\bar{y}^{k}}{\mathbf{Z}_2}\oplus\underset{\bar{x}\bar{y}^{k-1}}{\mathbf{Z}_2})\oplus\\ &\qquad\oplus\underset{\bar{y}^{k+1}}{\mathbf{Z}_2}\oplus\underset{\dots}{\dots}\oplus\underset{\bar{y}^{n-1}}{\mathbf{Z}_2}\\ (x)/(x^k) &\cong \underset{\bar{x}}{\mathbf{Z}_2}\oplus(\underset{\bar{x}^2}{\mathbf{Z}_2}\oplus\underset{\bar{x}\bar{y}}{\mathbf{Z}_2})\oplus\underset{\dots\dots\dots}{\dots\dots\dots}\oplus(\underset{\bar{x}^{k-1}}{\mathbf{Z}_2}\oplus\underset{\bar{x}\bar{y}^{k-2}}{\mathbf{Z}_2})\oplus\underset{\bar{x}\bar{y}^{k-1}}{\mathbf{Z}_2}\\ (z_2)/(z_{n+1}) &\cong \underset{\bar{z}_2}{\mathbf{Z}_2}\oplus\underset{\bar{z}_3}{\mathbf{Z}_2}\oplus\underset{\dots\dots\dots}{\dots\dots\dots}\oplus\underset{\bar{z}_n}{\mathbf{Z}_2} \end{aligned} \tag{5.20}$$

We are now going to extract as much information from the structure of $E_\infty(\Lambda)$ as we need in order to be able to complete the description of the ring $K^*(\mathrm{PSO}(2n))$. In this sense the following corollaries rest basically on (5.19).

Since the total space G of the fibre bundle Λ is of the homotopy type of a finite CW-complex the spectral sequence converges, i.e.

$E_\infty(\Lambda)\cong \mathrm{Gr}.K^*(G)$,

where $\mathrm{Gr}.K^*(G)$ is the graded ring associated to the usual filtration (see [2; p. 29]) of $K^*(G)$. There are no elements of finite order in $E_\infty^{0,*}(\Lambda)$ and no elements of infinite order in $\tilde{E}_\infty(\Lambda)$. Hence

$|\mathrm{Tors}.K^*(G)| = |\tilde{E}_\infty(\Lambda)|$.

(5.21) COROLLARY. *The number of elements of finite order in $K^*(G)$ is given by*

$|\mathrm{Tors}.K^*(G)| = 2^{(2n+4k-6)2^{n-2}}$

where $k=v_2(n)+2$.

Proof. Use (5.19) and (5.20).

(5.22). According to (5.19) the elements $1\otimes\beta_1,\dots,1\otimes\beta_{n-2}$, $1\oplus 2\gamma_{n-1}$, $1\otimes 2\gamma_n$, $\bar{x}\otimes 1$, $\bar{y}\otimes 1$, $\bar{x}\otimes\gamma_n$, $\bar{z}_2\otimes\gamma_{n-1}$ form a system of generators of the graded ring $E_\infty(G)\cong$ $\cong\mathrm{Gr}.K^*(G)$. (Recall that $(\bar{y}^r\otimes 1)\,(z_2\otimes\gamma_{n-1})=\bar{z}_{2+r}\otimes\gamma_{n-1}$.)

In the following table we record which elements of $K^*(G)$ represent the above generators of $E_\infty(\Lambda)$.

$K^*(G)$	$s=1,2,\ldots,n-2$ ν_s	ε_{n-1}	ε_n	ξ_1	ξ_2	τ	$\xi_2\varepsilon_{n-1}$
$E_\infty(G)$	$1\otimes\beta_s$	$1\otimes 2\gamma_{n-1}$	$1\otimes 2\gamma_n$	$\bar{x}\otimes 1$	$\bar{y}\otimes 1$	$\bar{x}\otimes\gamma_n$	$\bar{z}_2\otimes\gamma_{n-1}+v$

(5.23)

where in the right hand corner $v\in E_\infty^{4,*}(\Lambda)$ is an element of the form $v=\bar{x}\bar{y}\otimes\alpha_1+$ $+(\bar{x}\otimes\gamma_n)\cdot(\bar{y}\otimes\alpha_2)$; $\alpha_1, \alpha_2\in E_{\mathbf{Z}}(\beta_1,\ldots,\beta_{n-2})$.

