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Compact Quadratic s-Manifolds

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The definition of a Riemannian regular s-manifold (M, g, s) is similar to that of a Riemannian symmetric space but without the condition that symmetries have order two. A regularity condition (trivially satisfied for symmetric spaces) is imposed on the composition of symmetries. Such manifolds are known to be homogeneous ([2], [4]) and classification problems reduce essentially to the study of automorphisms of Lie groups. In this connection, recent work of Wolf and Gray [7] is of fundamental importance for cases when the symmetries have finite order.

A metrisable regular s-manifold (M, s) is defined by relaxing the unique choice of g in (M, g, s) to that of any g compatible with the s-structure. There is an obvious equivalence relation on the class of such manifolds, and we seek theorems which are valid up to this equivalence.

For any (M, s) there is an associated tensor field S of type $(1, 1)$ and (M, g, s) is symmetric if and only if S has linear minimal polynomial. We treat the case when S has a quadratic minimal polynomial; then (M, s) is called a quadratic s-manifold. Any such (M, s) admits an almost complex structure Φ canonically associated with S; moreover, either all symmetries have order 3 or Φ is integrable and there exists a metric g for which (M, g, s) is a Riemannian regular s-manifold and (M, g) is Hermitian symmetric with respect to Φ. This paper gives a classification up to equivalence of all compact quadratic (M, s).

§1 is mostly expository, but improves slightly some known results; it contains most basic definitions and properties for later use. In particular, it is easily seen that for any (M, s) the simply connected covering space $\tilde{M}$ of M admits a metrisable structure $(\tilde{M}, \tilde{s})$ whose symmetries cover those of (M, s). Then if (M, s) and (M', s') are covered by $(\tilde{M}, \tilde{s})$ the equivalence of (M, s) and (M', s') reduces to a study of certain deck transformations of $\tilde{M}$. We also develop for later use the relation between (M, s) and a triple (G, H, θ) where G is a Lie group acting transitively on M with isotropy group H, and θ is an automorphism of G determined by s. The section concludes with some remarks on the smoothness of the map s and tensor field S.

In §2 the notion of a quadratic s-manifold is defined and four theorems are stated

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giving the structure theory of such manifolds. Proofs of these theorems are given in §3, with the exception of certain details which form the two appendices.

§1. Preliminaries

DEFINITION 1.1. A *regular* s-manifold is a connected manifold M together with a map s from M into the group Diff$M$ of all diffeomorphisms of M with the following properties:

(i) for each $p \in M$, the point p is an isolated fixed point of the diffeomorphism $s(p)$ (written as s_p),

(ii) $s_p \circ s_q = s_{s_p(q)} \circ s_p$ for all $p, q \in M$,

(iii) the tensor field $S: M \to T_1^1(M)$ defined by $p \mapsto S_p = (s_{p*})_p$ is smooth.

The diffeomorphism s_p is referred to as the *symmetry* at p, and S as the *symmetry tensor field.* Any smooth map $x: M \to M$ is called *s-preserving* (resp. *S-preserving*) if $x \circ s_p = s_{x(p)} \circ x$ for all $p \in M$ (resp. $x_*(SX) = S(x_* X)$ for all $X \in \mathfrak{X}(M)$). Any tensor field on M is called *s-invariant* if it is invariant under the action of s_p for each $p \in M$.

DEFINITION 1.2. Let k be an integer $\geqslant 2$. A *k-symmetric* space is a regular s-manifold (M, s) for which each symmetry has order k; that is, for all $p \in M$, $(s_p)^k = \text{id}$ but $(s_p)^h \neq \text{id}$ for $0 < h < k$.

DEFINITION 1.3. The regular s-manifold (M, s) and the regular s'-manifold (M', s') are said to be *equivalent* if there exists a diffeomorphism $f: M \to M'$ such that $f \circ s_p = s'_{f(p)} \circ f$ for all $p \in M$.

DEFINITION 1.4. Let $\alpha: \bar{M} \to M$ be a covering space. Then $(\bar{M}, \bar{s})$ is said to *cover* (M, s) if $\alpha \circ \bar{s}_{\bar{p}} = s_{\alpha(\bar{p})} \circ \alpha$ for all $\bar{p} \in \bar{M}$.

