Comment. Math. Helvetici **50** (1975) 145–154 Birkhäuser Verlag, Basel

Unstable K-Theories of the Algebraic Closure of a Finite Field

ERIC M. FRIEDLANDER[1])

We show that the Brauer lifting of the standard modular representation of $GL_n(F_q)$ on $F^{\oplus n}$ to a virtual complex representation has a curious non-stable version (where F_q is a finite field of characteristic $p>0$ with algebraic closure F). Namely, the map $BGL_n(F_q) \to BGL_\infty^{top}(C)$ associated to this Brauer lifting factors through a map $BGL_n(F_q) \to BGL_n^{top}(C)$. This unstable lifting is also achieved for $BSO_n(F_q)$ and $BSp_{2k}(F_q)$. Using the induced maps $BGL_n(F)^+ \to BGL_n^{top}(C)$, $BSO_n(F)^+ \to BSO_n^{top}(C)$, $BSP_{2k}(F)^+ \to BSp_{2k}^{top}(C)$ we determine the unstable algebraic K-groups $\pi_i(BGL_n(F)^+)$, $\pi_i(BSO_n(F)^+)$, and $\pi_i(BSp_{2k}(F)^+)$ explicitly in terms of the homotopy groups of the corresponding classical groups.

In section 1, we exhibit natural map from $BG_n(F)$ to the prime-to-p pro-finite completion of $BG_n^{top}(C)$, where G_n denotes either GL_n, SL_n, O_n, SO_n or Sp_n (where $p = \mathrm{char}(F)$ is odd for O_n and $n=2k$ for Sp_n). This map induces isomorphisms in Z/lZ cohomology for l prime to p. Because $BG_n(F)$ is Z/pZ-acyclic, this map determines a map $\eta_n: BG_n(F) \to BG_n^{top}(C)$. We verify that the composition $BG_n(F_q) \to \to BG_n(F) \to BG_n^{top}(C) \to BG_\infty^{top}(C)$ corresponds to Brauer lifting.

In section 2, we show that $\pi_i(BG(F)^+)$ is directly computable from $\pi_i(BG_n^{top}(C))$ and $\pi_{i+1}(BG_n^{top}(C))$. More precisely, $\eta_n^+: BG_n(F)^+ \to BG_n^{top}(C)$ is shown to be the fibre of localization at $p = \mathrm{char}(F)$, $BG_n^{top}(C) \to BG_n^{top}(C)_{(p)}$. As a consequence, the sequences

$$\cdots \to BG_n(F)^+ \to BG_{n+1}(F)^+ \to BG_{n+2}(F)^+ \to \cdots$$

are seen to be "intrinsic spherical fibrations" with fibres prime-to-p torsion spheres.

We gratefully acknowledge several valuable conversations with G. Lusztig.

1. Unstable Brauer Lifting

We let G_n denote either GL_n, SL_n, O_n, SO_n, or Sp_n (with $n=2k$ for Sp_n), so that $G_{n,R}$ is the corresponding group scheme defined over $\mathrm{Spec}\, R$ for any (commutative with identity) ring R. We denote by $G_n^{top}(C)$ the corresponding classical topological

1) Partially supported by the N.S.F., by the U.S.-France Exchange of Scientists Program, and by I.H.E.S.

group. In the orthogonal case, we recall that $O_{n,R}$ is the algebraic sub-group scheme of $GL_{n,R}$ preserving the quadratic form $X_1X_{k+1}+\cdots+X_kX_{2k}$ if $n=2k$ and $X_1X_{k+1}+\cdots+X_kX_{2k}+X_{2k+1}^2$ if $n=2k+1$; $SO_{n,R}$ is the sub-group scheme of $GL_{n,R}$ defined over Spec $\mathbf{Z}$ as the reduction of the connected component of the identity of $O_{n,R}$.

We fix a prime p and let $\mathbf{F}$ denote the algebraic closure of the finite field $\mathbf{F}_p$. For any power of p, $q=p^d$, we let $\mathbf{F}_q$ denote the subfield of $\mathbf{F}$ with q elements. If $p=2$, we exclude the case $G_n=O_n$ so that $G_{n,\mathbf{F}}$ is an algebraic group. We denote by $W\{G_{n,\mathbf{F}}\}$ the simplicial algebraic variety with $(W\{G_{n,\mathbf{F}}\})_k=(G_{n,\mathbf{F}})^{\times k}$ and with face and degeneracy maps obtained by deleting and inserting a factor; we let $\bar{W}\{G_{n,\mathbf{F}}\}$ denote $W\{G_{n,\mathbf{F}}\}/G_{n,\mathbf{F}}$.

