Poincaré Duality and Groups of Type (FP).

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Poincaré Duality and Groups of Type (FP)

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0. Introduction

This paper continues our study of the groups $H^n(\Gamma, k\Gamma)$ begun in [3]. (Here Γ is a group and k is an arbitrary field.) There we generally restricted ourselves to the case $n=2$; here we allow n to be arbitrary, but usually require Γ to satisfy rather strong finiteness conditions.

In particular our main result (Theorem 1) applies only to *groups of type* (FP) *over* k. (See section 1 for the definition of this term.) It states that if the first non-vanishing $H^n(\Gamma, k\Gamma)$ contains a non-zero finite-dimensional (over k) sub-$k\Gamma$-module, then $H^n(\Gamma, k\Gamma)$ has dimension 1 and the remaining $H^i(\Gamma, k\Gamma)$ vanish.

As a consequence we obtain the following extension of some results from [3].

THEOREM 2. *If Γ is a finitely presented, torsion-free group, then any sub-$k\Gamma$-module of $H^2(\Gamma, k\Gamma)$ has dimension* 0, 1, *or* ∞.

Our second application shows that Γ satisfies Poincaré duality under weaker assumptions than were previously known. Namely Theorem 3 states the following. If Γ is a finitely presented group of type (FP) and the first non-vanishing $H^n(\Gamma, \mathbf{Z}\Gamma)$ is finitely generated (as an abelian group), then Γ is a Poincaré duality group.

This paper is an extension of some observations of A. Borel and J-P. Serre. They had obtained, previous to my work, the following facts about groups Γ of type (FP) such that $H^i(\Gamma, k\Gamma)=0$ for all $i\neq n$:

(a) $\dim H^n(\Gamma, k\Gamma)=0, 1$, or ∞;

(b) if $H^n(\Gamma, k\Gamma)$ has a proper $k\Gamma$-subspace of finite codimension, then $H^n(\Gamma, k\Gamma)$ has no non-zero finite-dimensional $k\Gamma$-subspace.

They had also obtained results in the case where k is replaced by $\mathbf{Z}$.

I wish to thank Professor Serre for communicating their results to me and for encouraging me in my own work.

1. Preliminaries

Notation. Throughout this paper k denotes an arbitrary field and Γ a group. Let V and W be two k-vector spaces, then the collection of linear transformations from V to

W is denoted by $\mathrm{Hom}(V, W)$, and $V \otimes W$ expresses the tensor product of V with W over k. If V and W are $k\Gamma$-modules, then $\mathrm{Hom}(V, W)$ and $V \otimes W$ are also $k\Gamma$-modules where the Γ-structures are defined by the equations

$$(\gamma \cdot f)(x) = \gamma f(\gamma^{-1} x), \quad \text{and} \quad \gamma \cdot (x \otimes y) = \gamma x \otimes \gamma y$$

for all $\gamma \in \Gamma$, $f \in \mathrm{Hom}(V, W)$, $x \in V$ and $y \in W$. If V is a $k\Gamma$-module (or k-vector space), then the *dimension of* V, abbreviated $\dim V$, refers to the dimension of the underlying k-vector space.

LEMMA 1. *If V and W are two $k\Gamma$-modules with W free and* $0 < \dim V < \infty$, *then* $\mathrm{Hom}(V, W)$ *is free. In fact,* $\mathrm{Hom}(V, W)$ *is $k\Gamma$-isomorphic to the direct sum of s-copies of W where* $s = \dim V$.