Remark 1.5. Given (M, s), let $\alpha: \tilde{M} \to M$ be the simply-connected covering space of M. Define for each $\tilde{p} \in \tilde{M}$ the symmetry $\tilde{s}_{\tilde{p}}$ as the lift of $s_{\alpha(\tilde{p})}$ which fixes $\tilde{p}$; then $(\tilde{M}, \tilde{s})$ is a regular $\tilde{s}$-manifold and covers (M, s). We call $(\tilde{M}, \tilde{s})$ the simply-connected covering space of (M, s).

For the converse problem of obtaining each (M, s) covered by $(\tilde{M}, \tilde{s})$ we have the following criterion.

PROPOSITION 1.6. *Let $(\tilde{M}, \tilde{s})$ be a simply-connected regular $\tilde{s}$-manifold and $\alpha: \tilde{M} \to M$ a covering space with group of deck transformations Γ. Then M admits a regular s-manifold structure (M, s) covered by $(\tilde{M}, \tilde{s})$ if and only if Γ is a group of $\tilde{s}$-preserving diffeomorphisms and each symmetry $\tilde{s}_{\tilde{p}}$ normalises Γ in Diff$\tilde{M}$.*

Proof. Let Γ be a group of $\tilde{s}$-preserving diffeomorphisms normalised by each $\tilde{s}_{\tilde{p}}$. For each $p \in M$ choose $\tilde{p} \in \alpha^{-1}(p)$; because $\tilde{s}_{\tilde{p}}$ normalises Γ, the relation $s_p \circ \alpha = \alpha \circ \tilde{s}_{\tilde{p}}$

defines a diffeomorphism $s_p: M \to M$. Moreover, s_p is well-defined (independently of the choice of $\tilde{p} \in \alpha^{-1}(p)$), because each element of Γ is $\tilde{s}$-preserving. Properties (i), (ii) and (iii) of Definition 1.1 are readily verified, and (M, s) is a regular s-manifold covered by $(\tilde{M}, \tilde{s})$.

Conversely, suppose (M, s) is covered by $(\tilde{M}, \tilde{s})$. Then for each $\tilde{p} \in \tilde{M}$ and $\gamma \in \Gamma$, $\alpha \circ \tilde{s}_{\tilde{p}} \circ \gamma \circ \tilde{s}_{\tilde{p}}^{-1} = \alpha$, whence $\tilde{s}_{\tilde{p}} \circ \gamma \circ \tilde{s}_{\tilde{p}}^{-1} \in \Gamma$; hence, the symmetries $\tilde{s}_{\tilde{p}}$ normalise Γ. Observe also that $\gamma \circ \tilde{s}_{\tilde{p}} \circ \gamma^{-1} = \tilde{s}_{\gamma(\tilde{p})}$, each being the lift of $s_{\alpha(\tilde{p})}$ which fixes $\gamma(\tilde{p})$; thus each $\gamma \in \Gamma$ is $\tilde{s}$-preserving.

DEFINITION 1.7. (a) A *metrisable regular* s-manifold is a regular s-manifold (M, s) which admits an s-invariant Riemannian metric.

(b) A *Riemannian regular* s-manifold (M, g, s) is a regular s-manifold (M, s) together with an s-invariant Riemannian metric g.

Remark 1.8. (a) Let (M_1, s_1) and (M_2, s_2) be metrisable, and let $s_1 \times s_2: M_1 \times M_2 \to M_1 \times M_2$ be the product map. Then $(M_1 \times M_2, s_1 \times s_2)$ is a metrisable $(s_1 \times s_2)$-manifold.

(b) Let $(\tilde{M}, \tilde{s})$ be the simply-connected covering space of (M, s) with covering map α. Then g is an s-invariant metric on M if and only if $\tilde{g} = \alpha^* g$ is an $\tilde{s}$-invariant, Γ-invariant metric on $\tilde{M}$; in that case we call $(\tilde{M}, \tilde{g}, \tilde{s})$ the simply-connected covering space of (M, g, s).