In [1], the rigid etale homotopy type of a noetherian scheme or noetherian simplicial scheme was introduced (denoted by $(\ )_{\mathrm{ret}}$), and the following was proved:

a) If $H\subset G_{n,\mathbf{F}}$ is a finite algebraic subgroup, then $(W\{G_{n,\mathbf{F}}\}/H)_{\mathrm{ret}}$ is naturally homotopy equivalent to BH, the classifying space of H viewed as a discrete group.

b) If $(\ )^{\Lambda}$ denotes pro-finite completion prime-to-char $(\mathbf{F})$ and if $G_{n,\mathbf{F}}$ is connected, then $(\bar{W}\{G_{n,\mathbf{F}}\}_{\mathrm{ret}})^{\Lambda}$ is weakly homotopy equivalent to $(BG_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda}$ via maps dependent only on a choice of embedding of the Witt vectors of $\mathbf{F}$ into $\mathbf{C}$.

We can actually verify that the maps determining the weak equivalence of b) depend only on a choice of embedding of $\mathbf{F}^*$ into $\mathbf{C}^*$ (using the fact that these maps are determined by their effect on cohomology and reducing to the case $n=1$).

In the case of $O_{n,\mathbf{F}}$ and char $(\mathbf{F})$ odd, property b) above remains valid: the natural maps relating $\bar{W}\{G_{n,\mathbf{F}}\}_{\mathrm{ret}}$ and $BG_n^{\mathrm{top}}(\mathbf{C})$ exist even if G_n is not connected; moreover, $W\{O_{n,\mathbf{F}}\}/SO_{n,\mathbf{F}}\to W\{O_{n,\mathbf{F}}\}/O_{n,\mathbf{F}}=\bar{W}\{O_{n,\mathbf{F}}\}$ is a double covering and $(W\{SO_{n,\mathbf{F}}\}/\times/SO_{n,\mathbf{F}})_{\mathrm{ret}}\to(W\{O_{n,\mathbf{F}}\}/SO_{n,\mathbf{F}})_{\mathrm{ret}}$ is a weak equivalence.

Properties a) and b) enable us to define

$$\chi_{n,q}: BG_n(\mathbf{F}_q)\to \varprojlim (BG_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda} \tag{1.1}$$

where $G_n(\mathbf{F}_q)$ is the discrete group of points of $G_{n,\mathbf{F}}$ rational over $\mathbf{F}_q$ and where $\varprojlim(\ )$ is the inverse limit of an inverse system in the homotopy category of spaces with finite homotopy groups as in [4]. Namely, to obtain $\chi_{n,q}$, compose the maps

$$(W\{G_{n,\mathbf{F}}\}/G_n(\mathbf{F}_q))_{\mathrm{ret}}\to(W\{G_{n,\mathbf{F}}\}/G_{n,\mathbf{F}})_{\mathrm{ret}}=\bar{W}\{G_{n,\mathbf{F}}\}_{\mathrm{ret}}$$
$$\bar{W}\{G_{n,\mathbf{F}}\}_{\mathrm{ret}}\to\varprojlim(\bar{W}\{G_{n,\mathbf{F}}\}_{\mathrm{ret}})^{\Lambda},$$

and the homotopy equivalence

$$\varprojlim(\bar{W}\{G_{n,\mathbf{F}}\}_{\mathrm{ret}})^{\Lambda}\to\varprojlim(BG_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda}$$

induced by the weak homotopy equivalence of b).

The following proposition will enable us to compute the Z/lZ cohomology of the discrete group $G_n(F)=\varinjlim G_n(F_q)$.

