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Minimal Varieties and Harmonic Maps in Tori

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Introduction

A little over a hundred years ago H. A. Schwarz constructed triply-periodic minimal surfaces in 3-space. This provides us with the first compact minimal surface in a flat real 3-torus. All compact complex submanifolds in complex tori provide further examples. More explicit examples of this type are mentioned in §3. We have made a systematic study of compact minimal submanifolds in flat tori and this is our account.

The first observation is that a harmonic 1-form on a flat torus is harmonic when restricted to a minimal submanifold (see Corollary 1). Since the submanifolds are compact, harmonic theory enters quite naturally and the Albanese apparatus introduced in §1 contains all the necessary harmonic theory. There is some advantage in presenting arguments valid for the weaker notion of a harmonic map into a flat torus, and this is the course taken in §1 and 2.

Our main results are:

i) that two minimal isometric (resp. holomorphic) immersions in a flat (complex) torus which are homotopic differ by a translation. (Theorem 2).

ii) any compact minimal submanifold in a torus has a torus which acts freely and equivariantly as its connected isometry group. The quotient manifold lies minimally in a flat torus and has negative Ricci curvature on an open dense set. (Theorem 3).

In the applications in §4 we pay particular attention to submanifolds with non-vanishing Euler number. This includes a result of Y. Matsushima [10] obtained by the method of theta functions. In another application to Riemannian geometry we show that the fundamental group of a compact manifold which admits a metric of negative Ricci curvature need not have exponential growth and can even be free abelian. This answers in the negative a question raised by Milnor [11].

We thank Professor Matsushima for the many times we have benefited from his opinion and the interest shown in this work; it was from his lectures in 1968 that the

[1]) Work supported by N.S.F. Grant GP 29662; these results were announced at the A.M.S. Summer Institute in Differential Geometry, Stanford University, August 1973, where we learned from S.T. Yau that he had obtained the result of Theorem 2 for complex submaniflods of certain complex tori. A.M.S. subject classification 53-xx, 53 Axx, 53 Cxx.

second author first learned about the Albanese map in Riemannian geometry. We are grateful to Professor Lichnerowicz who called to our attention his paper [9] in which this map is also developed.

§1. Harmonic Maps and the Albanese Map

The early work of Eells and Sampson [3] is of course the basic reference on harmonic maps.

Beginning with a smooth map f from a compact oriented n-dimensional Riemannian manifold M into another Riemannian manifold M' the tensor field ${}^t f_* f_*$ on M (f_* and ${}^t f_*$ denote the differential of f and its transpose) compares the quadratic form induced from M' with the metric on M. The energy density of f is

$$e(f)=\tfrac{1}{2}\operatorname{Tr}{}^t f_* f_*,$$

where Tr denotes trace, and the energy of f is

$$E(f)=\int e(f)\,dv$$

where dv is the Riemannian volume element on M.

If we denote the metrics on both M and M' by $\langle\ ,\ \rangle$, it will always be apparent from the context which one is involved. The Riemannian connexions on M and M' will be distinguished as ∇ and D. Let π denote the projection from the tangent bundle $T(M')$ of M' onto M'. The smooth maps $v: M\to T(M')$ with $\pi\circ v=f$ are called vector fields along f. Given a vector field X on M, we interpret $D_X v$ in the obvious way as a vector field along the map f. Let E be the vector bundle over M induced from $T(M')$ by f. This is a Riemannian vector bundle, i.e.

$$X\langle v, w\rangle=\langle D_X v, w\rangle+\langle v, D_X w\rangle$$

for any pair of vector fields v and w along f and any vector field X on M. If we set

$$\alpha(f)(X, Y)=D_X(f_* Y)-f_*(\nabla_X Y)$$

we can easily observe that $\alpha(f)$ is a symmetric E-valued quadratic tensor field on M. We call $\alpha(f)$ the second fundamental form of the map f. The tension field $\tau(f)$ of f defined by $\tau(f)=\operatorname{Tr}\alpha(f)$ is a vector field along the map f and its significance derives from Lemma 1.

Let f_t, $-\varepsilon<t<\varepsilon$, be a smooth variation of $f_0=f$; the variation vector field $v=df_t/dt|_{t=0}$ is a vector field along the map f.

LEMMA 1. *$(d/dt)(E(f_t))|_{t=0}=-\int_M\langle v, \tau(f)\rangle\,dv$, where $\tau(f)$ is the tension field of f, v is the variation vector field of f_t and dv is the volume element on M.*

Proof. $(d/dt)\,E(f_t)=\int_M (d/dt)\,(e(f_t))\,dv$.

If $\{e_1,\ldots,e_n\}$ is a local orthonormal frame field on a neighborhood of p in M, then

$$
\begin{aligned}
\frac{d}{dt}(e(f_t))(p)\Big|_{t=0} &= \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{n} \langle (f_t)_* e_i, (f_t)_* e_i\rangle_{f(p)}\Big|_{t=0} \\
&= \sum_{i=1}^{n} \langle D_{v(p)} f_* e_i, f_* e_i\rangle \\
&= \sum_{i=1}^{n} \langle D_{e_i} v, f_* e_i\rangle \qquad \text{(by a simple argument in local coordinates)} \\
&= \sum_{i=1}^{n} e_i \langle v, f_* e_i\rangle - \langle v, \sum_{i=1}^{n} D_{e_i} f_* e_i\rangle \\
&= \sum_{i=1}^{n} e_i \langle v, f_* e_i\rangle - \langle v, \sum_{i=1}^{n} f_*(\nabla_{e_i} e_i)\rangle - \langle v, \tau(f)\rangle \\
&= \delta w - \langle v, \tau(f)\rangle
\end{aligned}
$$

where w is the 1-form on M defined by

$$w(X)=\langle v, f_* X\rangle$$

and δ is the divergence. The lemma now follows from Green's theorem.

