Irregular Primes and Integrality Theorems for Manifolds.

Autor(en): **Frank, David**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **50 (1975)**

PDF erstellt am: **11.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-38812

Comment. Math. Helvetici 50 (1975) 321–326 Birkhäuser Verlag, Basel

Irregular Primes and Integrality Theorems for Manifolds

DAVID FRANK[1])

1. Introduction

Number theory has long played an important part in topology. We examine here a topological problem whose solution depends on the distinction between regular and irregular primes. (Recall that a prime p is said to be irregular if p is odd and p divides the ideal class number of the p-th cyclotomic field; the first irregular prime is 37.)

Let M be a closed oriented topological manifold of dimension $4k$, smooth on the complement of a point. A necessary condition for M itself to be smoothable is that the Pontrjagin number $p_k[M]$ be an integer. If, moreover, M is a spin manifold ($w_2M=0$), the integrality theorem of Atiyah-Borel-Hirzebruch-Singer [2], [3], [4] says an additional necessary condition is that $(1/a_k)\hat{A}[M]$ be an integer, where $\hat{A}$ is the $\hat{A}$-genus and $a_k=1$ (k even) or 2 (k odd).

We wish to know to what extent these necessary conditions for the smoothability of M are sufficient. If $k>1$, M is triangulable [11]; then the obstruction to extending the smoothing of (M-point) to M is an element Σ_M of the group of exotic spheres Γ^{4k-1}. There is a splitting [6], [9] $\Gamma^{4k-1}=bP\,(4k)\oplus\pi'_{4k-1}$, where $bP(4k)$ is the subgroup of exotic spheres bounding parallelizable manifolds and π'_{4k-1} is a certain complementary summand. The integrality of $p_k[M]$ and $\hat{A}[M]$ can provide no information on the component of Σ_M in π'_{4k-1}(Lemma 3.2 below). Hence we let Σ^b_M denote the component of Σ_M in $bP\,(4k)$, and pose the

Question. Let M be a closed oriented spin manifold of dimension $4k$, $k>1$, smooth on the complement of a point. Suppose M satisfies the integrality condition: $p_k[M]$ and $(1/a_k)\,\hat{A}\,[M]$ are integers. Is $\Sigma^b_M=0$ in $bP\,(4k)$?

THEOREM 1.1. *Suppose* $\dim M\leqslant 200$, $\dim M\neq 136$. *If M satisfies the integrality condition, then* $\Sigma^b_M=0$.

THEOREM 1.2. *There is a manifold N of dimension* 136 *satisfying the integrality condition with Σ_N of order* 37 *in bP* (136).

Since 37 is the first irregular prime, this suggests

[1]) Supported by NSF grant No. P029431000.

THEOREM 1.3. *Let p be a prime. The following are equivalent:*

i) *p is irregular*

ii) *for some* $k>1$, *there is a manifold M of dimension* $4k$ *satisfying the integrality condition with order* Σ_M^b *divisible by p.*

In each dimension $4k$, it is possible to determine which primes p (if any) can divide the order of a Σ_M^b (for some M satisfying the integrality condition). Table 1.4 lists all such values of k and p which can occur in the range $k<109$. If a value of k does not appear in the table, then for a manifold M of dimension $4k$, the integrality condition implies $\Sigma_M^b=0$.

Table 1.4

$k=\frac{1}{4}\dim M$	34	51	52	62	63	70	76	80	84	88	95	106
$p=$ order Σ_M^b	37	59	37	67	103	37	131	59	101	37	67	37

A reference for the number theory we use is [5]. However, we follow the notation of [10], which is standard in topology and differs from the notation of [5].

2. Proof of the Main Theorems

Let B_i be the i-th Bernoulli number [10]. For convenience, define B_0 to be 1. Kummer gave a numerical criterion for regularity.

ASSERTION 2.1. *A prime p is irregular if and only if p divides the numerator of* B_n, *for some* $n<\frac{1}{2}(p-1)$.

We give a table of all irregular primes less than 233. For each p, we give those $n<\frac{1}{2}(p-1)$ such that p divides num B_n. This information is in fact available for all $p\leqslant 4001$.

Table 2.2

p	37	59	67	101	103	131	149	157
n	16	22	29	34	12	11	65	31, 55

If k is an integer and p is a prime, let c $(=c(k,p))$ denote the residue class of k modulo $\frac{1}{2}(p-1)$. Thus $0\leqslant c<\frac{1}{2}(p-1)$. If $p=2$, set $c(k,2)=0$.