Only the last two entries of this table require some comment. By (4.3) one has $b_1^*(\tau)=\zeta_1\varepsilon_n^{(1)}\in K^*(G_1)$ and $b_2^*(\tau)=0$. The element $\zeta_1\varepsilon_n^{(1)}$ has exact filtration 2 and represents $w_1\otimes\gamma_n\in E_\infty(\Gamma_1)$. Hence the torsion element τ has also exact filtration 2. Looking at the homomorphisms $E_\infty^{2,*}(F_1)$ and $E_\infty^{2,*}(F_2)$ we then see that τ represents $\bar{x}\otimes\gamma_n$; (use (5.3) and (5.19)).

The filtration of $\xi_2\varepsilon_{n-1}$ is greater than 2, the reason being $(\bar{y}\otimes 1)\cdot(1\otimes 2\gamma_{n-1})=0$ in $E_\infty^{2,*}(\Lambda)$. On the other hand we have $b_2^*(\xi_2\varepsilon_{n-1})=b_2^*(\xi_2\cdot(b_{2*})\,\varepsilon_{n-1}^{(2)}))=\zeta_2\cdot 2\varepsilon_{n-1}^{(2)}$. Since $2\zeta_2\varepsilon_{n-1}^{(2)}=-\zeta_2^2\varepsilon_{n-1}^{(2)}$ has exact filtration 4, the same now holds for $\xi_2\varepsilon_{n-1}$. Hence $\xi_2\varepsilon_{n-1}$ represents an element $w\in E_\infty^{4,*}(\Lambda)$ such that $E_\infty(F_2)\,(w)=w_1^2\otimes\gamma_{n-1}$ and $E_\infty(F_1)\,(w)=0$ (recall that $b_1^*(\xi_2\varepsilon_{n-1})=0$) and the result again follows by looking at the homomorphisms $E_\infty^{4,*}(F_1)$ and $E_\infty^{4,*}(F_2)$.

(5.24) *Remark*. Note that in $E_\infty(\Lambda)$ we have $(\bar{y}\otimes 1)^{k-1}\cdot(\bar{x}\otimes\gamma_n)\neq 0$ and hence $\xi_2^{k-1}\tau\neq 0$. By (3.3), (4.4) and (4.5) we then conclude that the order of $\xi_2\tau$ is 2^{k-1}.

Since $K^*(G)$ has finite filtration we derive from (5.22) and (5.23):

(5.25) COROLLARY. *The elements* $\nu_1,\ldots,\nu_{n-2},\varepsilon_{n-1},\varepsilon_n,\xi_1,\xi_2$ *and* τ *generate the ring* $K^*(G)$.

By (5.19) we have $E_\infty^{p,*}(\Lambda)=0$ for $p>2n$ and hence we can identify $E_\infty^{2n,*}(\Lambda)$ with $K_{2n}^*(G)$, the subgroup of elements of filtration $2n$. Elements of $E_\infty^{2n,*}(\Lambda)$ are of the form $\bar{z}_n\otimes\gamma_{n-1}\beta=(\bar{y}^{n-2}\otimes 1)\,(\bar{z}_2\otimes\gamma_{n-1}+v)\,(1\otimes\beta)$, where $\beta\in E_{\mathbf{Z}}(\beta_1,\ldots,\beta_{n-2})$ and v is as in (5.23). (Note that $(\bar{y}^{n-2}\otimes 1)\cdot v=0$.) The latter element is represented by $\xi_2^{n-2}(\xi_2\varepsilon_{n-1})\,\nu=2^{n-2}\xi_2\varepsilon_{n-1}\nu$, where $\nu\in E_{\mathbf{Z}}(\nu_1,\ldots,\nu_{n-2})$. Consequently we may remark:

(5.26). Any element $\mu\in K^*(G)$ of filtration $2n$ is of the form

$$\mu=2^{n-2}\xi_2\varepsilon_{n-1}\nu,$$

where $\nu\in E_{\mathbf{Z}}(\nu_1,\ldots,\nu_{n-2})$.