DEFINITION. A smooth map f from a compact oriented Riemannian manifold M into a Riemannian manifold M' is called harmonic if the energy of f is critical, that is, if the tension field $\tau(f)$ vanishes identically on M. When f is an isometric immersion, this is equivalent to saying f is minimal.

LEMMA 2. *A harmonic map pulls a parallel* 1*-form back to a harmonic* 1*-form.*

Proof. Let w be a parallel 1-form on M'. Certainly $\mu=f^*w$ is closed. To calculate the divergence of μ at a point p of M we take a local frame field $\{e_1,\ldots,e_n\}$ on M in a neighborhood of p. Then

$$
\begin{aligned}
\delta\mu &= \sum_{i=1}^{n} (\nabla_{e_i}\mu)\, e_i \\
&= \sum_{i=1}^{n} e_i(\mu(e_i)) - \mu(\nabla_{e_i} e_i) \\
&= \sum_{i=1}^{n} e_i(w(f_* e_i)) - w(f_*(\nabla_{e_i} e_i)) \\
&= \sum_{i=1}^{n} (D_{f_* e_i} w)(f_* e_i) + \sum_{i=1}^{n} w(D_{e_i} f_* e_i - f_*(\nabla_{e_i} e_i)) \\
&= +w(\tau(f)),
\end{aligned}
$$

since w is parallel. Thus μ is harmonic if f is harmonic.

COROLLARY 1. *A smooth map f of M into a flat torus T is harmonic if and only if f pulls harmonic 1-forms back to harmonic 1-forms.*

Proof. A harmonic 1-form on T is parallel. Thus, by Lemma 2, its pull-back to M is harmonic if f is harmonic. If, conversely, the pull-back of every harmonic 1-form is harmonic, it is immediate from the proof of Lemma 2 that

$$w(\tau(f))=0$$

for every harmonic 1-form w on T. Consequently $\tau(f)=0$, i.e. f is harmonic.

COROLLARY 2. *If $f_1, f_2: M\to T$ are harmonic maps then so are the maps $f_1 \pm f_2$ and $\alpha\circ f_1$ for any homomorphism α of T into itself.*

Fortunately there is no question of existence of such maps, for every compact Riemannian manifold M with nonzero first Betti number enjoys a nontrivial canonical harmonic map into a canonical flat torus, known as the Albanese map and Albanese torus of M, respectively. As the Albanese map, and particularly the universal property of this map, are so effective in this work, we should begin by explaining these.

Let M be a compact connected oriented Riemannian manifold and $\mathfrak{h}$ the real vector space of all harmonic 1-forms on M. Let ϱ be the natural projection from the universal cover $\tilde{M}$ of M. Fix $x_0\in M$, say $\varrho(x_0)=p_0$. We define a smooth map

$$\tilde{a}:\tilde{M}\to\mathfrak{h}^*$$

by line integrals

$$\tilde{a}(x)(w)=\int_{x_0}^{x}\varrho^* w$$

where $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$. For $\sigma\in\pi_1(M)$

$$\tilde{a}(\sigma x)=\tilde{a}(x)+\psi(\sigma),$$

where $\psi(\sigma)(w)=\int_{x_0}^{\sigma x_0}\varrho^* w$, so that ψ is a homomorphism from $\pi_1(M)$ into $\mathfrak{h}^*$ as an additive group. It is a fact that $\Delta=\psi(\pi_1(M))$ is a lattice in the vector space $\mathfrak{h}^*$, and clearly this vector space has a natural Euclidean metric from the global inner product of forms on M. With the quotient metric, we call the torus $A(M)=\mathfrak{h}^*/\Delta$ the Albanese torus of the Riemannian manifold M. It is a simple matter to check that $\tilde{a}$ projects to a map $a:M\to A(M)$, called the Albanese map. From the very construction of a, it will be clear that the map it induces on fundamental groups

$$a:\pi_1(M)\to\pi_1(A)$$

is surjective and that a^* maps the space of harmonic 1-forms on $A=A(M)$ isomorphically onto $\mathfrak{h}$. So by Corollary 1, the Albanese map a is harmonic.

PROPOSITION 1. (Universality of the Albanese map). *Let $a:M\to A$ be the Albanese map of a compact oriented Riemannian manifold M. If $f:M\to T$ is any other harmonic map of M into any flat torus T, then there is a unique affine map*

$$g:A\to T$$

with $f=g\circ a$.

Proof. In the above construction $a(p_0)=1_A$. After a translation in T, we may assume $f(p_0)=1_T$. Recall a^* is an isomorphism. The map

$$(a^*)^{-1}\circ f^*:H^1(T,\mathbf{R})\to H^1(A,\mathbf{R})$$

is a homomorphism, thus it coincides with the codifferential g^* of some homomorphism,

$$g:A\to T.$$

Consequently $f^*=a^*\circ g^*$. Now f and $g\circ a$ are harmonic maps and so, by Corollary 2, is $f-g\circ a$. However the latter map must be constant, by the above, and is the identity at p_0. Thus $f=g\circ a$. The uniqueness is clear because $a(M)$ generates A as a group.

