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Complete Homogeneous Riemannian Manifolds of Negative Sectional Curvature

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1. Introduction

The classical uniformization theorem for Riemann surfaces says that, except few cases, every Riemann surface has the unit disc as its universal covering manifold. A natural question is to determine universal covering manifolds of complete Riemannian manifolds of negative sectional curvature. This is equivalent to classification of all simply connected complete Riemannian manifolds of negative sectional curvature. Here, we shall give a classification theorem of all complete homogeneous (necessarily simply connected) Riemannian manifolds with sectional curvature $K \leqq c < 0$, that is either the isometry group $I(M)$ of such a manifold M has a common fixed point in the boundary $M(\infty)$ of M or M is a noncompact symmetric space of rank one. It might be amusing to compare this theorem with the Riemann mapping theorem. This theorem also indicated that there may be other complete homogeneous Riemannian manifolds with negative curvature than the noncompact symmetric spaces. Consider now a simply connected, complete Riemannian manifold of negative sectional curvature. We can determine complete totally geodesic orbits of its isometry group. The second result is a classification theorem of connected Lie subgroups of the connected isometry group $I_0(M)$ of a noncompact symmetric space M of rank one. This has been done in [5] and [6]. Here, we give a unified argument. As a consequence, if G is a subgroup of $I_0(M)$ of a noncompact symmetric space M of rank one such that there is no point in $\bar{M}$ and no proper totally geodesic submanifold in M invariant under G, then G is either discrete or dense in $I_0(M)$. This theorem suggests an approach of construction of non-arithmetic discrete subgroups.

Let M be an n-dimensional, simply connected, complete Riemannian manifold with sectional curvature $K \leqq 0$. There is a natural compactification $\bar{M} = M \cup M(\infty)$ of M given by Eberlein and O'Neill [10]. $\bar{M}$ is homeomorphic to the closed unit ball in $\mathbf{R}^n$ and $M(\infty)$ is homeomorphic to S^{n-1}. For convenience, an n-dimensional, simply connected, complete Riemannian manifold M with sectional curvature $K \leqq 0$ will be called a Lobatchewsky manifold if any two distinct points in $M(\infty)$ can be joined by a unique geodesic in M. This condition is actually Axioms I and II used by Eberlein in [7] for geodesic flows. This condition holds for the following cases: (1) The sectional curvature $K \leqq c < 0$; (2) The geodesic flow of the compact quotient manifold M/Γ by a properly discontinuous group Γ of isometries of M is Anosov.

Section 2 states preliminaries of Lobatchewsky manifolds. Proofs and details can be found in [7], [9] and [10]. Basic facts of limit sets including a classification theorem of limit sets are given in Section 3. The following main lemma of this paper is also given there. If $I(M)$ acts effectively on a simply connected, complete Riemannian manifold with sectional curvature $K \leqq c < 0$ and if the totally geodesic hull $\langle L(G) \rangle$ of the limit set $L(G)$ of a subgroup G is M, then either G has a common fixed point in $M(\infty)$ or G is semisimple. Section 4 gives proofs of our results.

Our next project is to investigate the general case of nonpositive sectional curvature. This work will appear elsewhere.

2. Lobatchewsky Manifolds

For any Riemannian manifold M we denote the Riemannian structure by $\langle\ ,\ \rangle$, the Riemannian metric by d. We denote by $I(M)$ and $I_0(M)$ respectively the full isometry group of M and its identity component. If v and w are two unit tangent vectors at a point $p \in M$, the angle $\theta = \sphericalangle(v, w)$ is the unique number $0 \leqq \theta \leqq \pi$ such that $\langle v, w \rangle = \cos\theta$. If M is complete and v is a unit tangent vector, let $\gamma_v : \mathbf{R} \to M$ be the geodesic such that $\gamma_v'(0) = v$. All geodesics are assumed to have unit speed and to be defined on the entire real line.