PROPOSITION 1.9. *For any (M, g, s) the set of all s-preserving isometries is a closed subgroup of the group of all isometries $I(M, g)$ endowed with the compact-open topology.*

Proof. Let (x_n) be any sequence of s-preserving isometries which converges in $I(M, g)$ and let $x_n \to x$. Since M is connected, any isometry is s-preserving if and only if it is S-preserving. Hence each x_n is S-preserving. Since S is continuous, then x is S-preserving and therefore s-preserving. This proves closure.

DEFINITION 1.10. The Lie group $I(M, g, s)$ is the group of all s-preserving isometries of (M, g, s) endowed with the Lie group structure induced by inclusion in $I(M, g)$. Its identity component is denoted $I_0(M, g, s)$.

By the proof of Theorem 1 of [4] we have the following proposition and its immediate corollary:

PROPOSITION 1.11. *Given (M, g, s), any Lie transformation group G of M satisfying $s(M) \subset G$ is transitive on M. In particular, $I(M, g, s)$ is transitive on M.*

COROLLARY 1.12. *The symmetries of a metrisable regular s-manifold are determined by the symmetry at any one point.*

Proposition 1.11 shows that a metrisable regular s-manifold admits a transitive group of s-preserving diffeomorphisms; this yields the following useful criterion for the equivalence of two such manifolds covered by a given simply-connected one.

PROPOSITION 1.13. *Suppose* $(\tilde{M}, \tilde{s})$ *is the common simply-connected covering space of* (M_1, s_1) *and* (M_2, s_2) *where, for* $i=1, 2$, (M_i, s_i) *is metrisable. For* $i=1, 2$ *denote the covering by* $\alpha_i: \tilde{M} \to M_i$ *and let* Γ_i *be the group of deck transformations. Choose a base point* $\tilde{p} \in \tilde{M}$. *Then* (M_1, s_1) *and* (M_2, s_2) *are equivalent if and only if there exists an* $\tilde{s}$-*preserving diffeomorphism* $\tilde{f}: \tilde{M} \to \tilde{M}$ *such that* $\tilde{f}(\tilde{p}) = \tilde{p}$ *and* $\tilde{f}\Gamma_1\tilde{f}^{-1} = \Gamma_2$.

Proof. Suppose (M_1, s_1) and (M_2, s_2) are equivalent. Define $p_1 = \alpha_1(\tilde{p})$, $p_2 = \alpha_2(\tilde{p})$. By Proposition 1.11 some s_2-preserving group is transitive on M_2; it follows that there exists a diffeomorphism $f: M_1 \to M_2$ such that $f(p_1) = p_2$ and $f \circ (s_1)_q = (s_2)_{f(q)} \circ f$ for all $q \in M_1$. Define $\tilde{f}$ to be the lift of f which fixes $\tilde{p}$. Thus $f \circ \alpha_1 = \alpha_2 \circ \tilde{f}$, and consequently $\tilde{f} \circ \Gamma_1 \circ \tilde{f}^{-1} = \Gamma_2$. Also, for $\tilde{q} \in \tilde{M}$, $\tilde{f} \circ \tilde{s}_{\tilde{q}} = \tilde{s}_{\tilde{f}(\tilde{q})} \circ \tilde{f}$, each being the lift of $f \circ (s_1)_{\alpha_1(\tilde{q})}$ which maps $\tilde{q}$ to $\tilde{f}(\tilde{q})$; thus, $\tilde{f}$ is $\tilde{s}$-preserving.

Conversely, suppose given an $\tilde{s}$-preserving diffeomorphism $\tilde{f}: \tilde{M} \to \tilde{M}$ such that $\tilde{f}(\tilde{p}) = \tilde{p}$ and $\tilde{f}\Gamma_1\tilde{f}^{-1} = \Gamma_2$. Then the diffeomorphism $f: M_1 \to M_2$ is defined by $f \circ \alpha_1 = \alpha_2 \circ \tilde{f}$, and it follows that $f \circ (s_1)_q = (s_2)_{f(q)} \circ f$ for all $q \in M_1$. Thus, (M_1, s_1) and (M_2, s_2) are equivalent.

DEFINITION 1.14. For any (M, g, s) the symmetry group $\Sigma(M, g, s)$ is the topological Lie subgroup of $I(M, g)$ defined on the closure in $I(M, g)$ of the group generated by $s(M)$.