As an illustration of the efficacy of the Albanese map, we end this section with an application.

THEOREM 1. *Let f be a harmonic map from one compact orientable Riemannian manifold M into another M', which has nonnegative Ricci curvature. Then on cohomology*

$$f^*:H^1(M',\mathbf{R})\to H^1(M,\mathbf{R})$$

is injective if and only if $f(M)$ does not lie in an orthogonal trajectory of a parallel vector field on M'.

Remark. This result for minimal immersions is the subject of a paper by E. Kelly [6].

Proof. Let a and a' denote the respective Albanese maps of M and M' into their Albanese tori A and A'. If w is a harmonic 1-form on A', it is parallel; so, by Lemma 2, $(a')^*w$ is harmonic. But Bochner's theorem then assures us that $(a')^*w$ is parallel. Applying Lemma 2 once more to the harmonic map f, we find that $f^*(a')^*w$ is harmonic. By Corollary 1, $a'\circ f$ is a harmonic map. From universality, $a'\circ f=g\circ a$ for some affine map

$$g:A\to A'$$

Since $(a)^*$ and $(a')^*$ are isomorphisms, if f^* is not injective neither is g^*. This in turn means $g(A)$ is a proper closed subgroup of A'. Let w be a harmonic 1-form on A' perpendicular to $g(A)$; then $v=(a')^* w$ is a parallel 1-form on M' by the same reasoning as above. Now v can only be zero when $H^1(M', \mathbf{R})=0$ and f^* is then trivially injective. If $v \neq 0$ we still have $f^*v=0$ because $g^*w=0$; consequently $f(M)$ is in an orthogonal trajectory of the parallel field dual to v on M'.

§2. Rigidity of Harmonic Maps

LEMMA 3. *If two harmonic maps $f_1, f_2: M \to T$ of M into a flat torus are homotopic, then they differ by a translation.*

Proof. After translation, we may assume $f_1(p_0)=f_2(p_0)=1_T$ for some point $p_0 \in M$. Since f_1 and f_2 are homotopic, their induced maps on de Rham cohomology coincide. Thus for any harmonic 1-form w on T the forms f_1^*w and f_2^*w are cohomologous. But these are harmonic forms by Lemma 2. Hence $f_1^*w=f_2^*w$ for all harmonic 1-forms w on T. As a result, the maps f_1 and f_2 coincide in a neighborhood on p_0, and by connectedness this extends to all of M.

COROLLARY 3. *If two holomorphic maps of a compact Kahler manifold M into a complex torus are homotopic then they differ by a translation.*

Proof. Since the maps f_1 and f_2 are holomorphic each pulls the holomorphic 1-forms on the torus back to holomorphic 1-forms on M. But the real and imaginary parts of a holomorphic 1-form on a Kahler manifold are harmonic. Such maps are harmonic by Corollary 1, and applying Lemma 3 the result follows.

The next result follows from the work of Eells and Sampson [3], but we give an elementary proof.

COROLLARY 4. *Given a smooth map f from a compact Riemannian manifold M into a flat torus T there is a unique (to within translation) harmonic map homotopic to f.*

Proof. The map

$$(a^*)^{-1} \circ f^*: H^1(T, R) \to H^1(A, R)$$

is a homomorphism and so coincides with the codifferential of some homomorphism

$$g: A \to T.$$

Therefore $F=f-g \circ a$ induces the trivial map on first cohomology. Then if $(y', \ldots, y^m)$ are cartesian coordinates on the universal cover $\mathbf{R}^m$ of T the forms dy^i on T are pulled back to exact forms on M. Thus $F^*dy^i=dh_i$ for each i, where h_i is some smooth real function on M. Then the map $h=(h_1, \ldots, h_m)$ of M when projected into T must differ from F by at most a translation. Since $\mathbf{R}^m$ is contractible the map h is null-homotopic and the same will be true of $F=f-g\circ a$; in other words f is homotopic to the harmonic map $g\circ a$. The uniqueness follows from Lemma 3.

PROPOSITION 2. *A harmonic map $f:M\to T$ induces a homomorphism $F:I_0(M)\to T$ with the equivariance property*

$$f\circ\sigma=L_{F(\sigma)}\circ f$$

where σ is any element of the identity component $I_0(M)$ of the isometry group of M and L_τ customarily denotes left translation by τ. Moreover F is harmonic with respect to the bi-invariant metric on $I_0(M)$.

Proof. Suppose $f(p_0)=1_T$ for some $p_0\in M$. If $\sigma\in I_0(M)$ then f and $f\circ\sigma$ are homotopic harmonic maps and so, by Lemma 3, differ by a translation $L_{F(\sigma)}$. Clearly F is a Lie group homomorphism with the above equivariance property. While we can show F is harmonic by the theory of Lie groups, we prefer a more direct argument. Let ϱ denote the map of $I_0(M)$ onto the orbit of p_0. Let w be any harmonic 1-form on T and set $v=f^*w$. By equivariance the form v is invariant by $I_0(M)$, i.e. $\tau^*v=v$ for all $\tau\in I_0(M)$. Thus