From now on, let M denote a complete, simply connected Riemannian manifold with sectional curvature $K \leqq 0$. If $p \neq q$ in M, let γ_{pq} be the unique geodesic such that $\gamma_{pq}(0) = p$ and $\gamma_{pq}(t) = q$, where $t = d(p, q)$. The angle $\sphericalangle_p(m, n)$ subtended at p by points m, n of M distinct from p is $\sphericalangle(\gamma_{pm}'(0), \gamma_{pn}'(0))$. Any three noncollinear points of M determine a geodesic triangle and the law of cosines says that $c^2 \geqq a^2 + b^2 - 2ab \cos\theta$ where a, b, c are the sides of the triangle and θ is the angle opposite c.

Geodesics α and β in M are asymptotic if there exists a number $c > 0$ such that $d(\alpha(t), \beta(t)) \leqq c$ for all $t \geqq 0$. The asymptote relation is an equivalence relation on the set of all geodesics in M; the equivalence classes are called asymptote classes. If $\alpha : (-\infty, \infty) \to M$ is a geodesic, let $\alpha(\infty)$ be the asymptote class of α and let $\alpha(-\infty)$ be the asymptote class of the reverse curve $t \to \alpha(-t)$. A point at infinity for M is an asymptote class of geodesics of M. Let $M(\infty)$ be (the boundary of M) the set of points at infinity for M and let $\bar{M} = M \cup M(\infty)$. If $p \in M$ and $x \in M(\infty)$ are given there is a unique geodesic γ_{px} such that $\gamma_{px}(0) = p$ and $\gamma_{px}(\infty) = x$. If $p \in M$ is distinct from points a, b in $\bar{M}$ then the angle subtended at p by a and b is $\sphericalangle_p(a, b) = \sphericalangle(\gamma_{pa}'(0), \gamma_{pb}'(0))$. Let v be a unit tangent vector in M_p, the tangent space at p to M. Let $0 < \varepsilon < \pi$ and $r > 0$ be given. The cone $C(v, \varepsilon)$ of vertex p, axis v and angle ε is the set of $\{b \in \bar{M} \mid \sphericalangle_p(\gamma_v(\infty), b) < \varepsilon\}$ and the truncated cone of vertex p, axis v, angle ε and radius r is the set $T(v, \varepsilon, r) = C(v, \varepsilon) - \{q \in M \mid d(p, q) \leqq r\}$. The cone topology k of

Eberlein and O'Neill [7], [10] for $\bar{M}$ is defined by one of the following equivalent conditions: (1) The topology k of $\bar{M}$ extends the topology of M and M is a dense open subset of $\bar{M}$; (2) For each $x \in M(\infty)$ the set of cones containing x is a local basis for k at x; (3) For each $x \in M(\infty)$ and each $p \in M$ the set of truncated cones with the vertex p that contain x is a local basis for k at x.

The cone topology makes $\bar{M}$ homeomorphic to the closed unit ball in $\mathbf{R}^n$ and $M(\infty)$ homeomorphic to the sphere S^{n-1}. If $\alpha:(-\infty, \infty) \rightarrow M$ is a geodesic, then the natural asymptotic extension $\alpha:[-\infty, \infty] \rightarrow \bar{M}$ is continuous. If ϕ is an isometry of M and $x \in M(\infty)$, define $\phi(x) = (\phi \circ \alpha)(\infty)$ where α is any geodesic representing x. Since isometries preserve asymptotes, ϕ is well defined and becomes a homeomorphism of $\bar{M}$. Thus the isometry group $I(M)$ of M extends to a group of homeomorphisms of $\bar{M}$. If x and y are distinct points in $M(\infty)$, a geodesic α is said to join x to y if $\alpha(-\infty) = x$ and $\alpha(\infty) = y$.

DEFINITION 2.1. A Lobatchewsky manifold M is a complete, simply connected Riemannian manifold with sectional curvature $K \leqq 0$ satisfying the condition that any two distinct points in $M(\infty)$ can be joined by a unique geodesic.

The condition in the above definition is originally given in [7] as Axioms I and II.