THEOREM 2.3. *Let* $k>1$ *be an integer and p a prime. The following are equivalent:*

i) $p\leqslant 2k-1$ *and p divides* num B_c, *where c is the residue class of k modulo* $\frac{1}{2}(p-1)$,

ii) *there is a manifold M of dimension* $4k$ *satisfying the integrality condition with order* Σ_M^b *divisible by p.*

We show that Theorem 2.3, whose proof we defer to the next section, implies the results of the introduction. Note first that (2.3) implies Theorem 1.3. For if p is irregular, then p divides $\operatorname{num} B_c$ for some $c<\frac{1}{2}(p-1)$. Let $k=a(\frac{1}{2})(p-1)+c$, for any integer $a\geqslant 1$. Then $p\leqslant 2k-1$ and c is the residue class of $k \bmod \frac{1}{2}(p-1)$, so by (2.3) there is a manifold M with the desired properties. Conversely, if M exists, then p divides $\operatorname{num} B_c$, so p is irregular.

We now prove Theorems 1.1 and 1.2. Let $\dim M=4k$, $k\leqslant 50$. Suppose M satisfies the integrality condition and p divides the order of Σ_M^b. Then $p\leqslant 99$ and p is irregular, so $p=37$, 59, or 67. Suppose $p=59$. Then $k\geqslant 30$. Let c be the residue class of $k \bmod 29$. Then 59 divides $\operatorname{num} B_c$, and from Table 2.2 we find that $c=22$. Thus $k\equiv 22 \bmod 29$ and $30\leqslant k\leqslant 50$. Since no such k exists, we conclude that $p\neq 59$. Similarly, we may exclude $p=67$.

If $p=37$, then $k\geqslant 19$ and, from Table 2.2, $k\equiv 16 \bmod 18$. Thus $k=34$ is the only possibility, proving Theorems 1.1 and 1.2.

We leave the further verification of (1.4) to the reader.

3. Proof of Theorem 2.3

We will need

THEOREM (von Staudt). *Let p be a prime. Then p divides* $\operatorname{denom} B_k$ *if and only if $p-1$ divides $2k$.*

THEOREM (Kummer's Congruence). *Suppose $p-1$ does not divide $2k$, p a prime. Let $r=\frac{1}{2}(p-1)$. Then B_k/k is a p-integer (p does not divide its denominator), and in the ring of p-integers*

$$B_k/k\equiv \pm B_{k+r}/(k+r) \bmod p.$$

LEMMA 3.1. *The prime p divides* $\operatorname{num}(B_k/k)$ *if and only if p divides* $\operatorname{num} B_c$, *where c is the residue class of $k \bmod \frac{1}{2}(p-1)$.*

Proof. Suppose p divides $\operatorname{num}(B_k/k)$. Then by von Staudt's Theorem, $p-1$ does not divide $2k$. (In particular, p is odd.) Write $k=a(\frac{1}{2})(p-1)+c$, where $0\leqslant c<\frac{1}{2}(p-1)$. Then Kummer's Congruence shows p divides $\operatorname{num}(B_c/c)$. Hence p divides $\operatorname{num} B_c$.

Conversely, suppose p divides $\operatorname{num} B_c$, where $0\leqslant c<\frac{1}{2}(p-1)$. Since $2c<p-1$, we see that p divides $\operatorname{num}(B_c/c)$ and $p-1$ does not divide $2c$. Hence by Kummer, p divides $\operatorname{num}(B_k/k)$.

LEMMA 3.2. *Let M and N be manifolds of dimension $4k$, $k>1$. If $\Sigma_M^b=\Sigma_N^b$, then* $p_k[M]=p_k[N] \bmod 1$ *and* $\hat{A}[M]=\hat{A}[N] \bmod a_k$.

Proof. Let $\Sigma=\Sigma_N-\Sigma_M$. Then the $bP\,(4k)$-component of Σ is zero. It follows from [6] or [9] that there is a spin manifold V with $\Sigma_V=\Sigma$ such that all Pontrjagin numbers of V are zero. Let X be the connected sum of M, $-N$, and V. Then $\Sigma_X=\Sigma_M-\Sigma_N+\Sigma_V=0$, so X is smoothable. Thus $p_k[M]-p_k[N]=p_k[M]-p_k[N]+p_k[V]=p_k[X]$ is an integer.

A similar argument applies to the $\hat{A}$-genus.

Proof of Theorem 2.3. Let W be the closed Milnor manifold of dimension $4k$ and signature 8. Thus W-(point) is smooth and parallelizable. Then $\Sigma_W\in bP\,(4k)$. In fact, Σ_W generates $bP\,(4k)$, which is a cyclic group of order $N_k=a_k 2^{2k-2}(2^{2k-1}-1)$ num (B_k/k).