Proof. Our argument is modeled after that of Proposition 1 on page 149 of [8]. Since W is free, it contains a k-subspace X such that W can be expressed as the following direct sum.

$$W = \sum_{\gamma \in \Gamma} \gamma \cdot X .$$

Because $\dim V$ is finite, $\mathrm{Hom}(V, W)$ is the direct sum of the k-subspaces $\mathrm{Hom}(V, \gamma \cdot X)$; but $\mathrm{Hom}(V, \gamma \cdot X) = \mathrm{Hom}(\gamma^{-1} \cdot V, \gamma \cdot X) = \gamma \cdot \mathrm{Hom}(V, X)$. Hence if Y denotes $\mathrm{Hom}(V, X)$ given the trivial Γ-structure, then $\mathrm{Hom}(V, W)$ is $k\Gamma$-isomorphic to $k\Gamma \otimes Y$. If we also give X the trivial Γ-structure, then Y is isomorphic to s-copies of X. Therefore $\mathrm{Hom}(V, W)$ is $k\Gamma$-isomorphic to s-copies of $k\Gamma \otimes X$. But this completes our proof since W is $k\Gamma$-isomorphic to $k\Gamma \otimes X$.

LEMMA 2. *If V and W are two $k\Gamma$-modules, then*

$$\mathrm{Ext}^n_{k\Gamma}(V, W) \cong H^n(\Gamma, \mathrm{Hom}(V, W))$$

for all $n \geq 0$.

Proof. This lemma is well-known. (Compare [7], page 272, exercises 4–6.) Hence we only sketch its proof.

Denote the functors $A \mapsto H^n(\Gamma, \mathrm{Hom}(A, W))$ by $E^n(A)$. (Here A is a $k\Gamma$-module and $n \geq 0$.) Then the E^n satisfy the axiomatic description ([7], Theorem 10.1) of the functors $A \mapsto \mathrm{Ext}^n_{k\Gamma}(A, W)$.
The only axiom which is difficult to verify is that

$$E^n(F) = 0 \quad \text{for} \quad n > 0 \quad \text{and all free modules } F.$$

To do this one proves first, by an argument similar to that in the proof of Lemma 1, that $\mathrm{Hom}(F, W)$ is *co-induced over* k: that is, $k\Gamma$-isomorphic to $\mathrm{Hom}(k\Gamma, X)$ for some k-vector space X with trivial Γ-structure. Then one shows that $H^n(\Gamma, A)=0$ when A is co-induced over k and $n>0$. (Compare [8], Proposition 1, page 120.)

We next recall some well-known facts about dual modules. The dual of a $k\Gamma$-module M is the $k\Gamma$-module $M^*=\mathrm{Hom}_{k\Gamma}(M, k\Gamma)$. If P is a finitely generated, projective, right $k\Gamma$-module and A is a left $k\Gamma$-module, then P^* is finitely generated and projective, and

$$P\otimes_{k\Gamma} A \quad \text{and} \quad \mathrm{Hom}_{k\Gamma}(P^*, A)$$

are naturally isomorphic.

Given a chain complex of $k\Gamma$-modules of finite length $K: K_n \to K_{n-1} \to \cdots \to K_0$, where each K_i is finitely generated and projective, we can form its dual cochain complex $K^*: K_0^* \to K_1^* \to \cdots \to K_n^*$. Given, in addition, a $k\Gamma$-module A, we can form chain complexes

$$K\otimes_{k\Gamma} A: K_n\otimes_{k\Gamma} A \to K_{n-1}\otimes_{k\Gamma} A \to \cdots \to K_0\otimes_{k\Gamma} A,$$

and

$$\mathrm{Hom}_{k\Gamma}(K^*, A): \mathrm{Hom}_{k\Gamma}(K_n^*, A) \to \mathrm{Hom}_{k\Gamma}(K_{n-1}^*, A) \to \cdots \to \mathrm{Hom}_{k\Gamma}(K_0^*, A).$$

By the above remarks, $K\otimes_{k\Gamma} A$ and $\mathrm{Hom}_{k\Gamma}(K^*, A)$ are isomorphic chain complexes. Denote the i-th homology group of $K\otimes_{k\Gamma} A$ by C_i and the i-th cohomology group of K^* by C^i.

PROPOSITION 1. *Under the above assumptions, there exists a spectral sequence with*

$$E_2^{pq} \cong H^p(\Gamma, \mathrm{Hom}(C^{n-q}, A))$$

and converging to C_{n-p+q}.