$$\begin{aligned}(L_\tau)^*\,\varrho^*v&=(\varrho\circ L_\tau)^*\,v=(\tau\circ\varrho)^*\,v=\varrho^*\tau^*v=\varrho^*v\,.\\ (R_\tau)^*\,\varrho^*v&=(\varrho\circ R_\tau)^*\,v=(\tau\circ\varrho\circ\operatorname{ad}(\tau))^*\,v=\operatorname{ad}(\tau)^*\,\varrho^*\tau^*v\,,\\ &=\operatorname{ad}(\tau)^*\,\varrho^*f^*w\,,\quad\text{by the above,}\\ &=(f\circ\varrho\circ\operatorname{ad}(\tau))^*\,w=(f\circ\varrho)^*\,w=\varrho^*v\,,\end{aligned}$$

from equivariance and the fact that T is abelian. Since $\varrho^*v=(f\circ\varrho)^*\,w$ is a bi-invariant 1-form on $I_0(M)$ it is a harmonic 1-form in the bi-invariant metric. However $F=f\circ\varrho$ since $f(p_0)=1_T$. By Corollary 1, F is harmonic with respect to the bi-invariant metric.

Remark. We have noted a further property of such a harmonic map which, while it is of no use to us here is curious enough to be remarked upon. It is that the symmetric 2-form Ω on M induced by f from the metric on T is invariant by $I_0(M)$. Indeed, if σ is any isometry of M which is homotopic to the identity then, by Lemma 3, $f\circ\sigma=L_s\circ f$ for some $s\in T$. Hence

$$\Omega(\sigma_*X,\sigma_*Y)=\langle f_*\sigma_*X,f_*\sigma_*Y\rangle=\langle L_sf_*X,L_sf_*Y\rangle=\langle f_*X,f_*Y\rangle=\Omega(X,Y)$$

for smooth vector fields X and Y on M.

PROPOSITION 3. *If $f:M\to T$ is a harmonic immersion, then*

i) *F is an immersion,*

ii) *each nontrivial isometry of M homotopic to the identity has no fixed points,*

iii) $I_0(M)$ *is a torus acting freely on M and is contained in the isometry group of the Riemannian metric induced by f on M,*

iv) *The Albanese map of M is an immersion.*

Proof. i) If F is not an immersion, there is a nontrivial one-parameter subgroup $\{\sigma_t\}$ of $I_0(M)$ such that $F(\sigma_t)=1_T$ for all t. By equivariance

$$f\circ\sigma_t=f$$

for all t. Thus for each $p\in M$, the curve $\sigma_t(p)$ is mapped to the point $f(p)$. Since f is an immersion, $\sigma_t(p)=p$ for all t. Hence each σ_t is the identity, contradicting our assumption. Therefore F is an immersion.

ii) Let σ be an isometry of M homotopic to the identity and fixing a point $p_0\in M$. After applying a translation to f, if necessary, we may assume $f(p_0)=1_T$. Then by Lemma 3 the maps f and $f\circ\sigma$ differ by a translation. However, as they coincide at p_0 we must have $f\circ\sigma=f$. Taking differentials at p_0, since f_* is injective on $T_{p_0}(M)$ a simple argument with the exponential map at p_0 shows that σ is the identity.

iii) Since F is an immersion, the induced map on Lie algebras is injective. Thus the Lie algebra of $I_0(M)$ is abelian. So $I_0(M)$ is a compact abelian Lie group, that is, a torus. The remark preceeding Proposition 3 completes the proof.

iv) This follows at once on applying the universal property to f.

While the next result is well known a new proof may be of interest.

COROLLARY 5. *If M is a homogeneous space of a compact semisimple Lie group G, then the first Betti number of M is zero.*

Proof. Let a denote the Albanese map of M into its Albanese torus A. As above, we have a homomorphism

$$F:G\to A$$

and it is harmonic with respect to the bi-invariant metric on G, by Proposition 2. Since G is semisimple and A is abelian, F is constant. By equivariance, $a\circ\sigma=a$ for all $\sigma\in G$. Since M is homogeneous this implies that a is constant. It is easily seen that the Albanese map is constant only if the first Betti number of M vanishes.

§3. Minimal Submanifolds in a Torus

As mentioned in §1, a minimal immersion f of M in another Riemannian manifold is harmonic with respect to the metric induced on M by f. Our knowledge of minimal submanifolds in a flat torus goes deeper than anything obtained above for harmonic maps. But even the results of the previous section (Lemma 3) provide a very strong rigidity theorem for submanifolds.

THEOREM 2. (i) *If two minimal isometric immersions of a compact Riemannian manifold into a flat torus are homotopic then they differ by a translation.*

(ii) *If a compact complex manifold admits a holomorphic immersion in a complex torus then any holomorphic map homotopic to it differs from it by a translation.*

The only other class of submanifolds enjoying a comparable degree of rigidity would be nonsingular algebraic varieties in complex projective space [2].

Nor is there any want for examples; the Albanese map of a compact Kähler manifold M is holomorphic, so if it is an immersion it is a minimal immersion in a flat complex torus. The universal property makes this equivalent to M admitting a holomorphic immersion in some complex torus. It is very easily seen that it is also equivalent to the holomorphic cotangent bundle of M being ample [10]. Explicit examples would be

i) any compact Riemann surface of positive genus,

ii) the Fano surface of lines on a cubic in 4-dimensional complex projective space [16],

iii) a generic hyperplane section of an abelian variety in complex projective space.

