Closed Leaves in Foliations of Codimension One.

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Introduction

This paper is a study of the topological type of closed leaves that can occur in C^1 foliations of codimension one of closed n-manifolds. Of particular interest are the existence and properties of closed leaves of genus greater than one in 3-manifolds.

There are classical results in this area. For instance, Reeb's Stability Theorem [7] for foliations of codimension one on closed connected manifolds says that if one leaf is closed and has finite fundamental group, then all the leaves are closed with finite fundamental group. Also very important are S. P. Novikov's theorems [5] for 3-manifolds which give conditions on either the manifold or the foliation which imply the existence of a toral leaf (bounding a Reeb component). But little has been known about closed leaves of genus greater than one.

If a closed, oriented 3-manifold admits a smooth transversely oriented foliation with a closed surface T_g of genus g greater than one as a leaf, then a standard Euler class argument shows that this leaf cannot separate M.

Certainly some manifolds support foliations with leaves of genus greater than one (for example, $S^1 \times T_g$, $g > 1$); some other manifolds, such as S^3, can never have such a leaf (by the usual Euler class argument). A nontrivial construction of a foliation with leaves of genus greater than one is contained implicitly in [8]. It is natural to ask when such leaves can occur. A preliminary result of this paper (Theorem 1.4) shows that in order for a family of closed, disjoint, 2-sided $(n-1)$-submanifolds, each with non-vanishing real first cohomology, to be leaves of a foliation of a manifold M^n with $\chi(M)=0$, it is sufficient that the family not separate M. This improves a result of [2], [3]. For the special case of 3-manifolds, this theorem indicates a rich field of inquiry into closed leaves.

It is well-known that through any noncompact leaf of a foliation F, there passes a closed transversal to the foliation. The main result of this paper extends this to say that, for 3-manifolds, any leaf which is neither a torus nor a Klein bottle has a closed transversal meeting it (Theorem 2.2).

It should be noted that there are simple examples of foliations on T^3 which have nonseparating toral leaves and yet there is no closed transversal to the foliation meeting these leaves. Hence the property of having a closed transversal really follows from the fact that the leaf is nontoral, not merely from the fact that it does not separate M.

These theorems combine to give some interesting applications and examples of foliations, which are presented in section 3. There are a couple of particularly interesting applications of the closed transversal theorem (2.2). For instance, in the proof of Novikov's closed leaf theorem [5], it was shown that a leaf which has a vanishing cycle meets no closed transversal to F. From the above theorem, it follows immediately that such a leaf is a torus, a nonelementary fact established by Novikov through further appeal to properties of vanishing cycles.

The closed transversal property also enables one to open up all closed leaves which are neither tori or Klein bottles by a smooth homotopy through foliations while staying within the same concordance class [1].

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1. Preliminary Results

Let M^n be a closed manifold with $\chi(M)=0$ and $N_1, \ldots, N_k$ a family of closed, disjoint, 2-sided $(n-1)$-manifolds with $H^1(N_j; \mathbf{R}) \neq 0$. Let M_i, $1 \leqslant i \leqslant p$, denote the connected components of $M - \bigcup_{j=1}^{k} N_j$.

THEOREM 1.1. *The following conditions are necessary and sufficient for the existence of a foliation of M with each N_j as a leaf:*

(i) *if n is even, then $\chi(M_i)=0$ for each i,*

(ii) *if n is odd, then for each i there is a partition A_i, B_i of the N_j's in ∂M_i such that $\sum_{j \in A_i} \chi(N_j) = \sum_{j \in B_i} \chi(N_j)$.*

As is implicit in a recent paper of W. Thurston [12], this theorem follows readily from his results on homotoping plane fields (Theorem 1.3 stated below), via the following index theorem.

PROPOSITION 1.2 [6]. *Let M be a compact n-manifold with non-empty boundary. Let v be a vector field on M transverse to the boundary and with only isolated zeros in the interior of M. Finally, let $L_1, \ldots, L_k$ be the components of M along which v points outward and $L_{k+1}, \ldots, L_m$ those along which v points inward. Then the index of v is*

$$\text{index}(v) = \begin{cases} \frac{1}{2}\left\{\sum_{j=1}^{k} \chi(L_j) - \sum_{j=k+1}^{m} \chi(L_j)\right\} & \text{if } n \text{ is odd} \\ \chi(M) & \text{if } n \text{ is even} \end{cases}$$

THEOREM 1.3 [12]. *If M^n is compact and each boundary component $(\partial M)_i$ has $H^1((\partial M)_i;\ \mathbf{R})\neq 0$, then every $(n-1)$-plane field of M which is tangent to M is homotopic* $(\operatorname{rel}\partial M)$ *to a smooth integrable plane field.*

Proof of Theorem 1.1. The necessity of the conditions is immediate from Proposition 1.2.

