Rational Lie Algebras and p-Isomorphisms of Nilpotent Groups and Homotopy Types.

Comment. Math. Helvetici 50 (1975) 1–8 Birkhäuser Verlag, Basel

Rational Lie Algebras and p-Isomorphisms of Nilpotent Groups and Homotopy Types

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§ 1. Introduction

If $\varphi: G \to H$ is a homomorphism of finitely generated nilpotent groups and p is a prime, then φ is called a p-isomorphism (Hilton [7]) provided (a) $\ker \varphi$ consists of torsion prime to p and (b) given any $y \in H$, there exists $x \in G$ and an integer n prime to p such that $y^n = \varphi(x)$. Similarly, if, $f: X \to Y$ is a map of 1-connected CW-spaces of finite type, then f is called a p-isomorphism or p-equivalence if the induced maps $H_i(f): H_i(X) \to H_i(Y)$ of integral homology groups are all p-isomorphisms in the aforementioned sense.[2])

It seems natural to ask whether the existence of a p-isomorphism $\varphi: G \to H$, resp. p-equivalence $f: X \to Y$, necessarily implies the existence of a p-isomorphism $\psi: H \to G$, resp. p-equivalence $g: Y \to X$ in the opposite direction. Indeed, in the homotopy-theoretic context, this question has been answered in the negative by Mimura-Toda [11], who produced a suitable example with X and Y finite complexes. Because of the close relationship between the category of nilpotent groups and the homotopy category of 1-connected CW-spaces (see Bousfield-Kan [2], [3], [4]; Hilton-Mislin-Roitberg [8], [9], [10]; Roitberg [13]), the author was led to conjecture that there should also be a negative answer in the group-theoretic context. In fact, Milnor constructed an example of a p-isomorphism $\varphi: G \to H$ of finitely generated nilpotent groups such that no map $\psi: H \to G$ is a p-isomorphism, in response to the author's question. This example is presented in § 2.

Milnor's example is based on the construction of a rational nilpotent Lie algebra possessing a certain type of rigidity (originally observed by Joan Dyer [5] in a completely different context) and makes use of the well-known equivalence between the category of rational nilpotent Lie algebras and the category of rational (i.e. torsion-free, divisible) nilpotent groups. Quillen, in his fundamental paper on rational homotopy theory [12], has established a very similar equivalence between the "homotopy

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1) Supported in part by National Science Foundation grants GP36418X1, GP38043.

2) The definitions of p-isomorphism and p-equivalence make sense, of course, for arbitrary groups and spaces. However, we restrict our attention to groups and spaces of the given type since the question we discuss below is of interest primarily in these cases. See [14] for a fuller discussion of this point and for a survey of some related problems.

category" of reduced, rational, differential graded Lie algebras and the homotopy category of 1-connected rational CW-spaces. Following Milnor's lead, we are able to construct a reduced, rational, differential graded Lie algebra possessing properties quite analogous to the J. Dyer-Milnor Lie algebra example and then, via Quillen's theory, to obtain an example of the Mimura-Toda type. This program is carried out in §3.

Our approach in §3 is essentially algebraic and gives a quite systematic and perspicuous view of the Mimura-Toda phenomenon. In particular, we avoid completely the task of making extensive homotopy calculations, such as those in [11]. Moreover, the striking parallels between the constructions in §2 and §3 nicely illustrate the structural similarities between the category of nilpotent groups and the homotopy category of 1-connected CW-spaces.

I am greatly indebted to John Milnor for showing me his construction and allowing me to reproduce it here.

§2

As a general reference for the results required in this, as well as the next, section, we mention [12]. For a group G, or a Lie algebra L, we denote by G^n, or L^n, the nth term in the lower central series.

THEOREM 2.1. (J. Dyer). *There exists a finite-dimensional, rational nilpotent Lie algebra L with the property that any automorphism $\omega: L \to L$ is congruent, modulo L^2, to the identity: $\omega(x) \equiv x \pmod{L^2}$, $x \in L$.*

Proof. See J. Dyer [5].