PROPOSITION 1.2. *Let* $G_{n,F}$ *denote one of the following connected algebraic groups:* $GL_{n,F}$, $SL_{n,F}$, $SO_{n,F}$, *or* $Sp_{n,F}$ *(with* $n=2k$ *for* Sp_n*). Let* $\phi^q: G_{n,F}\to G_{n,F}$ *denote the geometric frobenius defined over* F_q, *with* q *a power of* $p=\mathrm{char}(F)$. *Let* l *be an integer not divisible by* p. *There exists an integer* $d>1$ *such that the map*

$$1\cdot\phi^q\cdots\phi^{q^{d-1}}: G_{n,F}\to G_{n,F}$$

defined as $\mu\circ(1\times\phi^q\times\cdots\times\phi^{q^{d-1}}): G_{n,F}\to(G_{n,F})^{\times d}\to G_{n,F}$ *induces the zero map in* Z/lZ *cohomology.*

Proof. We first assume that $G_{n,F}=GL_{n,F}$. Recall that

$$(1+\Psi^q+\cdots+\Psi^{q^{d-1}})^*: H^{2k}(BGL_\infty^{top}(C), Z/lZ)\to H^{2k}(BGL_\infty^{top}(C), Z/lZ)$$

where $+: BGL_\infty^{top}(C)\times BGL_\infty^{top}(C)\to BGL_\infty^{top}(C)$ is induced by tensor product of line bundles, is multiplication by $(1+q+\cdots+q^{d-1})^k$. Let d be the order of q in $(Z/l^{e+1}Z)^*$, where e is the exponent of l in $q-1$. Then l divides $1+q+\cdots+q^{d-1}$ so that $(1+\Psi^q+\cdots+\Psi^{q^{d-1}})^*$ is the zero map. Since the generators of the exterior algebra $H^*(GL_\infty^{top}(C), Z/lZ)$ totally transgress to the generators of the polynomial algebra $H^*(BGL_\infty^{top}(C), Z/lZ)$, we conclude that

$$\Omega(1+\Psi^q+\cdots+\Psi^{q^{d-1}})^*: H^*(GL_\infty^{top}(C), Z/lZ)\to H^*(GL_\infty^{top}(C), Z/lZ)$$

is the zero map.

We claim that the following diagram determines a commutative square in Z/lZ cohomology:

$$\begin{array}{ccc} (GL_{n,F})_{ret} & \xrightarrow{1\cdot\phi^q\cdots\phi^{q^{d-1}}} & (GL_{n,F})_{ret} \\ \downarrow\theta & & \downarrow\theta \\ GL_\infty^{top}(C)^\Lambda & \xrightarrow{\Omega(1+\Psi^q+\cdots+\Psi^{q^{d-1}})} & GL_\infty^{top}(C)^\Lambda \end{array} \tag{1.2.1}$$

where θ is determined by maps $(GL_{n,F})_{ret}\to(GL_{n,\mathrm{Witt}(F)})_{ret}$, $(GL_{n,C})_{ret}\to\times(GL_{n,\mathrm{Witt}(F)})_{ret}$, $GL_n^{top}(C)\to(GL_{n,C})_{ret}$, and $GL_n^{top}(C)\to GL_\infty^{top}(C)$; and where $(\ )^\Lambda$ is pro-finite, prime-to-p completion. We verify that (1.2.1) induces a commutative square in Z/lZ cohomology by observing that each of the maps above is induced by a homomorphism of groups in an appropriate category so that θ^Λ likewise commutes with the group structures on $((GL_{n,F})_{ret})^\Lambda$ and $GL_\infty^{top}(C)^\Lambda$; and by observing that $(\theta\circ\phi^q)^*=(\Omega(\Psi^q)^\Lambda\circ\theta)^*$, as can be seen by reducing to the case $n=1$ since $(BGL_1^{top}(C))^{\times n}\to BGL_n^{top}(C)$ induces an injection in cohomology. Therefore, the proposition follows for $G_{n,F}=GL_{n,F}$, by observing that θ induces a surjection in Z/lZ cohomology, since $\varprojlim(GL_{n,F})^{\Lambda}_{ret}\to\varprojlim(GL_n^{top}(C))^{\Lambda}$ is a homotopy equivalence.

We immediately conclude the proposition for $G_{n,F}=SL_{n,F}$ or $Sp_{n,F}$ (with $n=2k$ for Sp), or for $G_{n,F}$ and l odd since $SL_{n,F}\to GL_{n,F}$, $Sp_{n,F}\to GL_{n,F}$, and $SO_{n,F}\to GL_{n,F}$ are group homomorphisms commuting with frobenius and inducing surjections in Z/lZ cohomology (with l odd for $SO_{n,F}$).