Cutting M open along the N_j's, $1\leqslant j\leqslant k$, produces a manifold M' with boundary $N_1\cup\ldots\cup N_k\cup N_{k+1}\cup\ldots\cup N_{2k}$ where $N_j\cong N_{j+k}$ for $1\leqslant j\leqslant k$. If n is odd, choose a vector field on the boundary of each M_i (connected component of M') pointing inward along N_j if $j\in A_i$ and outward along N_j if $j\in B_i$, $1\leqslant j\leqslant 2k$. If n is even, choose any vector field transverse to ∂M_i. Such vector fields can always be extended over each M_i to a field v_i having only nondegenerate zeros.

By Proposition 1.2,

$$\operatorname{index}(v_i)=\begin{cases}\frac{1}{2}\left\{\sum\limits_{j\in A_i}\chi(N_j)-\sum\limits_{j\in B_i}\chi(N_j)\right\}=0 & \text{if } n \text{ is odd}\\ \chi(M_i)=0 \quad \text{if } n \text{ is even.}\end{cases}$$

Hence index $(v_i)=0$ in either case, and each zero of v_i of index -1 is matched by a zero of index $+1$. Since M_i is connected, each pair of zeros can be eliminated. Therefore, there exists a nonsingular vector field on each M_i transverse to ∂M_i; equivalently, a plane field on M_i tangent to ∂M_i. By Theorem 1.3, this plane field is homotopic $(\operatorname{rel}\partial M_i)$ to a smooth integrable plane field on M_i. Regluing along the N_j's gives the desired foliation of M. ∎

COROLLARY 1.4. *If $M-\bigcup_{j=1}^{k} N_j$ is connected, then there exists a foliation of M with each N_j as a leaf.*

Proof. Cut M open along the N_j's as before, producing a connected M' with boundary $N_1\cup\ldots\cup N_k\cup N_{k+1}\cup\ldots\cup N_{2k}$ where $N_j\cong N_{j+k}$. If n is even, $\partial M'$ is odd-dimensional; hence $\chi(\partial M')=0$ and $\chi(M')=\chi(M)=0$. If n is odd, there is a natural partition of the boundary components of M': $A=\{N_j\colon 1\leqslant j\leqslant k\}$, $B=\{N_j\colon k+1\leqslant j\leqslant 2k\}$. Notice $\sum_{j\in A}\chi(N_j)=\sum_{j\in B}\chi(N_j)$. The result now follows from Theorem 1.1. ∎

It is worthwhile noticing that there are examples which show that the condition that $M-\bigcup N_j$ be connected is not necessary. The manifold $T_g\times S^1$, foliated by $T_g\times\{\theta\}$ $(\theta\in S^1)$, provides a simple example (since any two of the leaves do separate the manifold). There is another more subtle example.

Let L_1, L_2, L_3 be diffeomorphic to the surfaces of genus g_1, g_2, g_3 respectively. Let them be imbedded in $\mathbf{R}^3$ in the standard manner and such that L_1 and L_2 lie within the bounded component of $\mathbf{R}^3-L_3$. The space bounded by L_1, L_2 and L_3 is a compact 3-manifold M' with boundary $L_1\cup L_2\cup L_3$. Gluing two copies of M' together along the common boundary produces a closed manifold M with an imbedding $i\colon L_1\cup L_2\cup$

$L_3 \to M$ which disconnects M. If, for instance, $g_1+g_2-g_3=1$, then $\chi(L_1)+\chi(L_2)=\chi(L_3)$ and Theorem 1.1 shows there is a foliation with the L_j's as leaves.

2. A Closed Transversal Theorem

Let F be a C^1 foliation of a closed oriented n-manifold M.

DEFINITION. Let L be a leaf of a transversely oriented foliation F. The *accessible set* of L, $A(L)$, is the set of points x in M for which there is a positive (nontrivial) transversal to the foliation from L to x.

There are several immediate consequences of this definition: (1) $A(L)$ is open and a union of leaves, (2) if a leaf $L' \subset A(L)$ then $A(L') \subset A(L)$, (3) a leaf L meets a closed transversal if and only if $L \subset A(L)$, and (4) if L and L' meet a closed transversal then $L \subset A(L')$ and $L' \subset A(L)$, in which case $A(L)=A(L')$.

PROPOSITION 2.1. *The accessible set $A(L)$ of any leaf L is either all of M or an open subset of M whose boundary consists of a finite union of closed leaves $L_1 \cup \ldots \cup L_k$ such that* (i) $\sum_{j=1}^{k} \chi(L_j)=0$ *and* (ii) $H^1(L_j;\mathbf{R}) \neq 0$ *for all j.*

Proof. If $A(L) \neq M$, then it is certainly an open subset of M with boundary. Let $L_j \subset \partial A(L)$. If $A(L)$ abuts L_j on the left, then, within a distinguished neighborhood of some point of L_j, a transversal from L can be extended slightly to cross L_j, contradicting the fact that $L_j \subset \partial A(L)$. Hence each boundary leaf must be oriented inward toward $A(L)$, i.e. $A(L)$ meets each boundary leaf on the right.

If L_j is noncompact, there is a closed transversal meeting it, which necessarily also meets leaves in $A(L)$; hence L_j is in $A(L)$, not in the boundary, a contradiction. Therefore each boundary leaf is closed. Also if $H^1(L_j;\mathbf{R})=0$, then by the Generalized Reeb Stability Theorem [10], all leaves are diffeomorphic to L_j and the foliation is a fibration over S^1. Therefore, there is a closed transversal meeting every leaf, a contradiction to $A(L) \neq M$.