PROPOSITION 1.15. *Let* g_1 *and* g_2 *be* s-*invariant metrics on* (M, s). *Then* $\Sigma(M, g_1, s) = \Sigma(M, g_2, s)$.

Proof. Let Ψ be the group generated by $s(M)$, and let (x_n) be a sequence in Ψ which converges in $I(M, g_1)$ to some element x. Then for each $p \in M$, $x_n(p) \to x(p)$. Since (x_n) is also a sequence in $I(M, g_2)$, then $x_n \to x$ in $I(M, g_2)$ (cf. Lemma 2.4 of Chapter IV in [3]). Thus $\Sigma(M, g_1, s) \subset \Sigma(M, g_2, s)$, and likewise $\Sigma(M, g_2\ s) \subset \Sigma(M, g_1, s)$; consequently, the two symmetry groups are equal as abstract groups. Each has the compact-open topology, so they coincide as Lie groups.

The following definition is now valid.

DEFINITION 1.16. The *symmetry group* $\Sigma(M, s)$ of a metrisable (M, s) is the symmetry group $\Sigma(M, g, s)$ where g is any s-invariant metric on M.

Note that since $s(M) \subset \Sigma(M, s)$, then by Proposition 1.11, $\Sigma(M, s)$ and its identity component $\Sigma_0(M, s)$ are transitive on M.

DEFINITION 1.17. Let G be a connected Lie group, H a closed subgroup of G, and $\theta \in \mathrm{Aut}\, G$. We call (G, H, θ) a *symmetric triple* if the following three conditions are satisfied:

(1) G acts effectively on the coset space G/H,

(2) $(G^\theta)_0 \subset H \subset G^\theta$ where G^θ is the closed subgroup of G defined by $G^\theta=$ $G^\theta = \{x \in G : \theta(x) = x\}$ and $(G^\theta)_0$ is its identity component,

(3) the subgroup of $\mathrm{Aut}\,\mathfrak{g}$ generated by $\mathrm{Ad}_G H$ and θ_* has compact closure Θ in $\mathrm{Aut}\,\mathfrak{g}$ where $\mathfrak{g}$ denotes the Lie algebra of G.

The next proposition shows how the metrisable regular s-manifolds and symmetric triples are related.

PROPOSITION 1.18. *Let (M, s) be a metrisable regular s-manifold with base point $p \in M$. Let G be any Lie group satisfying:*

(i) *G is a connected Lie group acting transitively on M,*

(ii) *G is normalised by s_p in $\mathrm{Diff}\, M$,*

(iii) *$G \subset I(M, g, s)$ for some s-invariant metric g on M.*

(Such G exist; for instance, $G = \Sigma_0(M, s)$ or $I_0(M, g, s)$.)

Let H be the isotropy subgroup of G at p, and $\nu : G \to M = G/H$ the natural projection. Then there exists a unique $\theta \in \mathrm{Aut}\, G$ such that $s_p \circ \nu = \nu \circ \theta$; moreover, (G, H, θ) is a symmetric triple.

Conversely, let (G, H, θ) be a symmetric triple. Define $M = G/H$; let $\nu : G \to M$ be the natural projection and set $p = \nu(H)$. Then M admits a unique metrisable regular s-manifold structure (M, s) such that

(a) $s_p \circ \nu = \nu \circ \theta$ and

(b) *each element of G is s-preserving; moreover, G satisfies conditions* (i), (ii) *and* (iii).

Proof. Since the Lie group G acts transitively on M and $G \subset I(M, g, s) \subset I(M, g)$, it follows (Remark 2, p. 176 of [3]) that G is a topological Lie subgroup of $I(M, g)$. Moreover, since s_p normalises G, the automorphism $\mathrm{ad}(s_p) \in \mathrm{Aut}(I(M, g))$ preserves G. Consequently, $\theta = \mathrm{ad}(s_p)|_G$ defines a Lie group automorphism of G. For $x \in G$, $(s_p \circ \nu)(x) = (s_p \circ x)(p) = (s_p \circ x \circ s_p^{-1})(p) = (\nu \circ \theta)(x)$; thus, $s_p \circ \nu = \nu \circ \theta$. Furthermore, since G acts effectively on M, then θ is the unique automorphism of G satisfying the relation $s_p \circ \nu = \nu \circ \theta$.