Isometries of M can be classified by their fixed points. Explicit forms of them can be determined up to conjugacy for rank one symmetric spaces ([5], [6] and [11]). Every isometry ϕ has at least one fixed point in $\bar{M}$, since $\bar{M}$ is a cell. ϕ is elliptic if it has a fixed point in M. If ϕ is not elliptic, then ϕ has at most two fixed points in $M(\infty)$ If ϕ has fixed points $x \neq y$ in $M(\infty)$, then ϕ translates the geodesic joining x to y and ϕ is called axial. If ϕ has one fixed point in $M(\infty)$, then ϕ is called parabolic.

3. Limit Sets and the Main Lemma

DEFINITION 3.1. Let G be a subgroup of the isometry group $I(M)$ of a Lobatchewsky manifold M. The limit set $L(G)$ of G is defined to be the intersection with $M(\infty)$ of the closure of any orbit of G in M. That is $L(G) = M(\infty) \cap \overline{\{g(p) \mid g \in G\}}$ where p is an arbitrary point in M.

The limit set is independent of p. In fact, if $p \neq p'$ in M and if $\{g_n\}$ is a sequence in G, then $\lim g_n(p) = \lim g_n(p')$, because g_n are isometries. $L(G)$ is a closed subset of $M(\infty)$ which is invariant under G. If $\bar{G}$ is the closure of G, then $L(\bar{G}) = L(G)$. The subgroups of $I(M)$ are divided into four classes, according as $L(G)$ is empty, contains exactly one point, contains exactly two points or contains more than two points.

THEOREM 3.1. *If A is a closed subset of $M(\infty)$ which contains more than one point and A is invariant under G, then $A \supset L(G)$.*

Proof. Let $z \in L(G)$ and $z_1, z_2 \in A$. There is a sequence $\{g_n\} \subset G$ so that $\lim g_n(p)$

$=z$ for any $p \in M$. At least one of the sequences $\{g_n(z_1)\}$, $\{g_n(z_2)\}$ has z as a limit point. If this is false, then there is a subsequence $\{g_{n_k}\}$ so that $\lim g_{n_k}(z_1)=x_1$ and $\lim g_{n_k}(z_2)=x_2$, where $x_1 \neq z$ and $x_2 \neq z$. Let λ be the geodesic with end points z_1, z_2 and let p be a point on λ. There are neighborhoods U_1 of x_1 and U_2 of x_2 so that $z \notin \bar{U}_1$ and $z \notin \bar{U}_2$. For large k, $g_{n_k}(z_1) \in U_1$ and $g_{n_k}(z_2) \in U_2$, while $g_{n_k}(p)$ is a point on the geodesic $g_{n_k}(\lambda)$. If the sequence $\{g_{n_k}(p)\}$ converges to a point of $M(\infty)$, this point must be in $\bar{U}_1$ or $\bar{U}_2$. This contradiction implies that the closed invariant set $A \supset L(G)$.

The following lemma ([7]) will be used to prove the following two theorems which have already been given in [7] and [10] for discrete subgroups. We refer to [7] and [10] for the similar proofs.

LEMMA 3.1. (Eberlein [7]) *Let $\{p_n\}$ be a sequence in a Lobatchewsky manifold M convergent to a point $x \in M(\infty)$. If W is any neighborhood of x in $\bar{M}$ then $\sphericalangle_{p_n}(\bar{M}-W) \to 0$ as $n \to \infty$.*

THEOREM 3.2. *Let G be a subgroup of the isometry group $I(M)$ of a Lobatchewsky manifold M. Then one of the following holds:*

(1) *$L(G)$ is empty,*
(2) *$L(G)$ contains one point,*
(3) *$L(G)$ contains two points,*
(4) *$L(G)$ is an infinite, perfect and nowhere dense subset of $M(\infty)$,*
(5) *$L(G)=M(\infty)$.*

THEOREM 3.3. *Let $z \in L(G)$ and let Gz contain more than one point. Then $\overline{Gz}= =L(G)$.*

We now investigate various cases of limit sets of subgroups.