Proof. Proposition 1 is a special case of the spectral universal coefficient theorem. (See [4], page 100, Theorem 5.4.1.) In order to fit with Godement's notation, let

$$L_i = K_{n-i}^*, \; M^0 = A, \quad \text{and} \quad M^i = 0 \quad \text{for all} \quad i \neq 0.$$

Then Theorem 5.4.1 of [4] posits the existence of a spectral sequence with $E_2^{pq}=\mathrm{Ext}_{k\Gamma}^p(C^{n-q}, A)$ and converging to $H^{p+q}(\mathrm{Hom}_{k\Gamma}(L, A))$. But Lemma 2 states that $\mathrm{Ext}_{k\Gamma}^p(C^{n-q}, A) \cong H^p(\Gamma, \mathrm{Hom}(C^{n-q}, A))$. On the other hand $H^{p+q}(\mathrm{Hom}_{k\Gamma}(L, A))$ and $H_{n-p+q}(\mathrm{Hom}_{k\Gamma}(K^*, A))$ are identical, and by the remarks preceding the statement of Proposition 1, $H_{n-p+q}(\mathrm{Hom}_{k\Gamma}(K^*, A))$ and C_{n-p+q} are isomorphic. Concatenating this information completes the proof of Proposition 1.

We say that Γ is a group *of type* $(n-\mathrm{FP})$ *over* k if k with the trivial Γ-structure has a resolution of finite length $0 \to P_s \to P_{s-1} \to \cdots \to P_0 \to k \to 0$ by projective $k\Gamma$-modules such that P_i is finitely generated for all $i \leqslant n$. When $n=\infty$ we say more simply that Γ

is a group *of type* (FP) *over* k. Moreover, if $n=\infty$ and k is replaced by $\mathbf{Z}$ in the above definition, then we say that Γ is a group *of type* (FP).

COROLLARY 1. *If Γ is a group of type* (FP) *over k and A is a $k\Gamma$-module, then there exists a spectral sequence (whose differentials d_r have bidegree $(r, r-1)$) with*

$$\mathscr{E}_2^{pq} \cong H^p(\Gamma, \operatorname{Hom}(H^q(\Gamma, k\Gamma), A))$$

and converging to $H_{q-p}(\Gamma, A)$.

Proof. Consider a resolution of k $0 \to K_n \to K_{n-1} \to \cdots \to K_0 \to k \to 0$ by finitely generated, projective modules K_i, and let K denote the chain complex $K_n \to K_{n-1} \to \cdots \to K_0$. Applying Proposition 1 to the complex K and the $k\Gamma$-module A, we obtain a spectral sequence with $E_2^{pq} \cong H^p(\Gamma, \operatorname{Hom}(H^{n-p}(\Gamma, k\Gamma), A))$ and converging to $H_{n-p+q}(\Gamma, A)$. Then let $\mathscr{E}_s^{pq}$ be $E_s^{p,n-q}$ and we are done.

The next corollary partially recovers the "inverse duality" discovered by Bieri. (See [1], Remark following Proposition 5.3.)

COROLLARY 2. *Let Γ be a group of type* (FP) *over k such that $H^i(\Gamma, k\Gamma)=0$ for all $i \neq n$. If C denotes $H^n(\Gamma, k\Gamma)$, then*

$$H_s(\Gamma, A) \cong H^{n-s}(\Gamma, \operatorname{Hom}(C, A))$$

for every integer s and every $k\Gamma$-module A.

Proof. Under the above assumptions, the spectral sequence of Corollary 1 collapses and yields that $H_{n-p}(\Gamma, A)$ is isomorphic to $H^p(\Gamma, \operatorname{Hom}(C, A))$. The result now follows by substituting $n-s$ for p in this isomorphism.