Now let qW be the connected sum of q copies of W. Then qW is smoothable on the complement of a point and $\Sigma_{qW}=q\Sigma_W$. Note that $\Sigma_{qW}=0$ if and only if N_k divides q. Suppose M is a manifold of dimension $4k$ which satisfies the integrality condition but $\Sigma_M^b\neq 0$. Then $\Sigma_M^b=\Sigma_{qW}$ for some q. Therefore by Lemma 3.2, the manifold qW satisfies the integrality condition. Using the Hirzebruch signature theorem for the L-genus and the definition of the $\hat{A}$-genus [10], we have

$$8=\text{signature } W=L_k[W]=\frac{2^{2k}(2^{2k-1}-1)\,B_k}{(2k)!}\,p_k[W]$$

$$\hat{A}[W]=\frac{-B_k}{2\,(2k)!}\,p_k[W].$$

We easily compute

$$p_k[qW]=q\,\frac{(2k)!}{2^{2k-3}(2^{2k-1}-1)\,B_k}$$

$$\hat{A}[qW]=\frac{-q}{2^{2k-2}(2^{2k-1}-1)}.$$

Now suppose the order of Σ_{qW} is ps, where p is prime and s is an integer. Then we can write

$$q=\frac{rN_k}{ps},\qquad r \text{ prime to } p.$$

Thus

$$p_k[qW]=\frac{4a_k r\,(2k-1)!\,\text{denom}\,(B_k/k)}{ps}$$

$$\hat{A}[qW]=\frac{(-a_k r)\,\text{num}\,(B_k/k)}{ps}.$$

Then the integrality condition for qW says that p divides $\mathrm{num}(B_k/k)$ and $p \leqslant 2k-1$. But if p divides num (B_k/k), then by Lemma 3.1, p divides $\mathrm{num}\, B_c$, which proves half of Theorem 2.3.

Conversely, if p divides $\mathrm{num}\, B_c$, then p divides $\mathrm{num}(B_k/k)$. Let $q = N_k/p$. Thus order $\Sigma_{qW} = p$. If $p \leqslant 2k-1$, then qW satisfies the integrality condition, concluding the proof of Theorem 2.3.

Remarks. 1. The manifold $N = qW$ $(q = N_k/p,\ p \leqslant 2k-1)$ in fact satisfies the stronger integrality theorem of [2], [3], [4]: $(1/a_k)\langle \hat{A}(N)\, ph(\gamma), [N]\rangle$ is an integer for every real vector bundle γ on N. Here ph is the Pontrjagin character. Nevertheless qW is not oriented cobordant to a smooth manifold. This does not contradict [12].

2. Note that $(1/a_k)\,\hat{A}[M]$ mod 1 is the Eells-Kuiper invariant μ of Σ_M [7]. Similarly $p_k[M]$ mod 1 is the Milnor invariant $\lambda(\Sigma_M)$ [13]. Thus our results say, for example, that if Σ is an exotic sphere with $\mu(\Sigma) = \lambda(\Sigma) = 0$, then Σ^b has order a product of irregular primes and $\dim \Sigma \geqslant 135$, or else $\Sigma^b = 0$.

3. There is a manifold N of dimension 436 satisfying the integrality condition with order Σ_N^b equal to the product of distinct primes (59) (157). This is the smallest dimension of such an example.

4. Information on the component of Σ_M in π'_{4k-1} can be obtained using the invariant of [8].

5. For the 136-dimensional manifold of Theorem 1.2 we may take the connected sum of $2^{66}(2^{67}-1)\,(1/37)\,\mathrm{num}(B_{34}/34)$ copies of W. Knowing B_{34} (Adams [1]) we find $(1/37)\,\mathrm{num}(B_{34}/34) =$

125, 235, 502, 160, 125, 163, 977, 598, 011, 460, 214, 000, 388, 469.

REFERENCES

[1] Adams, J. C., *Table of the first sixty-two numbers of Bernoulli*, J. reine angew. Math. (Crelle) *85* (1878), 269–272.

[2] Atiyah, M. F. and Hirzebruch, F., *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. *65* (1959), 276–281.

[3] Atiyah, M. F. and Singer, I. M., *The index of elliptic operators III*, Ann. of Math. *87* (1968), 546–604.

[4] Borel, A. and Hirzebruch, F., *Characteristic classes and homogeneous spaces III*, Amer. J. Math. *82* (1960), 491–504.

[5] Borevich, Z. I. and Shafarevich, I. R., *Number Theory*, Academic Press, 1966.

[6] Brumfiel, G., *On the homotopy groups of BPL and PL/O*, Ann. of Math. *88* (1968), 291–311.

[7] Eells, J. and Kuiper, N., *An invariant for certain smooth manifolds*, Annali di Matematica *60* (1962), 93–110.

[8] Frank, D., *An invariant for almost-closed manifolds*, Bull. Amer. Math. Soc. *74* (1968), 562–567.

[9] ——, *The signature defect and the homotopy of BPL and PL/O*, Comment. Math. Helv. *48* (1973), 525–530.

[10] Hirzebruch, F., *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.

[11] KIRBY, R. and SIEBENMANN, L., *Some theorems on topological manifolds*, Manifolds-Amsterdam 1970. Lecture Notes in Math. 197, Springer, 1971.
[12] STONG, R. E., *Relations among characteristic numbers II*, Topology *5* (1966), 133–148.
[13] MILNOR, J., *Differential structures on spheres*, Amer. J. Math. *81* (1959), 962–972.

State University of New York
at Stony Brook

Current address:
University of Minnesota

Received July 24, 1974