THEOREM 3.4. *If $L(G)=\phi$, then G has a common fixed point in M.*

Proof. $\bar{G}$ is a closed Lie subgroup of $I(M)$ and $L(\bar{G})=L(G)=\phi$. For any $p \in M$, the orbit $\bar{G}p$ is closed bounded set so that $\bar{G}p$ is compact. Let $K=\{g \in \bar{G} \mid g(p)=p\}$. K is a closed subgroup of the isotropy subgroup $I(M)_p$ at p. So K is compact. The factor space $\bar{G}/K$ is homeomorphic to $\bar{G}p$, so $\bar{G}/K$ is compact. $\bar{G}$ is a fibre space with compact base space $\bar{G}/K$ and compact fibre K. Therefore $\bar{G}$ is compact. By the well known theorem of Cartan, $\bar{G}$ has a common fixed point in M.

The following theorem is obvious.

THEOREM 3.5. *If $L(G)$ contains one point, then G has a common fixed point in $M(\infty)$ and G consists of parabolic elements and axial elements leaving that point fixed.*

Remark. G is a subgroup of the homeomorphism group of $\mathbf{R}^n$ leaving the infinity ∞ fixed.

THEOREM 3.6. *If $L(G)$ contains two points then G modulo a normal subgroup (isomorphic to a subgroup of $O(n-1)$) is a subgroup of the 1-parameter group of axial elements with these two given points as fixed points.*[1])

Proof. G leaves the geodesic λ joining the two points in $L(G)$ invariant. The kernel of the restriction map from G into $I(\lambda)$ is isomorphic to a subgroup of $0(n-1)$.

DEFINITION 3.2. The totally geodesic hull $\langle A\rangle$ of a cubset A in $M(\infty)$ is the intersection of all totally geodesic submanifolds in M whose boundaries contain A.

DEFINITION 3.3. A set N in M is called totally convex ([3]) if N contains every geodesic segment of M whose end points are in N. The totally convex hull $\{A\}$ of a subset A in $M(\infty)$ is the smallest totally convex set in M whose closure contains A.

From now on, we shall assume that M is a simply connected complete Riemannian manifold with $K \leqq c < 0$.

PROPOSITION 3.1. *If A contains more than two points, then any isometry ϕ of $I(M)$ which leaves A pointwise fixed, also leaves $\langle A\rangle$ pointwise fixed.*

Proof. We consider the totally convex hull $\{A\}$ of A. The set of tangents to $\{A\}$ at point p of $\{A\}$ is a convex cone in M_p (see [3], p. 7). If we complete all geodesic segments in $\{A\}$ and denote by N the totally convex hull of the union of those complete geodesics, then the set of tangents to N at p is a subspace N_p of M_p. Then $N = \operatorname{Exp} N_p$ and N is a totally geodesic submanifold of M. By definition, $N = \langle A\rangle$. If A contains more than two points, then any isometry ϕ of $I(M)$ which leaves A pointwise fixed, also leaves every geodesic joining any two points in A pointwise fixed (by the law of cosines). ϕ leaves $\langle A\rangle$ pointwise fixed.

Since $L(G)$ is invariant under G, $\langle L(G)\rangle$ is also invariant under G. In general, $\langle L(I(M))\rangle$ may not be the whole M. However if M is homogeneous, $L(I(M)) = M(\infty)$ and $\langle L(I(M))\rangle = M$. The following lemma is the main idea of this paper.

DEFINITION 3.4. The centralizer $Z(G, H)$ is the set $\{h \mid h \in H, gh = hg \text{ for all } g \in G\}$.

LEMMA 3.2. *Let M be a simply connected complete Riemannian manifold with $K \leqq c < 0$ such that $I(M)$ acts effectively on M. Suppose that G is a subgroup of $I(M)$ and $\langle L(G)\rangle = M$. If $L(G)$ contains more than two points, then the centralizer $Z(G, I(M))$ is trivial. If, in addition, G does not have a common fixed point in $M(\infty)$, then G is semisimple.*

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1) The factored out normal subgroup of G contains elliptic elements which leave the geodesic λ pointwise fixed but may rotate other in M.