Remark. Prior to my work, Borel and Serre had observed (private communication) that Bieri-Eckmann duality [2] could be recovered from a spectral sequence (constructed under the same hypotheses as Corollary 1) with $E_{pq}^2 \cong H_p(\Gamma, H^q(\Gamma, k\Gamma) \otimes A)$ and converging to $H^{q-p}(\Gamma, A)$. This spectral sequence is obtainable in a manner analogous to the one from Proposition 1 by making use of the spectral Künneth formula ([4], page 102, Theorem 5.5.1) together with the natural isomorphism between $P^* \otimes_{k\Gamma} A$ and $\operatorname{Hom}_{k\Gamma}(P, A)$ valid for any pair of left $k\Gamma$-modules, provided that P is finitely generated and projective.

2. The Main Theorem

We now come to the main result of this paper.

THEOREM 1. *Suppose that $H^i(\Gamma, k\Gamma)=0$ for all $i<n$ and that $H^n(\Gamma, k\Gamma)$ contains a non-zero finite-dimensional sub-$k\Gamma$-module. If Γ is of type* $(n-\text{FP})$ *over k, then we conclude the following*:

(a) Γ *is of type* (FP) *over* k;

(b) $H^i(\Gamma, k\Gamma)=0$ *for all* $i \neq n$;

(c) $\dim H^n(\Gamma, k\Gamma)=1$.

Proof. For $n=0$ this result is well-known. Hence we may assume that $n>0$. Consider a projective resolution of k with minimal length m

$$0 \to K_m \xrightarrow{d_m} K_{m-1} \to \cdots \to K_0 \to k \to 0,$$

where K_i is finitely generated for all $i \leqslant n$. Clearly $m \geqslant n$, and we intend to show that $m=n$. Let K be the chain complex $K_n \to K_{n-1} \to \cdots \to K_0$, and A be a $k\Gamma$-module. Applying Proposition 1 to this pair and noting that the resulting spectral sequence collapses, we obtain an isomorphism between $H^p(\Gamma, \mathrm{Hom}(C^n, A))$ and C_{n-p} for all p. Recall that C^i is the i-th cohomology group of K^* and that C_i is the i-th homology group of $K \otimes_{k\Gamma} A$. In particular, C_i and $H_i(\Gamma, A)$ are isomorphic for all $i<n$; consequently,

(i) $H^m(\Gamma, \mathrm{Hom}(C^n, A))=0$ if $m>n$, and

(ii) $H^n(\Gamma, \mathrm{Hom}(C^n, A)) \cong H_0(\Gamma, A)$.

By the hypotheses of Theorem 1, $H^n(\Gamma, k\Gamma)$ contains a sub-$k\Gamma$-module V such that $0<\dim V<\infty$. Since $H^n(\Gamma, k\Gamma)$ is a sub-$k\Gamma$-module of C^n, we see that V is also a sub-$k\Gamma$-module of C^n. Applying the functor $\mathrm{Hom}(\ , A)$ to the short exact sequence $0 \to V \to C^n \to C^n/V \to 0$, we obtain a new short exact sequence of $k\Gamma$-modules

$$0 \to \mathrm{Hom}(C^n/V, A) \to \mathrm{Hom}(C^n, A) \to \mathrm{Hom}(V, A) \to 0.$$

Now, applying the functor $H^*(\Gamma, \ )$ to this sequence, we obtain the exact sequence

$$H^m(\Gamma, \mathrm{Hom}(C^n, A)) \to H^m(\Gamma, \mathrm{Hom}(V, A)) \to H^{m+1}(\Gamma, \mathrm{Hom}(C^n/V, A)).$$

Since k has a projective resolution of length m, $H^{m+1}(\Gamma, \mathrm{Hom}(C^n/V, A))$ must vanish, and hence the above sequence degenerates into the following epimorphism:

(iii) $H^m(\Gamma, \mathrm{Hom}(C^n, A)) \to H^m(\Gamma, \mathrm{Hom}(V, A)) \to 0$.