While there are real examples, we have currently no way of generating an abundant supply. The simplest one is perhaps the following

iv) a surface constructed by H. A. Schwarz [15]. A regular tetrahedral frame with two opposite edges removed is taken as boundary and the solution of the Plateau problem for this boundary is reflected across each linear edge and the process repeated indefinitely. This determines an imbedded triply-periodic minimal surface in E^3 without singularities. The quotient by the period-lattice is a compact minimal surface in a flat 3-torus.

It is of interest to note that a minimal immersion of a torus in a flat torus must be a homomorphism with finite kernel. This can be readily seen from universality and the fact that the Albanese map of a torus T is the identity map on T no matter what the metric on T. In fact this proves that every harmonic map of a torus in a flat torus is a homomorphism.

We now begin the proof of the main theorem of this paper, Theorem 3 in this section. For obvious reasons we like to refer to it as the reduction theorem.

Let f be a minimal immersion of a compact orientable manifold M^n into a flat torus T^{n+r}. Consider M with the induced metric. In §1 we defined the second fundamental form α of f and now for each $p \in M$, we set

$$\mathfrak{n}_p = \{X \in T_p(M) \mid \alpha(X, Y) = 0 \quad \text{for all} \quad Y \in T_p(M)\},$$

calling it the *relative nullity space* at p.

For ξ normal to M at p

$$\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle$$

defines a symmetric endomorphism of $T_p(M)$. Because f is an immersion

$$\alpha(X, Y)=\sum_{\alpha=1}^{r} \langle A_\alpha X, Y\rangle \xi_\alpha, \tag{1}$$

where $\{\xi_1,\dots, \xi_r\}$ is an orthonormal frame for the normal space to M at p and $A_\alpha=A_{\xi_\alpha}$. In addition we have $\operatorname{Tr} A_\xi=0$ for all ξ, on account of f being minimal. We remark that the Gauss equation (see [7]) yields

$$S(X, Y)=\langle\sum_\alpha (\operatorname{Tr} A_\alpha\cdot A_\alpha-A_\alpha^2)\, X, Y\rangle$$

where S is the Ricci tensor of M. This reduces to

$$S(X, Y)=-\langle(\sum_\alpha A_\alpha^2)\, X, Y\rangle \tag{2}$$

since f is minimal.

LEMMA 4. *For an immersion f*

$$\begin{aligned} \mathfrak{n}_p&=\{X\in T_p(M) \mid A_\xi X=0 \quad \textit{for all } \xi \textit{ normal to } M \textit{ at } p\}\\ &=\{X\in T_p(M) \mid S(X, Y)=0 \quad \textit{for all} \quad Y\in T_p(M)\}, \end{aligned}$$

the latter holding because f is minimal.

Proof. The first identity follows from the fact that α only takes values normal to M when f is an immersion. From (1) we infer that $X\in\mathfrak{n}_p$ if and only if $X\in\bigcap_{\alpha=1}^r \operatorname{Ker} A_\alpha$. Since each A_α is symmetric, $\bigcap_{\alpha=1}^r \operatorname{Ker} A_\alpha=\operatorname{Ker}\sum_{\alpha=1}^r A_\alpha^2$ which by (2) is just the kernel of the Ricci tensor. This completes the proof.

Now assume $m=\min_{p\in M}\dim\mathfrak{n}_p\geq 1$ and denote the open set $\{p\in M \mid \dim\mathfrak{n}_p=m\}$ by G.

LEMMA 5. *$\mathfrak{n}$ is a totally geodesic foliation on G.*

Proof. The equation of Codazzi is written

$$(\tilde\nabla_X\alpha)(Y, Z)=(\tilde\nabla_Y\alpha)(X, Z), \tag{3}$$

where

$$(\tilde\nabla_X\alpha)(X, Z)=(D_X\alpha(Y, Z))^N-\alpha(\nabla_X Y, Z)-\alpha(Y, \nabla_X Z),$$

the superscript N being used here to indicate normal components are taken [7]. If X and Z are sections of the bundle $\mathfrak{n}$, and Y is any vector field on G, a simple computation shows that (3) reduces to

$$\alpha(Y, \nabla_X Z)=0,$$

that is, $\nabla_X Z$ is a section of $\mathfrak{n}$. Thus the leaves of $\mathfrak{n}$ are totally geodesic in M.

LEMMA 6. *With the metric induced from M the leaves of $\mathfrak{n}$ are complete.*

This lemma is due to P. Hartman and his proof applies verbatim when the ambient space is a flat torus [1, 5].

We now let N^m denote the leaf of $\mathfrak{n}$ passing through $p_0 \in G$. If X and Y are vector fields along N then

$$D_X f_* Y = f_*(\nabla_X Y) + \alpha(X, Y) = f_*(\nabla_X Y)$$

which is tangent to $f(N)$ since $\nabla_X Y$ is tangent to N on account of N being totally geodesic. Thus $f(N)$ is a totally geodesic submanifold of T^{n+p} and is complete by Lemma 6.