For p odd and l a power of 2, we observe that $(BO_1^{top}(C))^{\times n}\to BO_n^{top}(C)$ induces injections in Z/lZ cohomology and Ψ^q acts trivially on $BO_1^{top}(C)$. Since $H^*(BO_n^{top}(C), Z/lZ)\to H^*(BSO_n^{top}(C), Z/lZ)$ is surjective, we conclude that ϕ^q acts trivially on $H^*(SO_{n,F}, Z/lZ)$. Hence, if l divides d, $(1+\phi^q+\cdots+\phi^{q^{d-1}})^*$ is the zero map.

PROPOSITION 1.3. *Let* $G_{n,F}$ *denote one of the following algebraic groups:* $GL_{n,F}$, $SL_{n,F}$, $O_{n,F}(\mathrm{char}(F)$ *odd),* $SO_{n,F}$, *or* $Sp_{n,F}$ *(with* $n=2k$ *for* Sp_n*). Let* $G_n(k)$ *denote the discrete group of points of* $G_{n,F}$ *rational over* $k/\mathbf{F}_p$, *where* $p=\mathrm{char}(F)$. *Then the direct limit of the maps*

$$\chi_{n,q}: BG_n(F_q)\to\varprojlim BG_n^{top}(C))^{\Lambda}$$

of (1.1),

$$\chi_n: BG_n(F)\to\varprojlim(BG_n^{top}(C))^{\Lambda}$$

induce isomorphisms in Z/lZ *cohomology for all* l *relatively prime to* p, *where* $(\ )^{\Lambda}$ *denotes pro-finite, prime-to-p completion.*

Proof. Since $O_1(F)=O_1^{top}(C)=Z/2Z$ for p odd, the proposition for $G_{n,F}=O_{n,F}$ easily follows from the theorem for $G_{n,F}=SO_{n,F}$. Hence, we may assume $G_{n,F}$ to be connected.

Consider the following commutative diagram

$$\begin{array}{ccc} G_{n,F}/G_n(F_q) & \to & G_{n,F}/G_n(F_{q'}) \\ \downarrow & & \downarrow \\ W\{G_{n,F}\}/G_n(F_q) & \to & W\{G_{n,F}\}/G_n(F_{q'}) \\ \downarrow & & \downarrow \\ \bar{W}\{G_{n,F}\} & = & \bar{W}\{G_{n,F}\} \end{array} \tag{1.3.1}$$

where the top arrow fits in a commutative square

$$\begin{array}{ccc} G_{n,F}/G_n(F_q) & \longrightarrow & G_{n,F}/G_n(F_{q'}) \\ {\scriptstyle 1/\phi^q}\downarrow & & \downarrow{\scriptstyle 1/\phi^{q'}} \\ G_{n,F} & \xrightarrow{1\cdot\phi^q\ldots\phi^{q'/q}} & G_{n,F} \end{array}$$

where q' is a power of q. By a basic result of [1] (Corollary 2.6), (1.3.1) determines a map of "Serre spectral sequences"

$$E_2^{s,t}(\mathbf{F}_{q'}) = H^s(\bar{W}\{\mathrm{G}_{n,\mathbf{F}}\}_{\mathrm{ret}}, H^t((\mathrm{G}_{n,\mathbf{F}}/\mathrm{G}_n(\mathbf{F}_{q'}))_{\mathrm{ret}}, Z/lZ)) \Rightarrow H^{s+t}(\mathrm{BG}_n(\mathbf{F}_{q'}), Z/lZ)$$

$$E_2^{s,t}(\mathbf{F}_q) = H^s(\bar{W}\{\mathrm{G}_{n,\mathbf{F}}\}_{\mathrm{ret}}, H^t((\mathrm{G}_{n,\mathbf{F}}/\mathrm{G}_n(\mathbf{F}_q))_{\mathrm{ret}}, Z/lZ)) \Rightarrow H^{s+t}(\mathrm{BG}_n(\mathbf{F}_q), Z/lZ) \qquad (1.3.3)$$

where we have used the homotopy equivalence between $\mathrm{BG}_n(\mathbf{F}_q)$ and $(W\{\mathrm{G}_{n,\mathbf{F}}\}/\mathrm{G}_n(\mathbf{F}_q))_{\mathrm{ret}}$. Because $E_2^{s,t}(\mathbf{F}_q)$ is finite for all s, t and all powers q of p, (1.3.3) determines an inverse limit spectral sequence

$$E_2^{s,t} = \varprojlim H^s(\bar{W}\{\mathrm{G}_{n,\mathbf{F}}\}_{\mathrm{ret}}, H^t(\mathrm{G}_{n,\mathbf{F}}/\mathrm{G}_n(\mathbf{F}_q), Z/lZ)) \Rightarrow \varprojlim H^{s+t}(\mathrm{BG}_n(\mathbf{F}_q), Z/lZ).$$