Suppose that $m>n$. (We intend to show that this assumption leads to a contradiction.) Then, by (i) and (iii), $H^m(\Gamma, \mathrm{Hom}(V, A))=0$ for every $k\Gamma$-module A. This fact, in conjunction with Lemma 1, yields that $H^m(\Gamma, W)=0$ for every free (hence, also every projective) module W. In particular $H^m(\Gamma, K_m)$ vanishes, which implies that $d_m: K_m \to K_{m-1}$ is a split-$k\Gamma$-monomorphism. Therefore $K_{m-1}/d_m K_m$ is projective (and finitely generated if $m-1=n$), and

$$0 \to K_{m-1}/d_m K_m \to K_{m-2} \to \cdots \to K_0 \to k \to 0$$

is a projective resolution of k with length $m-1$ whose first $n+1$-terms (starting with K_0) are finitely generated. But this is a contradiction. Hence $m=n$, which proves assertions (a) and (b) of Theorem 1.

Since $H_0(\Gamma, k\Gamma)=k$ we obtain, using (ii) and (iii), the following inequality:

(iv) $\dim H^n(\Gamma, \mathrm{Hom}(V, k\Gamma))\leqslant 1$. But Lemma 1 states that $\mathrm{Hom}(V, k\Gamma)$ is the direct sum of s-copies of $k\Gamma$ where $s=\dim V$. This fact, together with the inequality (iv), implies that $\dim H^n(\Gamma, k\Gamma)=1$, which completes the proof of Theorem 1.

One says that Γ is a group *of type* (VFP) *over* k if Γ contains a subgroup of finite index of type (FP) over k.

ADDENDUM. *If we replace in the hypotheses of Theorem* 1 $(n-\mathrm{FP})$ *by* (VFP), *then conclusions* (b) *and* (c) *remain true.*

Proof. This is a consequence of the following well-known fact: *If Γ' is a subgroup of finite index in Γ, then $H^i(\Gamma, k\Gamma)$ and $H^i(\Gamma', k\Gamma')$ are isomorphic $k\Gamma'$-modules for all integers i.*

3. Applications

Our first application of Theorem 1 is to extend some results from [3].

THEOREM 2. *If Γ is a finitely presented, torsion-free group, then any sub-$k\Gamma$-module of $H^2(\Gamma, k\Gamma)$ has dimension* 0, 1, *or* ∞.

The proof of Theorem 2 depends on the following elementary lemma.

LEMMA 3. *Let l be a subfield of k, and A a $l\Gamma$-module. If $A\otimes_l k$ contains a sub-$k\Gamma$-module V such that*

$$0<\dim_k V<\infty,$$

then A contains a sub-$l\Gamma$-module W such that

$$\dim_k V\leqslant \dim_l W<\infty.$$

Proof. Regarding k as a vector space over l, let $f: k\to l$ be a non-zero linear functional. Then define a $l\Gamma$-homomorphism $g: A\otimes_l k\to A$ by composing $\mathrm{id}\otimes f: A\otimes_l k\to A\otimes_l l$ with the natural isomorphism from $A\otimes_l l$ to A. Let $W=g(V)$, then one easily checks that W satisfies the conclusion of Lemma 3.

Proof of Theorem 2. Because of Theorem 5.1 of [3], it suffices to consider the case where k has characteristic 0. Since Γ is finitely presented, $H^2(\Gamma, k\Gamma)$ and $H^2(\Gamma, \mathbf{Q}\Gamma)\otimes_{\mathbf{Q}} k$ are isomorphic $k\Gamma$-modules. (Here $\mathbf{Q}$ denotes the rational numbers.) Let V be a sub-$k\Gamma$-module of $H^2(\Gamma, k\Gamma)$ such that $0<\dim_k V<\infty$. By Lemma 3, $H^2(\Gamma, \mathbf{Q}\Gamma)$ contains a sub-$\mathbf{Q}\Gamma$-module W such that $\dim_k V\leqslant\dim_{\mathbf{Q}} W<\infty$; hence to prove Theorem 2, we need only show that $\dim_{\mathbf{Q}} W=1$.