Finally we derive from $E_\infty(\Lambda)$ the following relation involving the (non-zero) element $2^{k-1}\tau\in K^1(G)$.

(5.27) COROLLARY. *There is an element* $v\in E_{\mathbf{Z}}(v_1,\ldots,v_{n-2})\subset K^*(G)$ *such that*

$$2^{k-1}\tau=2^{n-2}\xi_2\varepsilon_{n-1}v.$$

Proof. Note that $2^{k-1}\tau=\xi_1^{k-1}\tau$ (see (4.4) and (4.5)). In $E_\infty(\Lambda)$ we have $(\bar{x}\otimes 1)^{k-1}\cdot(\bar{x}\otimes\gamma_n)=0\in E_\infty^{2k,*}(\Lambda)$ and we conclude that $\xi_1^{k-1}\tau\in K^1(G)$ has filtration greater than $2k$. This in turn implies that $\xi_1^{k-1}\tau$ represents a non-zero element $t\in E_\infty^{2s,*}(\Lambda)$ for some s with $k+1\leqslant s\leqslant n$. Since $b_2^*(\xi_1^{k-1}\tau)=0$ we infer that $E_\infty^{2s,*}(F_2)(t)=0$. But $E_\infty^{2s,*}(F_2)$ is an isomorphism for $k+1\leqslant s\leqslant n-1$; (see (5.3) and (5.19)). Hence $t\in E_\infty^{2n,*}(\Lambda)$, i.e. $\xi_1^{k-1}\tau$ has exact filtration $2n$, and the corollary follows from (5.26).

6. The Ring $K^*(\mathrm{PSO}(2n))$; n even.

In this section we state the main theorem – for the "non cyclic" case – and complete its proof.

For this purpose define the $\mathbf{Z}_2$-graded commutative ring $T^*(G)=T^0(G)\oplus T^1(G)$ to be the subring of $K^*(G)$ generated by 1, ξ_1, ξ_2 and $\tau\in K^*(G)$.

Referring to (3.3), (4.4), (4.5) and (5.24) we get:

(6.1) *The subring* $T^*(G)\subset K^*(G)$ *is subject to the following relations*

(i) *The elements* ξ_1, $\xi_1\xi_2$ *and* $\tau\xi_2$ *are of order* 2^{k-1}, *the element* τ *is of order* 2^k, *where* $k=v_2(n)+2$. *The element* ξ_2 *is of order* 2^{n-1}.

(ii) $\xi_i^2+2\xi_i=0$, $(i=1,2)$, $\tau^2=0$ *and* $\xi_1\tau+2\tau=0$.

(6.2) THEOREM (Non-cyclic case). *Let* $G=\mathrm{PSO}(2n)$, *where* $n\geqslant 6$ *is an even integer. Then the canonical homomorphism*

$$E_{\mathbf{Z}}(v_1,\ldots,v_{n-2},\varepsilon_{n-1},\varepsilon_n)\otimes T^*(G)\to K^*(G)$$

induces a ring isomorphism

$$\{E_{\mathbf{Z}}(v_1,\ldots,v_{n-2},\varepsilon_{n-1},\varepsilon_n)\otimes T^*(G)\}/S(G)\cong K^*(G),$$

where $S(G)$ *is the ideal generated by the elements*

$$\varepsilon_{n-1}\otimes\xi_1,\ \varepsilon_n\otimes\xi_2,\ \varepsilon_{n-1}\otimes\tau,\ \varepsilon_n\otimes\tau,\ \varepsilon_{n-1}\otimes 2^{n-2}\xi_2-1\otimes 2^{k-1}\tau$$