By Proposition 1.2 and the fact that the vertical arrows of (1.3.2) are isomorphisms (the "Lang isomorphisms" of [1]), there exists $d>1$ such that $E_2^{s,t}(\mathbf{F}_{q'}) \to E_2^{s,t}(\mathbf{F}_q)$ has image 0 for $t>0$, any s, and $q'=q^d$. Moreover,

$$H^{s+t}(\mathrm{BG}_n(\mathbf{F}), Z/lZ) \simeq \varprojlim H^{s+t}(\mathrm{BG}_n(\mathbf{F}_q), Z/lZ).$$

The proposition now follows from the degeneration of the spectral sequence $E^{s,t}$ and the fact that $(\bar{W}\{\mathrm{G}_{n,\mathbf{F}}\})_{\mathrm{ret}} \to \varprojlim(\bar{W}\{\mathrm{G}_{n,\mathbf{F}}\}_{\mathrm{ret}})$ induces isomorphisms in Z/lZ cohomology (since

$$H^*(\varprojlim(\mathrm{BGL}_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda}, Z/lZ) \simeq H^*(\mathrm{BGL}_n^{\mathrm{top}}(\mathbf{C}), Z/lZ) \quad [4]).$$

We record for future use the following proposition.

PROPOSITION 1.4. *Let* $\mathrm{G}_n(\mathbf{F})$ *denote the discrete group of rational points of* $\mathrm{G}_{n,\mathbf{F}}$, *where* $\mathrm{G}_{n,\mathbf{F}}$ *denotes either* $\mathrm{GL}_{n,\mathbf{F}}$, $\mathrm{SL}_{n,\mathbf{F}}$, $\mathrm{O}_{n,\mathbf{F}}$, $\mathrm{SO}_{n,\mathbf{F}}$, *or* $\mathrm{Sp}_{n,\mathbf{F}}$ *(with* $\mathrm{char}(\mathbf{F})=p$ *odd for* O_n *and* $n=2k$ *for* Sp_n*). Then, for every* $i>0$,

$$H^i(\mathrm{BG}_n(\mathbf{F}), Z/pZ) = 0 = H^i(\mathrm{BG}_n(\mathbf{F}), \mathbf{Q}).$$

Proof. For $\mathbf{Q}$ coefficients, the proposition is an easy consequence of the fact that $\mathrm{BG}_n(\mathbf{F})$ is the direct limit of classifying spaces of finite groups. For Z/pZ coefficients, the proposition is the stable version of the vanishing theorem given in [3]; a more elementary proof is carried out in detail in [1] for all cases except $\mathrm{SO}_n(\mathbf{F})$ with n odd or $p=2$.

In the following theorem, we see that χ_n of Proposition 1.3 provides a sharper form of Brauer lifting as generalized by Quillen to orthogonal and symplectic representations [2].

THEOREM 1.5. *Let* G_n *denote either* GL_n, SL_n, O_n, SO_n, *or* Sp_n *(with* $p = \mathrm{char}(F)$ *odd for* O_n *and* $n=2k$ *for* Sp_n*). The maps* χ_n *of Proposition* 1.3 *determine maps (uniquely up to homotopy)*

$$\eta_n : BG_n(\mathbf{F}) \to BG_n^{\mathrm{top}}(\mathbf{C})$$

which induce isomorphisms in Z/lZ *cohomology for* $(l, p) = 1$. *Furthermore, the composition of* η_n *with the natural inclusions*

$$BG_n(\mathbf{F}_q) \to BG_n(\mathbf{F}) \to BG_n^{\mathrm{top}}(\mathbf{C}) \to BG_\infty^{\mathrm{top}}(\mathbf{C})$$

corresponds to the virtual complex bundle on $BG_n(\mathbf{F}_q)$ *obtained as the Brauer lifting of the standard modular representation of* $G_n(\mathbf{F}_q)$ *on* $F^{\oplus n}$ *for the embedding* $\mathbf{F}^* \to \mathbf{C}^*$ *chosen for* χ_n.