If there are an infinite number of boundary leaves of $A(L)$, then the compactness of M guarantees that some two of them, say L_m and L_n, intersect a distinguished neighborhood. Since $A(L)$ meets each boundary leaf on the right, there exists a point of $A(L)$ in this neighborhood and between L_m and L_n. Then a transversal to this point from L can be extended within this neighborhood to cross either L_m or L_n (whichever lies to the right), a contradiction.

Therefore $\partial A(L)$ consists of a finite union of closed leaves, each oriented inward toward $A(L)$. Since the boundary of $A(L)$ is homologically trivial, the Euler class evaluates trivially on it, i.e. $\sum_{j=1}^{k} \chi(L_j)=0$. ∎

We now turn to the special case of 3-manifolds.

THEOREM 2.2. *Let F be a foliation of codimension one of a closed 3-manifold M. Through any leaf which is neither a torus nor a Klein bottle, there passes a closed transversal to the foliation.*

Proof. Assume first that M is oriented and F transversely oriented. Let L_0 be a leaf of F with no closed transversal through it. Then $A(L_0) \neq M$; in fact, $L_0 \subset \partial A(L_0)$. By Proposition 2.1, the boundary consists of finitely many closed leaves $L_0 \cup L_1 \cup \ldots \cup L_k$ such that $\sum_{j=0}^{k} \chi(L_j)=0$ and $H^1(L_j; \mathbf{R}) \neq 0$ for all j. For closed 2-manifolds, nonzero cohomology implies that $\chi(L_j) \leqslant 0$ for all j, and $\sum_{j=0}^{k} \chi(L_j)=0$ then implies that $\chi(L_j)=0$ for all j. Hence L_0 is a torus. Therefore any nontoral leaf has a closed transversal passing through it.

The conditions of orientability of M and transverse orientability of F are easily eliminated. There is a finite covering of M which is orientable and such that the lifted foliation is transversely orientable. By the above argument, there is a closed transversal there through any nontoral leaf. Projecting to M gives a closed transversal through any leaf of F which is neither a torus or a Klein bottle. ▌

The closed transversal produced in the theorem may not be an imbedding, but it can be perturbed sl ghtly to produce an imbedded circle. Further, it can be modified to one which meets the leaf exactly once, if the leaf is closed.

3. Corollaries

For orientable manifolds, Theorems 1.1 and 2.2 combine to give some results on the existence of leaves diffeomorphic to $T_{g,k}$, the surface obtained by deleting k points from T_g. The proofs depend upon a standard modification of a foliation of an oriented 3-manifold about a closed transversal. Let t be a smoothly imbedded circle transverse to F. There is a normal neighborhood U of t, diffeomorphic to $S^1 \times D^2$, chosen sufficiently small so that $F \mid U$ is the product foliation. Within U, the foliation F may be modified to a foliation F^1 agreeing with F near the boundary of U, but having a Reeb component along the core of U, with the leaves which were met by t now punctured and approaching the Reeb component asymptotically. Throughout this section, the dimension of M is assumed to be 3.

COROLLARY 3.1. *If $H_2(M; \mathbf{Z})$ is nontrivial (for instance, if M fibers over S^1), then for each sufficiently large g and each $k \geqslant 0$, there is a foliation of M having a leaf diffeomorphic to $T_{g,k}$.*

Proof. Since M is orientable, every element of $H_2(M; \mathbf{Z})$ is representable by an orientable submanifold [9]. Hence some nonzero element is representable by a nonseparating imbedding of $T_{g'}$, some $g' \geqslant 0$. In order to apply both Theorems 1.1 and 2.2, the surface must have genus greater than one. A handle can be added to the imbedded surface within a small Euclidean neighborhood in M of any point of the surface. Thus for any $g > 1$ and $g \geqslant g'$, there is a nonseparating imbedding of T_g in M.

By Theorem 1.1, there is a foliation F of M with a leaf diffeomorphic to T_g, and by Theorem 2.2, there is a closed transversal t to F passing through T_g (which may be assumed to be an imbedded circle meeting T_g exactly once). The standard modification of a foliation can then be performed about t. If this modification is done through k points of T_g, the resulting foliation has a leaf diffeomorphic to $T_{g,k}$. ∎

COROLLARY 3.2. *If the Hurewicz homomorphism* $\pi_2(M) \to H_2(M; \mathbf{Z})$ *is non-trivial (for instance, if M is obtained from a closed, connected manifold by surgery on a* 0*-sphere) then for any* $g > 1$ *and* $k \geqslant 0$*, there is a foliation of M with a leaf diffeomorphic to* $T_{g,k}$.

Proof. The nontriviality of the Hurewicz homomorphism guarantees that M admits a nonseparating imbedding of S^2 [13]. The proof now follows as for the first corollary, adding a sufficient number of handles to apply the theorems (two or more) and then modifying the foliation given by Theorem 1.1. ∎

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