Remark. An important feature of the Lie algebra L in Th. 2.1 is that it is not homogeneous. (We say that a Lie algebra M is homogeneous if there is a presentation $M = F/R$ where F is a free Lie algebra and where R is homogeneous in the sense that it is generated by homogeneous elements.) It is, in fact, clear (compare [5]) that any homogeneous Lie algebra admits automorphisms which are not congruent to the identity modulo L^2, for example, dilatations.

The fact that L is not homogeneous is relevant to the construction of a graded analog of L in §3 and is further discussed there.

From Th. 2.1, we may now deduce the existence of a "non-invertible" p-isomorphism.

THEOREM 2.2. (Milnor) *There exist finitely generated nilpotent groups G, H and a p-isomorphism $\varphi: G \to H$ such that no map $\psi: H \to G$ is a p-isomorphism.*

Proof. Let L be the Lie algebra guaranteed by Th. 2.1 and let N be the rational nilpotent group corresponding to L under the log-exp equivalence([12]; see also G. Baumslag [1]) between the category of rational nilpotent Lie algebras and the category of rational nilpotent groups. More precisely, N may be thought of as the group generated by elements $\exp a$, $a \in L$, with multiplication given by the Baker-Campbell-Hausdorff formula

$$\exp a \cdot \text{exo}\, b = \exp\left(a + b + \tfrac{1}{2}[a, b] + \cdots\right), \tag{2.3}$$

the remaining terms in the right hand side of (2.3) involving higher order brackets in L. From this description of N, it is clear that $N/N^2 \cong L/L^2$.

Let $x_1, \ldots, x_m \in N$ map onto a basis of the vector space N/N^2 and let H be the subgroup of N generated by $x_1, \ldots, x_m$. We clearly have a surjection $\alpha: H \twoheadrightarrow \mathbf{Z}^m$ of H onto a lattice in N/N^2 and we define G to be the kernel of $\sigma \circ \alpha$, where $\sigma: \mathbf{Z}^m \twoheadrightarrow \mathbf{Z}/q$ is any surjection onto the cyclic group of order q, q being any prime $\neq p$. Thus G is a proper (normal) subgroup of H such that $\alpha(G)$ is a proper subgroup of $\alpha(H) = \mathbf{Z}^m$: in fact, $[H:G] = [\alpha(H): \alpha(G)] = q$. If we write $\varphi: G \to H$ for the inclusion, it follows that φ is a p-isomorphism, in fact an l-isomorphism for all primes $l \neq q$.

If there were a p-isomorphism $\psi: H \to G$, then we could form the composition $\chi = \varphi \circ \psi: H \to H$, which would then also be a p-isomorphism. By the manner in which G is constructed, we note that χ, and more importantly, the induced map $\bar{\chi}: H/H^2 \to H/H^2$, is not the identity. It follows that the rationalization χ_0: $N \to N$ of χ is an automorphism (compare [7]) and that $\bar{\chi}_0: N/N^2 \to N/N^2$ is not the identity. (The notation $\bar{\chi}_0$ is unambiguous by virtue of Th. 5.6 of [7].) Associated to the automorphism $\chi_0: N \to N$, we have an automorphism $\omega: L \to L$ of its associated Lie algebra. But the induced map $\bar{\omega}: L/L^2 \to L/L^2$ is, by Th. 2.1, the identity. By a remark above, $N/N^2 \cong L/L^2$, and furthermore, we may identity $\bar{\chi}_0$ with $\bar{\omega}$, thus arriving at a contradiction.

§3

We begin by stating the graded analog of Th. 2.1 which will be required. Recall that a graded Lie algebra is said to be reduced if all its non-0 elements have positive degree.