and

$$1\otimes\tau\xi_2-\varepsilon_n\otimes\xi_1+1\otimes 2\tau.$$

Proof. Let us first establish the relation $2^{k-1}\tau=2^{n-2}\xi_2\varepsilon_{n-1}$ in $K^*(G)$. Reverting to (5.27) we recall that we have already shown $2^{k-1}\tau=2^{n-2}\xi_2\varepsilon_{n-1}\nu$, for some $\nu\in E_{\mathbf{Z}}(\nu_1,\ldots,\nu_{n-2})$. In order to verify that actually $\nu\equiv 1 \pmod 2$ and hence $2^{n-2}\xi_2\varepsilon_{n-1}\nu=2^{n-2}\xi_2\varepsilon_{n-1}$, we propose to look at the homomorphism $g^*:K^*(G)\to\to K^*(G)\otimes K^*(G_0)$ which is induced by the obvious action map $g:G\times G_0\to G$. We then easily calculate that

$$g^*(2^{k-1}\tau)=2^{k-1}\tau\otimes 1\,.$$

On the other hand – since ν_s, $(s=1,\ldots,n-2)$, is primitive modulo torsion and since $2^{n-2}\xi_2\cdot\text{Tors.}K^*(G)=0$ – it is not hard to show that

$$g^*(2^{n-2}\xi_2\varepsilon_{n-1}\nu)=2^{n-2}\xi_2\varepsilon_{n-1}\nu\otimes 1+\alpha(\nu),$$

where $\alpha(\nu)\neq 0$ unless $\nu\equiv 1 \pmod 2$. Hence the relation $2^{k-1}\tau=2^{n-1}\xi_2\varepsilon_{n-1}$ is established.

Next we observe that we have $\varepsilon_{n-1}\xi_1=0$ and $\varepsilon_n\xi_2=0$. (Use the fact that $\varepsilon_{n-1}=(b_2)_*(\varepsilon_{n-1}^{(2)})$, $\varepsilon_n=(b_1)_*(\varepsilon_n^{(1)})$, (see (2.12)), $b^*(\xi_1)=0$, $b_1^*(\xi_2)=0$, (see (3.3)), and the 'Frobenius law'.) The validity of the above relations together with (3.3), (4.4) and (4.5) then imply that the canonical homomorphism $h:E_{\mathbf{Z}}(\nu_1,\ldots,\nu_{n-2},\varepsilon_{n-1},\varepsilon_n)\otimes T^*(G)\to K^*(G)$ factors through $\{E_{\mathbf{Z}}(\nu_1,\ldots,\nu_{n-2},\varepsilon_{n-1},\varepsilon_n)\otimes T^*(G)\}/S(G)$. On the other hand h is an epimorphism by (5.25) and the order of the torsion subgroup of $\{E_{\mathbf{Z}}(\nu_1,\ldots,\nu_{n-2},\varepsilon_{n-1},\varepsilon_n)\otimes T^*(G)\}/S(G)$ is the same as $|\text{Tors.}K^*(G)|$ (see (5.21)). Therefore h is an isomorphism and the theorem is proved.

II. THE CYCLIC CASE; $\pi_1(\mathrm{PSO}(2n)\cong\mathbf{Z}_4$

7. The Ring $K^*(\mathrm{PSO}(2n))$; n odd.

If $n\geqslant 5$ is an *odd* integer then the centre π of $G_0=\mathrm{Spin}(2n)$ is isomorphic to $\mathbf{Z}_4$. In order to determine the ring structure of $K^*(G)$, where $G=G_0/\pi$, one has to analyze the spectral sequence of the fibration

$$\Lambda=(G_0\xrightarrow{u}G\xrightarrow{c}B_\pi)$$

where $\pi\cong\mathbf{Z}_4$, $G_0\xrightarrow{u}G$ the universal 4-fold covering of G and c is its classifying map. The structure of the spectral sequence of Λ can be worked out essentially along the lines of [8]. It turns out that the only non-trivial differentials are d_6^Λ and d_{2n}^Λ. The reason for that may be indicated as follows.

Let $j:\pi\subsetneq G_0$ be the inclusion of the centre. Then $R(\pi)/J\cong\mathbf{Z}\oplus\mathbf{Z}_{2^{n-1}}\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$, where J is the ideal generated by $j^*(I_{G_0})$ and the cyclic summands of $R(\pi)/J$ are generated by 1, $\bar{\sigma}$, $\bar{\sigma}^2+2\bar{\sigma}$ and $\bar{\sigma}^3+2\bar{\sigma}^2$, with $1+\sigma$ being the canonical representation of π.