LEMMA 7. *$f(N)$ is a subtorus of T.*

Proof. Assume $f(N)$ is not compact. From the above we know it is a complete totally geodesic submanifold of T. Therefore its closure T_0' in T is a subtorus of dimension $l > m$. Since M is compact, $f(M) \supset T_0'$. Since f is locally an imbedding, it is permissible and convenient to think of M as sitting in T containing a totally geodesic subspace N of T passing through $p_0 \in M$ and the closure N' of N in T again lies in M. For $p \in N'$ there exists $X \in \mathfrak{n}_{p_0}$ such that $\dot\gamma(t) = \exp tX$ comes arbitrarily close to p. Noting that the velocity vector $\dot\gamma(t)$ is always in the distribution we have

$$D_{\dot\gamma(t)} Y = \nabla_{\dot\gamma(t)} Y + \alpha(\dot\gamma(t), Y) = \nabla_{\dot\gamma(t)} Y$$

for all vector fields Y tangent to M along γ. Thus the tangent space to M is parallel in T along the curve $\gamma(t)$; which by the above choice of γ means that $T_p(M)$ is parallel to $T_{p_0}(M)$ in T. Consequently the normal space to M in T is constant along N'. This means that we may choose locally an orthonormal frame field $\{\xi_1, \dots, \xi_r\}$ for the normal bundle to M in T such that $D_X \xi_\alpha = 0$, $1 \leqslant \alpha \leqslant r$, when X is tangent to N'. Hence, for each X tangent to N' we have $A_\alpha X = 0$, $1 \leqslant \alpha \leqslant r$. In particular $\mathfrak{n}_{p_0}$ must have dimension $\geqslant l > m$. This contradiction proves the lemma.

LEMMA 8. *The spaces $f_*(\mathfrak{n}_p)$, for all $p \in G$, are parallel to each other in T. In particular $\mathfrak{n}$ is a parallel foliation on G.*

Proof. The Lie algebra $\mathfrak{t}$ of T is identified with the tangent space to T at the identity. By Lemma 7, $f_*(\mathfrak{n}_p)$ is parallel to an m-dimensional subalgebra V_p of the Lie algebra $\mathfrak{t}$ which determines a compact subgroup of T. But the set of such subalgebras is countable so V_p is independent of $p \in G$, that is, the $f_*(\mathfrak{n}_p)$ are all parallel to each other in T. In particular the spaces $\mathfrak{n}_p$ are all parallel to each other in the metric of M.

LEMMA 9. *G is an open dense subset of M.*

Proof. Let $(x^1, \ldots, x^n)$ be normal coordinates in a neighborhood U of any point p_0 in M. The coordinate functions $f_1, \ldots, f_{n+r}$ are analytic functions of the variables $(x^1, \ldots, x^n)$ on the region U (see [13] for example). In particular the same is true of the second fundamental form and, by the equation (2) above, the Ricci tensor S of M. If U is an open set in the complement of G, then the analytic operator $\wedge^{n-m} S$ (denoting the $(n-m)$ exterior power of S) vanishes on U and, by analyticity, on all of M. We have proved then that G is dense. Openness was clear from the outset.

LEMMA 10. *For $p \in G$ and $u \in \mathfrak{n}_p$ there is a parallel (and therefore Killing) vector field X on M with $X_p = u$. In other words $\mathfrak{n}$ extends from G to a parallel foliation on all of M.*

Proof. Given $u \in \mathfrak{n}_p$, $p \in G$, it determines a parallel vector field X on T tangent to M at each point of G. Since G is an open dense set, X is tangent to M everywhere and since it is paıallel on T its restriction is a parallel vector field on M.

As we have constructed m linearly independent parallel fields, $\dim I_0(M) \geqslant m$. On the other hand a classical result of Bochner on compact Riemannian manifolds with negative semidefinite Ricci tensor tells us that $\dim I_0(M) \leqslant m$, seeing as the maximal rank of the Ricci tensor is $n-m$. So $I_0(M)$ is an m-dimensional torus acting freely on M and its orbits are the leaves of $\mathfrak{n}$. We note at this point that we have a local splitting for the Riemannian manifold M, that the leaves of the foliation orthogonal to $\mathfrak{n}$ are not necessarily compact and that G is an open dense subset of M.

Let us assume $f(p_0) = 1_T$ for some $p_0 \in M$. With the notation of the previous section, we have the diagram

$$\begin{array}{ccc} I_0 & \xrightarrow{F} & F(I_0) \subset T \\ {\scriptstyle \varrho}\downarrow & & \downarrow{\scriptstyle i} \\ M & \xrightarrow{f} & T \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_T} \\ M/I_0 & \xrightarrow{f_1} & T/F(I_0). \end{array}$$

The tori I_0 and $F(I_0)$ act freely on M and T respectively so that M/I_0 and $T/F(I_0)$ are compact manifolds of dimension $n-m$ and $n-m+r$, respectively, and the projections π_M and π_T have maximal rank. Clearly there is a natural map

$$f_1 : M/I_0 = M_1 \to T_1 = T/F(I_0)$$

making the above diagram commute. Evidently the map f_1 is an immersion and it is equally clear that it is a minimal isometric immersion with respect to natural metrics on M_1 and T_1 (that is, the metrics which make π_M and π_T, restricted to the horizontal spaces, into isometries).

THEOREM 3. *Let f be a minimal isometric immersion of a compact orientable Riemannian manifold M^n in a flat torus T^{n+r}. Then*

i) *The identity component $I_0(M)$ of the isometry group of M is a torus acting freely on M and the orbits are parallel in M (in fact correspond to the flat factor in the local Riemannian decomposition).*

ii) *This action makes M a principal torus bundle over a compact Riemannian manifold $M_1 = M/I_0(M)$. Moreover f induces a minimal isometric immersion of M_1 in the torus $T_1 = T/F(I_0)$, where F is the homomorphism from I_0 to T induced by f, also with codimension r.*

iii) *The Ricci tensor of M_1 is negative definite on an open dense set. In particular the isometry group of M_1 is finite.*

§4. Applications

The reduction theorem of the previous section gives information bearing directly on minimal submanifolds in tori and supplies us with enough detail on a certain class of minimal submanifolds to be helpful in questions of a purely Riemannian nature. First the direct applications.