THEOREM 3.1. *There exists a rational, reduced, differential graded Lie algebra L of finite type with the property that any "weak automorphism" $\omega: L \to L$ (that is, the induced homology map $H(\omega): H(L) \to H(L)$ is an automorphism) is congruent, modulo L^2, to the identity.*

L may furthermore be chosen to be of totally finite dimension, that is, if L_n is the set of all elements in L of degree $\leqq n$, then $L = L_n$ for sufficiently large n.

(In fact, for the L we construct, it is the case that any weak automorphism $\omega: L \to L$ is actually equal to the identity. We state the theorem in the above form since that is all that is required for Th. 3.2 below.)

We postpone the proof of Th. 3.1 to the end of this section in order to first state and prove the main result, rederiving an example of the Mimura-Toda type.

THEOREM 3.2. *There exist* 1-*connected* CW-*spaces of finite type* X, Y *and a* p-*equivalence* $f: X \to Y$ *such that no map* $g: Y \to X$ *is a* p-*equivalence.*

X *and* Y *may furthermore be chosen so that either* (a) *they both have only finitely many non*-0 *homotopy groups or* (b) *they both are finite complexes.*

Proof. Let L be the differential graded Lie algebra guaranteed by Th. 3.1 and let W be the rational (that is, $H_i(W)$, $i>0$, is a rational group) CW-space corresponding to L under Quillen's equivalence ([12]) between the category consisting of reduced, rational, differential graded Lie algebras localized with respect to its family of weak isomorphisms $\mu: M \to M'$ (that is, $H(\mu): H(M) \to H(M')$ is an isomorphism) and the homotopy category of 1-connected rational CW-spaces. We do not propose to recall the manner in which W is constructed from L since the details are not needed here; we do have to know, however, that there is a functorial isomorphism $\Pi(W) \cong H(L)$ of the Whitehead product Lie algebra $\Pi(W)$ ($\Pi_i(W) \cong \pi_{i+1}(W)$) and the homology Lie algebra $H(L)$ ([12]).

Since W is 1-connected, it admits a decomposition into "local cells" (see Sullivan [15]) and a simple argument ([15; 3.71, 3.72]) proves the existence of a CW-space Y of finite type whose rationalization is precisely W. (Moreover, if W is precisely m-connected, $m \geqq 1$, it may be assumed that the $(m+2)$-skeleton Y_{m+2} of Y has the form

$$Y_{m+2} = S^{m+1} \vee \dots \vee S^{m+1} \vee S^{m+2} \vee \dots \vee S^{m+2}, \tag{3.3}$$

since attaching $(m+2)$-cells nontrivially to $(m+1)$-cells would only contribute (irrelevant) torsion to the situation). We now construct X and $f: X \to Y$ by a skeletal induction as follows. Put $X_{m+2} = Y_{m+2}$ (see (3.3)) and let $f_{m+2}: X_{m+2} \to Y_{m+2}$ be an arbitrary p-equivalence which induces a nontrivial homomorphism of $\Pi_m = \pi_{m+1}$. For definiteness, we take $f_{m+2} = q \cdot \text{identity}$, q a prime $\neq p$, using the suspension structure on X_{m+2}. Assuming now that we already have a map $f_n: X_n \to Y_n$, $n \geqq m+2$, extending f_{m+2} and which is an l-equivalence for all primes $l \neq q$, we proceed to construct a map $f_{n+1}: X_{n+1} \to Y_{n+1}$ extending f_n and which is again an l-equivalence for all primes $l \neq q$. We have a cofibration

$$\vee S_i^n \xrightarrow{\vee u_i} Y_n \to Y_{n+1},$$

arising from the cellular description of Y. Thinking of the attaching maps $u_i: S_i^n \to Y_n$ as elements of $\pi_n(Y_n)$, we may infer, using the inductive assumption that f_n is an l-equivalence for all primes $l \neq q$, the existence of maps $v_i: S_i^n \to X_n$ and "integers" $q_i: S_i^n \to S_i^n$, each q_i being a power of q, such that

$$f_n \circ v_i \simeq u_i \circ q_i.$$

Thus, if we define X_{n+1} to be the cofibre of $\vee v_i$, we obtain a morphism of cofibrations