Proof. Suppose that c is an element of the centralizer and $z_0 \in L(G)$. There is a sequence $\{g_n\} \subset G$, so that $\lim g_n(p) = z_0$ for any $p \in M$. Then $z_0 = \lim g_n(p) = \lim cg_nc^{-1}(p) = \lim cg_n(p) = c(z_0)$. Thus every point in $L(G)$ is a fixed point of c. Proposition 3.1 implies that every point in $M = \langle L(G) \rangle$ is fixed by c. Since $I(M)$ acts on M effectively, $c = 1$.

Now suppose that H is a commutative, normal subgroup of G. If $L(H) = \phi$, then H has a common fixed point $p \in M$. Suppose that $z_0 \in L(G)$, $\{g_k\}$ is a sequence in G, so that $\lim g_k(p) = z_0$ and $h \in H$. The point $g_k(p)$ is fixed under $g_k h g_k^{-1}$. Since the mapping $h \to g_k h g_k^{-1}$ maps H onto itself, $g_k(p)$ is a common fixed point of H. It follows that z_0 is a common fixed point of H. Thus the elements of H leave every point of $L(G)$ and $\langle L(G) \rangle$ fixed. It follows that $H = \{1\}$.

If $L(H) \neq \phi$, we shall show that $L(H) = L(G)$. Suppose that $z_0 \in L(H)$, $\{h_k\} \subset H$, $\lim h_k(p) = z_0$ and $g \in G$. Then $\lim g h_k g^{-1}(p) = \lim g h_k(p) = g(z_0)$. Thus $L(H)$ is invariant under G. If $L(H)$ consists of exactly one point, then this point is invariant under G, contradicting the hypothesis. Therefore $L(H)$ is a closed subset of $M(\infty)$, which contains more than one point and $L(H)$ is invariant under G. Theorem 3.1 implies that $L(H) \supset L(G)$. Since $H \subset G$, $L(H) = L(G)$. Thus $\langle L(H) \rangle = \langle L(G) \rangle = M$. The centralizer $Z(H, I(M))$ is trivial. Since H is commutative it is a subgroup of this centralizer, and $H = \{1\}$.

4. Complete Homogeneous Riemannian Manifolds With $K \leqq c < 0$

In the sequel, we shall always assume that the isometry group $I(M)$ acts effectively on the homogeneous manifold M.

LEMMA 4.1. (Mostow [17]) *Let $\tilde{\mathscr{G}}$ be a real noncompact semisimple Lie algebra and $\mathscr{G}$ be a noncompact semisimple subalgebra. Let $\mathscr{G} = \mathscr{K} \oplus \mathscr{P}$ be a Cartan decomposition of $\mathscr{G}$. Then there is a Cartan decomposition $\tilde{\mathscr{K}} \oplus \tilde{\mathscr{P}}$ for $\tilde{\mathscr{G}}$ with $\mathscr{K} \subseteq \tilde{\mathscr{K}}$ and $\mathscr{P} \subseteq \tilde{\mathscr{P}}$.*

THEOREM 4.1. *Let M be a complete homogeneous Riemannian manifold with $K \leqq c < 0$. Then*

(1) *Either $I(M)$ has a common fixed point in $M(\infty)$ or $I(M)$ is a noncompact semisimple Lie group of rank one and M is a noncompact symmetric space of rank one.*

(2) *Let G be a Lie subgroup of $I_0(M)$ such that $M = \langle L(G) \rangle$. Suppose that $L(G)$ contains more than two points and G does not have a common fixed point in $M(\infty)$. Then either G is discrete or $G = I_0(M)$.*