Proof. By Proposition 1.4, $H_i(BG_n(\mathbf{F}), Z)$ is a prime-to-p torsion abelian group for $i > 0$. If H is such a prime-to-p torsion abelian group and A is a finitely generated abelian group, then

$$\mathrm{Hom}(H, A) = \mathrm{Hom}(H, {}_{\mathbf{p}-p}(A)) = \mathrm{Hom}(H, \varprojlim (A)^{\Lambda})$$

where ${}_{\mathbf{p}-p}(A)$ is the prime-to-p torsion subgroup of A and $\varprojlim (A)^{\Lambda}$ is the discrete group given as the inverse limit of the prime-to-p completion of A. Moreover,

$$\mathrm{Ext}^1_Z(H, A) \simeq \mathrm{Ext}^1_Z(H, \varprojlim (A)^{\Lambda})$$

since $\mathrm{Ext}^1_Z(H, Z) = \mathrm{Hom}(H, \mathbf{Q}/Z) = \mathrm{Hom}(H, \oplus_{l \neq p}(\mathbf{Q}_l/Z_l)) = \mathrm{Ext}^1_Z(H, \varprojlim (Z^{\Lambda}))$.

Since

$$\varprojlim (\pi_i(BG_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda}) \simeq \pi_i(\varprojlim (BG_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda})$$

we conclude that for all $i, j > 0$

$$H^i(BG_n(\mathbf{F}), \pi_j(BG_n^{\mathrm{top}}(\mathbf{C}))) \simeq H^i(BG_n(\mathbf{F}), \pi_j(\varprojlim (BG_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda})). \tag{1.5.1}$$

The existence and uniqueness of η_n is then obtained by inductively working up the Postnikov towers of $BG_n^{\mathrm{top}}(\mathbf{C})$ and $\varprojlim (BG_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda}$, since (1.5.1) implies that χ_n determines maps into the Postnikov truncations of $BG_n^{\mathrm{top}}(\mathbf{C})$ which are homotopy compatible via homotopy compatible homotopies. Since $BG_n^{\mathrm{top}}(\mathbf{C}) \to \varprojlim (BG_n^{\mathrm{top}}(\mathbf{C}))^{\Lambda}$ induces isomorphisms in Z/lZ cohomology for $(l, p) = 1$, so does η_n by Proposition 1.3.

To prove the second assertion comparing η_n with Brauer lifting, it suffices to consider $G_n = GL_n$, since $G_n \to GL_n$ induces homotopy commutative squares

$$\begin{array}{ccc} BG_n(F) & \xrightarrow{\eta_n} & BG_n^{top}(C) \\ \downarrow & & \downarrow \\ BGL_n(F) & \xrightarrow{\eta_n} & BGL_n^{top}(C) \end{array}$$

and since $[BG_n(F_q), BG_\infty^{top}(C)] \to [BG_n(F_q), BGL_\infty^{top}(C)]$ is injective ([2], 5.1.6). Let η'_n denote the composition $BGL_n(F) \to BGL_n^{top}(C) \to BGL_\infty^{top}(C)$ and let β_n denote the map $BGL_n(F) \to BGL_\infty^{top}(C)$ determined by Brauer lifting. One readily verifies that η'_1 and β_1 are induced by the chosen embedding

$$F^* = GL_1(F) \to GL_1^{top}(C) = C^*.$$

Since $B(GL_1(F)^{\times n}) \to BGL_n(F)$ induces an injection in Z/lZ cohomology by Proposition 1.3 for $(l, p) = 1$, we conclude

$$(\eta'_n)^* = \beta_n^* : H^*(BGL_\infty^{top}(C), Z/lZ) \to H^*(BGL_n(F)\, Z/lZ).$$