But because of Theorem 5.3 of [3], we may assume that Γ is a group of type $(2-\text{FP})$ over **Q**. Since $H^0(\Gamma, \mathbf{Q}\Gamma)=0$, Theorem 1 implies Theorem 2 provided we can show that $H^1(\Gamma, \mathbf{Q}\Gamma)$ vanishes.

To do this we assume its opposite, i.e. $H^1(\Gamma, \mathbf{Q}\Gamma)\neq 0$, and show that this assumption leads to a contradiction. As a consequence of Lemma 3.5 of [10] and section 5.1 of [9], Γ has infinitely many ends. Hence by the Main Theorem of [9], Γ is a non-trivial free product of subgroups Γ_1 and Γ_2; both of which are finitely presented by a result of Stallings ([11], Lemma 1.3). By the "Mayer-Vietoris" sequence ([6] or [10], Theorem 2.3), $H^2(\Gamma, \mathbf{Q}\Gamma)$ is $\mathbf{Q}\Gamma$-isomorphic to the direct sum of $H^2(\Gamma_1, \mathbf{Q}\Gamma)$ and $H^2(\Gamma_2, \mathbf{Q}\Gamma)$. Therefore one of these modules, say $H^2(\Gamma_1, \mathbf{Q}\Gamma)$ to be specific, contains a non-zero finite-dimensional sub-$\mathbf{Q}\Gamma$-module. But this is impossible, since

$$H^2(\Gamma_1, \mathbf{Q}\Gamma)\cong H^2(\Gamma_1, \mathbf{Q}\Gamma_1)\otimes_{\mathbf{Q}\Gamma_1} \mathbf{Q}\Gamma$$

as $\mathbf{Q}\Gamma$-modules. This completes the proof of Theorem 2.

One says that Γ is *virtually torsion-free* if Γ contains a torsion-free subgroup of finite index. Then the following extension of Theorem 2 is easily proven.

ADDENDUM. *If Γ is finitely presented and virtually torsion-free, then any sub-$k\Gamma$-module of $H^2(\Gamma, k\Gamma)$ has dimension* 0, 1, *or* ∞.

Our second application is the following result.

THEOREM 3. *Suppose that Γ is a finitely presented group of type* (FP), *and let n be the smallest integer such that $H^n(\Gamma, \mathbf{Z}\Gamma)\neq 0$. If $H^n(\Gamma, \mathbf{Z}\Gamma)$ is a finitely generated abelian group, then Γ is an n-dimensional Poincaré duality group.*

Remark. Such an integer n exists, since for groups of type (FP) $H^i(\Gamma, \mathbf{Z}\Gamma)$ cannot vanish for all i.

Proof. Since Γ is a group of type (FP), it is also of type (FP) over k. Furthermore $H^i(\Gamma, k\Gamma)$ is k-isomorphic to the direct sum of $H^i(\Gamma, \mathbf{Z}\Gamma)\otimes k$ and $\mathrm{Tor}(H^{i+1}(\Gamma, \mathbf{Z}\Gamma), k)$, and $H^i(\Gamma, \mathbf{Z}\Gamma)\otimes k$ is embedded as a sub-$k\Gamma$-module of $H^i(\Gamma, k\Gamma)$ via this isomorphism. (Here, and for the rest of this paper, $\otimes$ and Tor are over **Z**.)

Suppose $H^n(\Gamma, \mathbf{Z}\Gamma)$ has p-torsion for some prime p. Then by the above discussion, we have the following facts:

(a)$H^i(\Gamma, \mathbf{Z}_p\Gamma)=0$ for all $i<n-1$;

(b) $H^n(\Gamma, \mathbf{Z}_p\Gamma)\neq 0$; and

(c) $0<\dim_{\mathbf{Z}_p} H^{n-1}(\Gamma, \mathbf{Z}_p\Gamma)<\infty$.