$$\begin{array}{ccccc} VS_i^n & \xrightarrow{Vv_i} & X_n & \longrightarrow & X_{n+1} \\ \downarrow{\scriptstyle Vq_i} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ VS_i^n & \xrightarrow{Vu_i} & Y_n & \longrightarrow & Y_{n+1} \end{array},$$

f_{n+1} being, of course, the induced map on cofibres. It is an immediate consequence of the Puppe sequence that f_{n+1} has the desired properties. To conclude the construction, we simply take $X=\varinjlim X_n$, $f=\varinjlim f_n$. Obviously, $f:X\to Y$ is a p-equivalence, in fact an l-equivalence for all primes $l\neq q$.

If there were a p-equivalence $g:Y\to X$, then, as in the proof of Th. 2.2, we could form the composition $h=f\circ g:Y\to Y$, which would then also be a p-equivalence. By the manner in which X is constructed, we note that $\Pi_m(h):\Pi_m(Y)\to\Pi_m(Y)$ is not the identity. It follows that the rationalization $h_0:W\to W$ of h is a homotopy equivalence and that $\Pi_m(h_0):\Pi_m(W)\to\Pi_m(W)$ is not the identity. (It is necessary to observe that $\Pi_m(h_0)=\Pi_m(h)_0$.) Associated to the homotopy equivalence $h_0:W\to W$, we have a weak automorphism $\omega:L\to L$ of its associated Lie algebra. Since, by Th. 3.1, $\omega(x)\equiv x \pmod{L^2}$, it follows that the restriction of ω to the elements of smallest degree, namely m,[1]) is precisely the identity. Hence the induced homology map $H_m(\omega):H_m(L)\to H_m(L)$ is also the identity. But by a remark above, we may identify $H_m(\omega)$ with $\Pi_m(h_0)$ and thus arrive at a contradiction.

The final assertion of Th. 3.2 may be verified by appealing to the final assertion of Th. 3.1 and to the fact that the p-isomorphism $f:X\to Y$ we have constructed restricts to a p-isomorphism on each skeleton of X.

We turn now to the proof of Th. 3.1. The construction is motivated by the computations in [11] but we work in an entirely algebraic framework, basing ourselves on the theory of basic commutators in graded Lie algebras; see Hilton [6].[2])

Let F be the free, graded Lie algebra over $\mathbf{Q}$ generated by elements a, b, c having degrees 1, 3, 2 respectively. A differential $d:F\to F$ is imposed by defining d on the generators as follows:

$$da=0,\quad db=aa,\quad dc=0.$$

Here we write aa for $[a, a]$. In general, we use an abbreviated notation for iterated brackets, writing for example acb in place of $[a,[c, b]]$ and $(ab)\,cab$ in place of $[[a,b],[c,[a,b]]]$.

[1]) It is important to point out that the Lie algebra L we construct has the property that not all its elements of smallest degree are boundaries.

[2]) It is perhaps worth mentioning that the proof of Th. 2.1 is based on the theory of basic commutators for free (ungraded) Lie algebras. I am indebted to Joan Dyer for a helpful conversation concerning the theory of basic commutators.

The basic commutators of weight 1 in F are a, b and c and we order these by declaring $a<c<b$. Now the basic commutators of weight n in F are the products xy where x and y are basic commutators satisfying:

(i) $x<y$ (The basic commutators may be ordered arbitrarily except that we insist that $x<y$ whenever weight $(x)<$weight (y).),

(ii) weight $(x)+$weight $(y)=n$,

(iii) if $y=y_1y_2$, then $x\geqq y_1$.