Next we check that (G, H, θ) is a symmetric triple. Condition (1) of Definition 1.17 is satisfied, because $G \subset I(M, g)$. Consider $y \in H$; then y is s-preserving, $y(p) = p$, and so $\theta(y) = s_p \circ y \circ s_p^{-1} = s_p \circ s_{y(p)}^{-1} \circ y = y$. Thus, $H \subset G^\theta$. Suppose now $\theta_* X = X$ for some $X \in \mathfrak{g}$, and let $Y = \nu_* X$. Then, $Y = \nu_*(\theta_* X) = (s_p)_* Y$, and so $Y = 0$ because p is an isolated fixed point of s_p, an isometry of (M, g). Consequently, $X \in \ker \nu_* = \mathfrak{h}$ (the Lie algebra of H), and the inclusion $(G^\theta)_0 \subset H$ follows. Thus, (2) of Definition 1.17 is satisfied. Define the Lie group G' as the closure in $I(M, g)$ of the group generated by G and s_p. Then G' has compact isotropy subgroup K at p. Moreover, for each $k \in K$,

G is invariant by $\mathrm{ad}(k)\in\mathrm{Aut}\,G'$; also the action $\lambda:K\times G\to G$ defined by $\lambda(k,x)=\mathrm{ad}(k)x=kxk^{-1}$ is smooth. It follows that the homomorphism $\mu:K\to\mathrm{Aut}\,\mathfrak{g}$ defined by $\mu(k)=\mathrm{Ad}(k)|_{\mathfrak{g}}$ is smooth; (here $\mathrm{Ad}(k)$ denotes the automorphism induced on the Lie algebra of G' by $\mathrm{ad}(k)\in\mathrm{Aut}\,G'$). Consider now the group Θ, the closure of the group generated by $\mathrm{Ad}_G(H)$ and $\theta_*=\mathrm{Ad}(s_p)|_{\mathfrak{g}}$. Since $H\subset K$ and $s_p\in K$, then Θ is closed in the compact group $\mu(K)$. Consequently, Θ is compact. Thus, (3) of Definition 1.17 is also satisfied, and (G, H, θ) is a symmetric triple.

We now turn to the converse of Proposition 1.18, and consider a given symmetric triple (G, H, θ). By (2) of Definition 1.17 we have $\theta_*(\mathfrak{h})=\mathfrak{h}$ (where $\mathfrak{h}$ is the Lie algebra of H), and hence $\Theta(\mathfrak{h})=\mathfrak{h}$. Therefore, as a consequence of (3) of Definition 1.17, there exists a direct sum decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ with $\Theta(\mathfrak{m})=\mathfrak{m}$, and a Θ-invariant positive definite quadratic form B on $\mathfrak{m}$ Let g_p be the corresponding quadratic form induced by ν on the tangent space M_p to $M=G/H$ at $p=\nu(H)$. Then g_p is invariant under the action of H on M_p, and so g_p extends uniquely to a G-invariant Riemannian metric g on M.

Define $s_p\in\mathrm{Diff}\,M$ by $s_p\circ\nu=\nu\circ\theta$. Then s_p is an isometry of (M, g) because B is θ_*-invariant. For each $q\in M$ choose $x\in\nu^{-1}(q)$ and define $s_q=x\circ s_p\circ x^{-1}$, an isometry of (M, g). By (2) of Definition 1.17, s_q is well-defined, and if $X\in M_p$ is non-zero then $(s_p)_*X\neq X$. Since s_p is an isometry it is immediate that p is an isolated fixed point of s_p. It follows that q is an isolated fixed point of s_q for all $q\in M$. Thus (i) of Definition 1.1 is satisfied.

Observing that for $x, y\in G$,
$(s_{x(p)}\circ y)(p)=(x\circ s_p\circ x^{-1}\circ y)(p)=(x\circ s_p\circ\nu)(x^{-1}y)=(x\circ\theta(x^{-1}y))(p)$, a short computation shows that, for all $q, q'\in M$, $s_q\circ s_{q'}=s_{s_q(q')}\circ s_q$. This establishes (ii) of Definition 1.1.