(Here $\mathbf{Z}_p$ denotes the field with p-elements.) But these facts contradict Theorem 1. Thus $H^n(\Gamma, \mathbf{Z}\Gamma)$ is a free abelian group of rank s where $0<s<\infty$.

Therefore $H^i(\Gamma, k\Gamma)=0$ for all $i<n$; and $H^n(\Gamma, k\Gamma)$ contains a sub-$k\Gamma$-module of

dimension s. Now by a second application of Theorem 1, we have $\dim H^n(\Gamma, k\Gamma)=1$ and $H^i(\Gamma, k\Gamma)=0$ for all $i\neq n$. Consequently we have $s=1$ and both $H^i(\Gamma, \mathbf{Z}\Gamma)\otimes k$ and $\mathrm{Tor}(H^i(\Gamma, \mathbf{Z}\Gamma), k)$ vanish for all $i\neq n$. By setting k equal to $\mathbf{Q}$ and $\mathbf{Z}_p$ respectively, we see that $H^i(\Gamma, \mathbf{Z}\Gamma)=0$ for all $i\neq n$. And since $s=1$, $H^n(\Gamma, \mathbf{Z}\Gamma)$ is infinite cyclic. Hence Γ satisfies the conditions of [5] to be an n-dimensional Poincaré duality group.

ADDENDUM. *The conclusion of Theorem* 3 *remains true when the hypothesis*

"*$H^n(\Gamma, \mathbf{Z}\Gamma)$ is a finitely generated abelian group*"

is replaced by the following two assumptions:

(a) *$H^n(\Gamma, \mathbf{Z}\Gamma)$ contains a non-zero finitely generated (as an abelian group) sub-Γ-module, and*

(b) *$H^n(\Gamma, \mathbf{Z}\Gamma)$ is a free abelian group.*

Proof. Let A be a non-zero sub-Γ-module of $H^n(\Gamma, \mathbf{Z}\Gamma)$ such that A is finitely generated as an abelian group. By assumption (b), $A\otimes\mathbf{Q}$ is a non-zero finite-dimensional sub-$\mathbf{Q}\Gamma$-module of $H^n(\Gamma, \mathbf{Z}\Gamma)\otimes\mathbf{Q}$. But $H^i(\Gamma, \mathbf{Z}\Gamma)\otimes\mathbf{Q}$ and $H^i(\Gamma, \mathbf{Q}\Gamma)$ are isomorphic $\mathbf{Q}\Gamma$-modules for all $i\geq 0$. Therefore Theorem 1 implies $\dim_{\mathbf{Q}} H^n(\Gamma, \mathbf{Z}\Gamma)\otimes\mathbf{Q}=1$. This fact, together with (b), yields that $H^n(\Gamma, \mathbf{Z}\Gamma)$ is infinite cyclic. Now apply Theorem 3 to complete the proof.

4. Appendix

We mention a consequence of Theorem 2.

COROLLARY 3. *If Γ is finitely presented and virtually torsion-free, then any sub-Γ-module of $H^2(\Gamma, \mathbf{Z}\Gamma)$ is either*

(a) *zero,*

(b) *an infinite cyclic abelian group, or*

(c) *not finitely generated as an abelian group.*

(This result extends Corollary 5.2 of [3].)

Proof. Corollary 3.7 of [10] implies that $H^2(\Gamma, \mathbf{Z}\Gamma)$ is a torsion-free abelian group. Thus it suffices to show that $\dim_{\mathbf{Q}} A\otimes\mathbf{Q}=1$ when A is a non-zero finitely generated (as an abelian group) sub-Γ-module of $H^2(\Gamma, \mathbf{Z}\Gamma)$. But this follows from the addendum to Theorem 2 where we specify k to be $\mathbf{Q}$.

Note added in proof: 1) There are analogues to our results in the theory of homology manifolds, namely in the work of P. E. Conner and E. E. Floyd (Michigan Math. J. *6* (1959), 33–43).

2) K. Brown has recently found an elegant new proof for Theorem 1 which avoids the use of spectral sequences.

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