In particular, the degree 10 elements $cccaac$, $(ac)\,(ac)\,ab$ and $(ab)\,cab$ are all basic commutators, provided only that we require $ac<ab$, which we do. Furthermore, each of these elements is a cycle with respect to the differential operator d. In fact, $d(ab)= =da\cdot b-a\cdot db=-aaa=0$ (by the Jacobi identity; recall we are working over $\mathbf{Q}$) and since also $da=0=dc$, it readily follows that the three elements in question are themselves cycles. Let R be the ideal generated by

$$z=cccaac+(ac)\,(ac)\,ab+(ab)\,cab.$$

By what we have just said, it follows that R is, in fact, a differential ideal. Thus the quotient $L=F/R$ is itself a differential graded Lie algebra and it is plainly reduced and of finite type.

Now let $\omega:L\to L$ be a weak automorphism. We have, for any morphism ω,

$$\omega(a)=ra,\quad \omega(b)=sb+u(ac),\quad \omega(c)=tc+v(aa), \tag{3.4}$$

where r, s, t, u, $v\in\mathbf{Q}$. Writing α, γ for the homology classes of a, c respectively, we obtain from the first and third equations in (3.4),

$$H_1(\omega)\,(\alpha)=r\alpha,\quad H_2(\omega)\,(\gamma)=t\gamma,$$

since $\alpha\alpha=0$. Since ω is assumed to be a weak automorphism, it follows that $r\neq 0$, $t\neq 0$. Moreover, from the first and second equations in (3.4) and the relation $\omega(db)=d\omega(b)$, we deduce, since $aa\neq 0$, that $s=r^2$. We thus have

$$\begin{aligned}
\omega(cccaac)&\equiv r^2t^4(cccaac)\pmod{L^7},\\
\omega((ac)\,(ac)\,ab)&\equiv r^5t^2((ac)\,(ac)\,ab)\pmod{L^7},\\
\omega((ab)\,cab)&\equiv r^6t((ab)\,cab)+r^4tu((aac)\,cab)+r^4tu((ab)\,caac)-\\
&\quad -2r^6v((ab)\,aaab)\pmod{L^7}.
\end{aligned}\tag{3.5}$$

(In the third equation in (3.5), we have used $(aa)\,ab=-2aaab$.) Equations (3.5) express the images under ω of the three summands comprising z ($=0$ in L) in terms of basic commutators, provided only that we require $aac<cab$, which we do. Hence, since the basic commutators are linearly independent ([6]) and since $\omega(z)=\omega(0)=0$,

we are forced to conclude that $r^2t^4 = r^5t^2 = r^6t$, and $r^4tu = 0 = r^6v$. As $r \neq 0$, $t \neq 0$, we deduce $r = t = 1$, $u = v = 0$, that is, $\omega =$ identity.

To verify the final assertion of Th. 3.1, it is only necessary to replace L, by, say, L/L^7.

Remarks. (1) It would be interesting to see if an example validating Th. 3.2 exists in which the differential is trivial. There is no apparent way of generalizing the examples in the ungraded case. For instance, if we were simply to take such an ungraded Lie algebra and assign the same positive degree to each of the generators, the non-homogeneous relations (see Remark after Th. 2.1) would split up into several relations and the rigidity of the Lie algebra would very likely disappear.

(2) For the maps $\varphi: G \to H$, $f: X \to Y$ of Ths. 2.2, 2.3, we may p-localize to obtain $\varphi_p: G_p \cong H_p$, $f_p: X_p \simeq Y_p$. Thus, an isomorphism between the p-localizations of two finitely generated nilpotent groups, resp. 1-connected CW-spaces of finite type, need not lift to a p-isomorphism of the groups, resp. spaces, at least not in both directions. On the other hand, given $G_p \cong H_p$, resp. $X_p \simeq Y_p$, there always exist nilpotent groups K, K', resp. 1-connected spaces Z, Z', together with p-isomorphisms

$$G \to K \leftarrow H, \quad G \leftarrow K' \to H, \quad \text{resp.} \quad X \to Z \leftarrow Y, \quad X \leftarrow Z' \to Y.$$

In the group-theoretic context, this is Th. 6.8 of [7] and the argument of [7] applies equally well in the homotopy-theoretic context.

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Received April 18, 1974