A compact complex manifold M admits a holomorphic immersion in a complex torus if and only if M has ample holomorphic cotangent bundle and admits a Kähler metric, where by ample we mean that the holomorphic 1-forms on M determine the full cotangent space at each point of M. This follows directly from the universality of the Albanese map of M and is verified in [10].

COROLLARY 6. *Let M be a compact Kähler manifold with ample holomorphic cotangent bundle. Then the identity component* $\mathrm{Aut}_0(M)$ *of the Lie group of all holomorphic transformations of M is a complex torus acting freely on M. The quotient $M_1 = M/\mathrm{Aut}_0(M)$ is a compact complex manifold with ample holomorphic cotangent bundle and admits a Kähler metric with negative Ricci curvature on an open dense set. In particular* $\mathrm{Aut}_0(M_1)$ *is trivial.*

Proof. The Albanese map a of M into its Albanese torus $A(M)$ is a holomorphic immersion because the holomorphic cotangent bundle is ample. The metric induced on M by this map is of the form $\sum_\alpha w^\alpha \bar{w}^\alpha$, where $w^1, \ldots, w^q$ is a basis for the space of all holomorphic 1-forms on M. Thus any holomorphic transformation of M will leave this metric invariant so that $\mathrm{Aut}_0(M)$ coincides with the identity component of the isometry group of M. Since a holomorphic immersion is minimal we may apply

Theorem 3 to conclude that $\mathrm{Aut}_0(M)$ is a complex torus, of complex dimension s say, acting freely on M. The quotient compact complex manifold $M_1 = M/\mathrm{Aut}_0(M)$ admits a holomorphic immersion f_1 in a complex torus T_1 by the argument of Theorem 3. In particular the holomorphic cotangent bundle of M_1 is ample. Since f_1 is minimal we can see from Eq. 2 in §3 that the Ricci curvature of the metric it induces on M_1 is nonpositive. Supposing that the Ricci curvature is not negative on an open dense set we deduce from Theorem 3 that $\mathrm{Aut}_0(M_1)$ is nontrivial. But if we follow the commutative diagram at the end of the preceding section – with a and $A(M)$ replacing f and T – we find that $a(M)$ is invariant by a complex subtorus of $A(M)$ of dimension larger than s. This implies $\dim \mathrm{Aut}_0(M) > s$, which is a contradiction.

COROLLARY 7. *Let M be a compact orientable manifold minimally immersed in a flat torus. The following are equivalent for the induced metric*

i) *$I(M)$ is finite,*

ii) *$S<0$ on an open dense set in M.*

Moreover these hold if the Euler number $\chi(M) \neq 0$.

Proof. This is contained in Theorem 3.

COROLLARY 8. *Let M^n be a complex hypersurface in a flat complex torus. The following are equivalent*

i) *$\chi(M) \neq 0$,*

ii) *$\mathrm{Aut}_0(M)$ is the identity,*

iii) *$S<0$ on an open dense set,*

iv) *the Gauss map $\Gamma: M^n \to P^n(C)$ is nonsingular on an open dense set and surjective.*

Remark. This result was obtained by a very different approach by Matsushima [10].

Proof. By Corollary 5, or rather its complex counterpart, we have i)$\Rightarrow$ii)$\Rightarrow$iii). The Euler number is, to within a nonzero multiple, the integral of the Jacobian of Γ, which is really the determinant of the second fundamental form [12]. The integrand therefore never changes sign and both iii) and iv) imply it is nonzero somewhere, that is, iii) or iv)$\Rightarrow$i). Now i)$\Rightarrow$iii) and it follows from iii) and equation (2) in the previous section that Γ is an immersion on an open dense set. It suffices to add that the Gauss map is onto; for if we are given a parallel vector holomorphic plane field $\{v, Jv\}$ (here J is the complex structure on T) which does not lie in $\Gamma(M)$, then the vector field v is never normal to M. Thus v determines a nonvanishing vector field on M, which is impossible by i). In other words Γ is onto, so i)$\Rightarrow$iv).

Our investigation has led us to a rich source of examples in Riemannian geometry. Beginning with an abelian variety T^{n+1}, that is, a complex torus holomorphically imbedded in complex projective space we let V^n denote any nonsingular hyperplane section. By Lefschetz' theorem on hyperplane sections,

$$\pi_i(V^n) \to \pi_i(T^{n+1})$$

is a bijection for $i<n$, where π_i denotes the ith homotopy group. In particular for $n>1$, $\pi_1(V)=\oplus^{2n+2} Z$. If the Euler number $\chi(V)$ were zero, Corollary 4 would say that V admits a parallel vector field. Since our hyperplane section is generic, we can certainly assume $\chi(V)\neq 0$ from the outset. As a matter of fact it turns out to be no assumption. Matsushima [10] shows $\chi(V)\neq 0$. If $\pi_i(V)=0$ for all $i>1$ then V is homotopically equivalent to a real $(2n+2)$-torus since they have the same homotopy groups; for reasons of dimension alone this is not possible. Hence $\pi_i(V)\neq 0$ for some $i>1$. By Theorem 3 the Ricci tensor S is negative definite on an open dense subset of V. As remarked earlier, the Gauss map (see §5)

$$\Gamma: V^n \to P^n(C)$$

is an immersion precisely on the subset of V where S is negative definite. This subset cannot be V itself for then Γ would be a covering map, i.e., V would be simply connected, which is obviously absurd.