Since $H^*(BGL_n(F), Z_l)$ has no torsion for any prime $l \neq p$ by Proposition 1.3, the compositions

$$BGL_n(F) \rightrightarrows BGL_\infty^{top}(C) \to (\varprojlim (BGL_\infty^{top}(C))^{\hat{l}})^{(i)}$$

determined by η'_n and β_n are therefore homotopic for all i, where $(\ )^{\hat{l}}$ denotes pro-l completion and $(\ )^{(i)}$ denotes the i-th Postnikov truncation. As argued above for the existence and uniqueness of η_n, η'_n and β_n thus determine homotopic maps from $BGL_n(F)$ to $(BGL_\infty^{top}(C))^{(i)}$ for all i. This implies that the compositions

$$BGL_n(F_q) \to BGL_n(F) \rightrightarrows BGL_\infty^{top}(C)$$

determined by η'_n and B_n are homotopic as asserted, since the groups $H^i(BGL_n(F_q)$, $\pi_j(BGL_\infty^{top}(C))$ are finite for all $i, j > 0$ so that maps from $BGL_n(F_q)$ into $BGL_\infty^{top}(C)$ are determined by maps into the Postnikov tower of $BGL_\infty^{top}(C)$.

2. $P-p$ Torsion Completion and Unstable K-theories

As in section 1, we fix a prime p and let F denote the algebraic closure of F_p. We let $P-p$ denote the set of all primes except p.

DEFINITION 2.1. Let X be a connected nilpotent space with $\pi_1(X)$ a $P-p$

torsion abelian group. Then the $\mathbf{P}-p$ torsion completion of X,

$$t:\tau_{\mathbf{P}-p}(X)\to X$$

is the homotopy fibre of localization at p, $X\to X_{(p)}$.

Using obstruction theory, one can readily verify that any map from a CW complex Z with $\mathbf{P}-p$ torsion homology groups to X factors uniquely (up to homotopy) through $t:\tau_{\mathbf{P}-p}(X)\to X$ (the obstruction to lifting a map "up along the Postnikov tower", the obstruction to lifting a homotopy between two such liftings, and the obstruction to lifting a homotopy between two homotopies all vanish), if the torsion subgroup ${}_{\mathbf{P}-p}(\pi_i(X))$ is a direct summand of $\pi_i(X)$ for all $i>0$.

Our principal result is the following theorem.

THEOREM 2.2. *Let* G_n *denote either* GL_n, SL_n, O_n, SO_n, *or* Sp_n *(with* $p=\mathrm{char}(\mathbf{F})$ *odd for* O_n *and* $n=2k$ *for* Sp_n*). Let* $\mathrm{BG}_n(\mathbf{F})\to\mathrm{BG}_n(\mathbf{F})^+$ *denote Quillen's plus construction with respect to the commutator subgroup* $[\mathrm{G}_n(\mathbf{F}),\mathrm{G}_n(\mathbf{F})]$ *of* $\pi_1(\mathrm{BG}_n(\mathbf{F}))=\mathrm{G}_n(\mathbf{F})$. *Then the map*

$$\eta_n^+:\mathrm{BG}_n(\mathbf{F})^+\to\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})$$

induced by η_n *of Theorem* 1.5 *is the* $\mathbf{P}-p$ *torsion completion of* $\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})$.

Proof. Since $H_i(\mathrm{BG}_n(\mathbf{F}))\simeq H_i(\mathrm{BG}_n(\mathbf{F})^+)$ is $\mathbf{P}-p$ torsion for $i>0$ by Proposition 1.4, η_n^+ factors uniquely as $\eta_n^+=t\circ\varrho_n^+$, where t is $\mathbf{P}-p$ torsion completion and

$$\varrho_n^+:\mathrm{BG}_n(\mathbf{F})^+\to\tau_{\mathbf{P}-p}(\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})).$$

By Theorem 1.5, η_n^+ induces isomorphisms in Z/lZ cohomology for $(l,p)=1$. Moreover, t induces isomorphisms in Z/lZ cohomology because the Serre spectral sequence for the fibration

$$\tau_{\mathbf{P}-p}(\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C}))\to\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})\to\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})_{(p)}\tag{2.2.1}$$

and Z/lZ coefficients degenerates and $\pi_1(\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})_{(p)})=0$. Therefore ϱ_n^+ induces isomorphisms in Z/lZ cohomology so that

$$(\varrho_n^+)_*:H_*(\mathrm{BG}_n(\mathbf{F})^+)\simeq H_*(\tau_{\mathbf{P}-p}(\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C}))).\tag{2.2.2}$$

(for $i>0$, $H_i(\tau_{\mathbf{P}-p}(\mathrm{BG}_n^{\mathrm{top}}(\mathbf{C})))$ is $\mathbf{P}-p$ torsion by the degeneration of the Serre spectral sequence for (2.2.1) with Z/pZ and $\mathbf{Q}$ coefficients).