Now the symmetry tensor field S is G-invariant since G is s-preserving. Then smoothness of S follows by using a local cross-section in G; alternatively, one observes that if T is the smooth right-invariant tensor field on G with value θ_* at the identity, then T and S are ν-related, and hence S is smooth. This establishes (iii) of Definition 1.1.

Thus, (M, s) is a regular s-manifold. By construction, g is an s-invariant metric, so (M, s) is metrisable; moreover, G satisfies properties (i), (ii) and (iii). Finally, the conditions that $s_p\circ\nu=\nu\circ\theta$ and that G be s-preserving clearly determine the s-manifold structure on M uniquely.

Remarks 1.19. (a) Given a symmetric triple (G, H, θ) and the corresponding metrisable (M, s) as in Proposition 1.18, we say that (G, H, θ) and (M, s) are *related.*

(b) For later use we make the following observation. Consider a metrisable (M, s) related to a symmetric triple (G, H, θ), and let $\tau\in\mathrm{Aut}\,G$. Define $H'=\tau(H)$ and $\theta'=\tau\theta\tau^{-1}$. Then (G, H', θ') is a symmetric triple, and so determines a related metrisable (M', s'). Define the diffeomorphism $f:M\to M'$ by $f(xH)=\tau(x)H'$ for $xH\in G/H=$

M. From the relation between s, s' and θ, θ' (see (a) of Proposition 1.18), it follows that $f \circ s_{xH} = s'_{f(xH)} \circ f$ for all $xH \in G/H$. Thus, (M, s) and (M', s') are equivalent.

We conclude this section by showing that, for the metrisable case, (iii) of Definition 1.1 may be replaced by other equivalent smoothness conditions.

PROPOSITION 1.20. *Let (M, g) be a connected Riemannian manifold with a map $s: M \to I(M, g)$ satisfying* (i) *and* (ii) *of Definition* 1.1. *Define S as in* (iii) *of Definition* 1.1 *and define a map $\mu: M \times M \to M$ by $\mu(p, q) = s_p(q)$. Then the smoothness of s, S or μ implies the smoothness of all three.*

Proof. Consider the following smooth maps:

$\alpha: I(M, g) \times M \to M$ (Lie group action),
$(0, X): M \to TM \times TM$, defined by $p \mapsto (0_p, X_p)$

where 0 is the zero vector field and X any smooth vector field on M.

Now $\mu = \alpha \circ (s \times \mathrm{id}_M)$, hence s smooth implies μ smooth. Again, if μ is smooth, then $S \circ X = \mu_* \circ (0, X)$, whence $S \circ X$ is smooth for each smooth vector field X and so S is smooth.

Finally, suppose S smooth. Then (M, g, s) is a Riemannian regular s-manifold. Write $G = I(M, g, s)$, and define the following smooth maps:

$i: G \to I(M, g)$ (inclusion)
$\beta: G \times G \to G$ (group multiplication)
$l_{s_p}: G \to G$ (left multiplication by s_p)
$\tau: G \to G$ (group inversion)
$\Delta: G \to G \times G$ (diagonal map).

Now by Proposition 1.11 the Lie group G acts transitively on M, and so for any $p \in M$ there is a smooth cross-section $\lambda: U \to G$ for some open neighbourhood U of p. Then,

$$s|_u = i \circ \beta \circ (\mathrm{id}_G \times (l_{s_p} \circ \tau)) \circ \Delta \circ \lambda.$$

Thus, s is smooth. This completes the proof.

We may regard μ as a multiplication on M, and write $\mu(p, q) = p \cdot q$. Then a smooth map $x: M \to M$ is s-preserving if and only if it is a homomorphism of the multiplication μ, that is

$$x(p \cdot q) = x(p) \cdot x(q) \quad \text{for all } p, q \in M.$$

By (ii) of Definition 1.1 each symmetry s_p is such a homomorphism.

§2. Structure Theory of Quadratic s-Manifolds

The Riemannian symmetric spaces studied by Cartan form the class of Riemannian regular s-manifolds with symmetries of order two; equivalently, the minimal polynomial of the symmetry tensor field S is linear (necessarily then $S+I=0$, since S is orthogonal and $S\neq I$).