To summarize, we have constructed for each $n>1$:

A Hodge manifold V^n with negative Ricci curvature on an open dense set $\neq V$ satisfying

(i) $\chi(V)\neq 0$,

(ii) $\pi_1(V)=\oplus^{2n+2} Z$,

(iii) $\pi_i(V)\neq 0$, for some integer $i>1$.

To this we may add that this Kähler metric on V can be deformed into a Riemannian metric having strictly negative Ricci curvature everywhere, for P. Ehrlich has shown in his thesis [4] that such a deformation is always possible starting from a compact Riemannian manifold with non-positive Ricci curvature provided the Ricci tensor is negative definite at some point.

These examples show that very basic results on the topology of Riemannian manifolds of nonpositive sectional curvature vanish once the curvature condition is replaced by the corresponding condition on the Ricci curvature. For example:

(a) The universal cover of a compact Riemannian manifold of nonpositive curvature is a cell by the theorem of Hadamard and Cartan. The universal cover of V is not a cell.

(b) If a compact Riemannian manifold of nonpositive curvature has abelian fundamental group, then it is flat [8]. The fundamental group of V is abelian and V is far from flat.

(c) The fundamental group of a compact Riemannian manifold of negative curvature has exponential growth. This was first proved by Milnor [11] and he raised the question of whether it might still hold for manifolds with strictly negative Ricci curvature. But V carries a Riemannian metric of negative Ricci curvature while its fundamental group is free abelian. This answers Milnor's question in the negative.

This indicates that the topological implications of the existence of metrics of negative Ricci curvature may be remote, except of course in dimension $\leqq 3$. A decidedly more difficult question than Milnor's is whether spheres can carry Riemannian metrics of negative Ricci curvature.

§5. The Density of the Gauss Map

As there is an absolute parallelism in a flat torus we can define a Gauss mapping from a submanifold M^n in T^{n+r} into the Grassmann manifold of unoriented n-planes at the point $1_T \in T$. This map will be denoted Γ. The work of Ruh and Vilms [14] applies, telling us that Γ is harmonic when M is minimal. In the special case that M^n is a complex manifold holomorphically immersed in a complex torus T^{n+r}, the map Γ sends a point of M to the complex n-dimensional plane tangent to M at that point. Thus the natural range for Γ is the Grassmann manifold of all complex n-planes. The Gauss map is then holomorphic.

THEOREM 4. *Let M^n be a compact complex manifold holomorphically immersed in a flat complex torus T^{n+r}. If $\chi(M)\neq 0$, every parallel field of holomorphic planes on T^{n+r} is normal to M somewhere. Moreover, when $r=1$, the Gauss map is onto if and only if the Euler number of M is nonzero.*

Proof. If a constant vector field a is nowhere normal to M, then its component tangential to M is a nonvanishing vector field on M. This is not possible when $\chi(M)\neq 0$. Now whenever a is normal to M, so also is Ja where J stands for the complex structure of T. Here we have used the fact that M is a complex submanifold. The holomorphic plane field $\{a, Ja\}$ is therefore normal to M at some point. When $r=1$, this is but another way of saying that $\Gamma: M^n \to P^n(C)$ is onto; the converse is already contained in Corollary 8.

Of course for compact real hypersurfaces M^n of a torus T^{n+1}, whether minimal or not, the Gauss map $\Gamma: M^n \to S^n$ is onto if $\chi(M)\neq 0$. However, as of writing, we do not have a complete description of $\Gamma(M)$ for general real minimal hypersurfaces. By virtue of the reduction theorem (Theorem 3) we need only concern ourselves with those submanifolds whose Ricci tensor is negative definite on an open dense set. In this respect we add

PROPOSITION 4. *Let M^n be a compact minimal hypersurface in a flat torus T^{n+1}. If the isometry group of M is finite, then the image of the Gauss map*

$$\Gamma: M^n \to S^n$$

lies in no closed hemisphere of S^n.

Proof. We may suppose M is orientable and denote by ξ a unit normal vector field along M. Assume $\Gamma(M)$ lies in a closed hemisphere of S^n, that is to say, there is a constant unit vector field a on T^{n+1} such that $\langle \xi, a \rangle \geqslant 0$ on M. A routine computation for the Laplacian of $\langle \xi, a \rangle$ gives

$$\Delta \langle \xi, a \rangle = -\mathrm{Tr} A^2 \langle \xi, a \rangle$$

where A is the second fundamental form. In the previous notation $\langle AX, Y \rangle = \langle \alpha(X, Y), \xi \rangle$. By Green's theorem, $\mathrm{Tr} A^2 \langle \xi, a \rangle \equiv 0$. However if the isometry group of M is finite, then $\mathrm{Tr} A^2$ is negative on an open dense set. This is by Theorem 3. Thus $\langle \xi, a \rangle \equiv 0$. However this means that a is everywhere tangent to M, that is, M has a parallel vector field. This again is impossible since the isometry group of M is finite.

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