For G_n equal to SL_n, SO_n, or Sp_n, (2.2.2) plus the Whitehead theorem imply that ϱ_n^+ is a homotopy equivalence. Moreover, (2.2.2) implies that ϱ_n^+ induces isomorphisms of fundamental groups for G_n equal to GL_n or O_n. Since the map on universal coverings induced by ϱ_n^+ for $G_n = GL_n$ (respectively, $G_n = O_n$) is ϱ_n^+ for SL_n (resp., SO_n), ϱ_n^+ is also a homotopy equivalence for G_n equal to GL_n or O_n.

The fact that localization of spaces localizes homotopy groups enables us to immediately derive the following evaluation of unstable algebraic K-groups.

COROLLARY 2.3. *Let* G_n *be as in Theorem* 2.2. *Then*

$$\pi_i(BG_n(F)^+) \simeq_{\mathbf{P}-p} (\pi_i(BG_n^{top}(C))) \oplus (\pi_{i+1}(BG_n^{top}(C)) \otimes \bigoplus_{l \neq p} (\mathbf{Q}_l/Z_l)).$$

We conclude with the following proposition asserting that the sequences

$$\cdots \to BG_n(F)^+ \to BG_{n+1}(F)^+ \to BG_{n+2}(F)^+ \to \cdots$$

determine "intrinsic" $\mathbf{P}-p$ torsion spherical fibrations in the sense of Sullivan ([4]). For notational simplicity, we state and prove the proposition for $G_n = GL_n$.

PROPOSITION 2.4. *The homotopy fibre of*

$$i_n : BGL_n(F)^+ \to BGL_{n+1}(F)^+$$

is $\tau_{\mathbf{P}-p}(S^{2n+1})$. *Moreover, the pullback of the* $\tau_{\mathbf{P}-p}(S^{2n+3})$ *fibration* i_{n+1} *via the map* $i_{n+1} \circ i_n$ *is fibre homotopy equivalent to the fibre-wise join of* i_{n-1} *and the trivial* $\tau_{\mathbf{P}-p}(S^3)$ *fibration over* $BGL_n(F)^+$.

Proof. Because localization is an exact functor on abelian groups, localization preserves fibrations with simply connected base and total space. Thus $\tau_{\mathbf{P}-p}(\ )$ also preserves such fibrations, implying that the fibre of $i_n : BGL_n(F)^+ \to BGL_{n+1}(F)^+$ is $\tau_{\mathbf{P}-p}(S^{2n+1})$.

Let S^i, S^j be spheres of dimension $i, j > 1$. Any representatives of $\mathbf{P}-p$ torsion completions

$$\tau_{\mathbf{P}-p}(S^i) \to S^i, \qquad \tau_{\mathbf{P}-p}(S^j) \to S^j$$

determine a map of joins

$$\phi : \tau_{\mathbf{P}-p}(S^i) * \tau_{\mathbf{P}-p}(S^j) \to \tau_{\mathbf{P}-p}(S^i * S^j).$$

Using the equality of excisive pairs, $(CS^i, S^i) \times (CS^j, S^j) = (CS^i \times CS^j, S^i * S^j)$, we conclude that ϕ induces isomorphisms in Z/lZ cohomology. Thus, ϕ is a homotopy equivalence. Consequently, $\tau_{\mathbf{P}-p}(\ )$ preserves the fibre-wise join of sphere fibrations over a simply connected base. Since $\tau_{\mathbf{P}-p}(\ )$ preserves fibrations and thus homotopy theoretic fibre products, the second assertion now follows from the well known corresponding "intrinsic" property of the sphere fibrations $BG_n^{top}(C) \to BG_{n+1}^{top}(C)$.

REFERENCES

[1] FRIEDLANDER, E., *Computations of K-theories of finite fields*, to appear
[2] QUILLEN, D., *The Adams conjecture*, Topology *10* (1971), 67–80.
[3] ——, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. *96* (1972), 552–586.
[4] SULLIVAN, D., *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. *100* (1974), 1–79.

Department of Mathematics
Princeton University

Received September 27, 1974.