Proof. Let G be a Lie subgroup of $I_0(M)$ satisfying the assumptions in (2). Suppose that G is not discrete, and let G_0 be the identity component of G. G_0 is a nontrivial normal subgroup of G. As in the proof of Lemma 3.2, $L(G_0) = L(G)$. Furthermore, G_0 has no common fixed point in the closure $\overline{\langle L(G) \rangle}$. For if z is such a point and $g \in G$, then $g(z)$ is also a common fixed point of G_0. But the orbit Gz is dense in $L(G)$ so every point of $L(G)$ and $\langle L(G)\rangle$ is fixed under G_0. Thus $G_0=\{1\}$. Lemma 3.2 implies that G_0 is semisimple with trivial centralizer. We can apply the above fact to $I_0(M)$, since $M=\langle L(I_0(M))\rangle$. If $I_0(M)$ does not have a common fixed point, then $I_0(M)$ is semisimple with trivial centralizer. Hence M is symmetric. Since noncompact symmetric spaces with rank greater than one do not satisfy the condition that any two boundary points can be joined by a unique geodesic (see [9]). M must be a noncompact symmetric space of rank one. Let $\tilde{\mathscr{G}}$ be the Lie algebra of $I_0(M)$ and let $\tilde{\mathscr{G}}=\tilde{\mathscr{K}}\oplus\tilde{\mathscr{P}}$ be the corresponding Cartan decomposition. Then the axial elements of $I_0(M)$ are given by $\exp X$, where $X\in\tilde{\mathscr{P}}$. We now go back to the given group G. The existence of such a subgroup G implies that $I(M)$ does not have a common fixed point in $M(\infty)$ and M is a symmetric space. Since G_0 is semisimple, $G_0=I_0(M)$ otherwise $\langle L(G)\rangle=\langle L(G_0)\rangle$ would be a proper totally geodesic submanifold of M by Lemma 4.1. Consequently either G is discrete or $G=I_0(M)$.

THEOREM 4.2. ([5], [6]) *Let M be a noncompact symmetric space of rank one. Let G be a connected Lie subgroup of $I_0(M)$. Then one of the following holds:*

(1) *G has a common fixed point in M,*

(2) *G has a common fixed point in $M(\infty)$,*

(3) *G modulo a normal subgroup (isomorphic to a subgroup of $O(n-1)$, $n=\dim M$) is the 1-parameter group of axial elements,*

(4) *G modulo a normal subgroup (isomorphic to a subgroup of $O(n-m)$, $n=\dim M$, $m=\dim\langle L(G)\rangle$) is the connected isometry group $I_0(\langle L(G)\rangle)$ of the totally geodesic submanifold $\langle L(G)\rangle$ which is a noncompact symmetric space of rank one,*

(5) *G is $I_0(M)$.*

Proof. If G has a common fixed point in $\bar{M}$, then we have (1) or (2). If $L(G)$ contains two points, then the geodesic $\langle L(G)\rangle$ is invariant under G. The restriction homomorphism from G into $I_0(\langle L(G)\rangle)$ has kernel K which is a subgroup leaving $\langle L(G)\rangle$ pointwise fixed and is isomorphic to a subgroup of $O(n-1)$. Since the image of G in $I_0(\langle L(G)\rangle)$ is a connected Lie subgroup, it must be the 1-parameter group of axial elements. If $L(G)$ contains more than two points but $\langle L(G)\rangle\subset M$, then $\langle L(G)\rangle$ is invariant under G. The restriction homomorphism from G into $I_0(\langle L(G)\rangle)$ has kernel K which is a subgroup leaving $\langle L(G)\rangle$ pointwise fixed and is isomorphic to a subgroup of $O(n-m)$ $(m=\dim\langle L(G)\rangle)$. We can apply Theorem 4.1. to get (4). If $\langle L(G)\rangle=M$, then Theorem 4.1. again gives (5).

THEOREM 4.3. *Let M be a noncompact symmetric space of rank one and G be a subgroup of $I_0(M)$. If there is no point in $\bar{M}$ and no proper totally geodesic submanifold in M invariant under G, then G is discrete or dense in $I_0(M)$.*

Proof. $\bar{G}$ is a closed subgroup and $L(\bar{G})=L(G)$. If $(\bar{G})_0$ is trivial, then G is discrete.

If $(\bar{G})_0$ is not trivial, then it is a connected Lie subgroup. Thus this theorem follows from Theorem 4.2.

COROLLARY. *Let M be a noncompact symmetric space of rank one. Let there be no point in $\bar{M}$ and no proper totally geodesic submanifold in M invariant under G. If there is an open subset U of $I_0(M)$ such that $U \cap G = \phi$, then G is discrete.*

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