The fact that $J\subset I_\pi^3$ but $J\not\subset I_\pi^4$ together with [8; (5.5)] implies that $d_6^A\neq 0$.

The non-triviality of d_{2n}^A then is worked out by comparing the spectral sequence of A with the spectral sequence of $\Gamma_2=(G_0\xrightarrow{a_2}SO(2n)\xrightarrow{c_2}B_{Z_2})$.

From the $E_\infty(A)$ term we derive that

$$T^*(G)=T^0(G)=\operatorname{im}\{K^*(B_\pi)\xrightarrow{c^*}K^*(G)\}\cong R(\pi)/J. \tag{7.1}$$

Let $1+\xi\in K^0(G)$ represent the line bundle associated to the (cyclic) covering $G_0\xrightarrow{u}G$. Clearly $\xi\in T^0(G)$ and moreover it corresponds to the generator $\bar{\sigma}$ under the above isomorphism $T^0(G)\cong R(\pi)/J$. In particular ξ generates $\tilde{T}^0(G)$ and it is subject to the relations

$$2^{n-1}\xi=0,\ (1+\xi)^4=1 \quad\text{and}\quad 2(\xi^2+2\xi)=0.$$

As in the "non-cyclic" case there are elements $v_1,\ldots,v_{n-2},\varepsilon_{n-1},\varepsilon_n\in K^1(G)$ generating an exterior algebra in $K^*(G)$ which is isomorphic to $K^*(G)/\text{Tors.}K^*(G)$.

Summarizing all the information we get from the spectral sequence of A and from the transfer maps of the coverings involved, we arrive at the following description of the ring $K^*(G)$.

(7.2) THEOREM (Cyclic case). *Let $G=PSO(2n)$, where $n\geqslant 5$ is an odd integer. Then the canonical homomorphism*

$$E_Z(v_1,\ldots,v_{n-2},\varepsilon_{n-1},\varepsilon_n)\otimes T^*(G)\to K^*(G)$$

induces a ring isomorphism

$$\{E_Z(v_1,\ldots,v_{n-2},\varepsilon_{n-1},\varepsilon_n)\otimes T^*(G)\}/S(G)\cong K^*(G),$$

where $T^*(G)=T^0(G)\cong R(\pi)/(j^*(I_{G_0}))$ and $S(G)$ is the ideal generated by $\varepsilon_n\otimes 2\xi$, $\varepsilon_{n-1}\varepsilon_n\otimes\xi$, $\varepsilon_n\otimes\xi^3$ and $\varepsilon_{n-1}\otimes(\xi^2+2\xi)$.

BIBLIOGRAPHY

[1] Atiyah, M. F., *Thom Complexes*, Proc. London Math. Soc. (1961), pp. 291–310.
[2] ——, *Characters and cohomology of finite groups*, Publ. Math. Inst. Hautes Etudes Scientifiques, (1961).
[3] ——, *K-theory*, (Benjamin, 1967).
[4] Baum, P. F. and Browder, W. *The cohomology of quotients of classical groups*, Topology *3* (1965), 305–336.
[5] Borel, A., *Seminar on transformation groups*, Princeton University Press 1960.
[6] Gitler, S. and Lam, K. Y., *The K-theory of Stiefel manifolds*, Springer Lecture Notes in Mathematics, *168*, pp. 35–66.

[7] HELD, R. P. und SUTER, U., *Die Bestimmung der unitären K-Theorie von* SO(*n*) *mit Hilfe der Atiyah-Hirzebruch Spektralreihe*, Math. Z. *122* (1971), 33–52.
[8] ——, *On the unitary K-theory of compact Lie groups with finite fundamental group*, Quart. J. Math. Oxford (2), *24* (1973), 343–356.
[9] HODGKIN, L., *On the K-theory of Lie groups*, Topology, *6* (1967), 1–36.
[10] KAHN, D. S. and PRIDDY, S. B., *Applications of the transfer to stable homotopy theory* (preprint, Northwestern University, Evanston, Illinois).
[11] TITS, J., *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Springer Lecture Notes in Mathematics, *40* (1967).

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