Consider now a Riemannian regular s-manifold (M, g, s) for which S has quadratic minimal polynomial, say $\xi^2+\alpha\xi+\beta$. Thus, for each $p\in M$, $S_p^2+\alpha S_p+\beta I_p=0$. Because $I(M, g, s)$ is transitive and S preserving, α and β are constants on M. Now each S_p is orthogonal so its eigenvalues must have modulus one, and since they are roots of the quadratic minimal polynomial these eigenvalues are either real or form a complex conjugate pair. Since (M, g, s) is a regular s-manifold, S_p has no eigenvalue $+1$, so if S had real eigenvalues we would have $S=-I$, contradicting S having quadratic minimal polynomial. Thus S has two eigenvalues $e^{\pm i\phi}$ with $\phi\in]0, \pi[$. Since these are roots of $\xi^2+\alpha\xi+\beta=0$ we have $\xi^2+\alpha\xi+\beta=(\xi-e^{i\phi})(\xi-e^{-i\phi})$, whence $\alpha=-2\cos\phi$ and $\beta=1$.

The next definition introduces the manifolds which form the principal objects of study in this paper.

DEFINITION 2.1. Let $\phi\in]0, \pi[$. A *quadratic s-manifold* (M, s, ϕ) *with angular parameter* ϕ is a metrisable regular s-manifold (M, s) whose symmetry tensor field has quadratic minimal polynomial $\xi^2-2(\cos\phi)\xi+1$.

DEFINITION 2.2. For any (M, s, ϕ) the almost complex structure $\Phi=(\sin\phi)^{-1}\{S-(\cos\phi)I\}$ is called the *canonical almost complex structure.*

The next proposition is immediate.

PROPOSITION 2.3. *Let* (M_1, s_1, ϕ_1) *and* (M_2, s_2, ϕ_2) *be quadratic. Then* $(M_1\times M_2, s_1\times s_2)$ *is quadratic with angular parameter* ϕ *if and only if* $\phi=\phi_1=\phi_2$.

Remark 2.4. (Recall Definition 1.2 of a k-symmetric space.) A quadratic s-manifold $(M, s, 2\pi/3)$ is a metrisable 3-symmetric space, and conversely. For $k\geqslant 3$, any quadratic s-manifold $(M, s, 2m\pi/k)$, where m and k are relatively prime integers, is a metrisable k-symmetric space. However, not every metrisable k-symmetric space is quadratic. For instance, let $(M_1, s_1, 2\pi/k)$ and $(M_2, s_2, 2m\pi/k)$ be quadratic with $1<m<k$ (such (M_i, s_i) exist – cf. Theorem B below); then $(M_1\times M_2, s_1\times s_2)$ is metrisable k-symmetric, but (by Proposition 2.3) is not quadratic.

We now state four theorems which describe the structure of compact quadratic s-manifolds. (Proofs are given in Section 3.) Before stating the theorems we introduce some notation.

$\mathbf{C}^n$ denotes the complex vector space of n-tuples $(z_1,\ldots, z_n)$, and writing each

$z_\alpha = x_\alpha + iy_\alpha$ one has the underlying real vector space $\mathbf{R}^{2n}$ of $2n$-tuples $(x_1, y_1, \ldots, x_n, y_n)$. The natural complex structure $\tilde{J}_0$ on $\mathbf{C}^n$ (or more precisely on $\mathbf{R}^{2n}$) is that induced by scalar multiplication by i on $\mathbf{C}^n$. The natural basis $\{\varepsilon_\alpha, \tilde{J}_0(\varepsilon_\alpha)\}_{\alpha=1,2\ldots,n}$ is defined by the relation

$$(x_1, y_1, \ldots, x_n, y_n) = \sum_{\alpha=1}^n x_\alpha \varepsilon_\alpha + \sum_{\alpha=1}^n y_\alpha \tilde{J}_0(\varepsilon_\alpha);$$

the Euclidean metric on $\mathbf{C}^n$ is defined by the condition that the natural basis be orthonormal. We define the following two real lattices:

Σ^n = the lattice generated by $\{\varepsilon_\alpha, \tilde{J}_0(\varepsilon_\alpha)\}_{\alpha=1,2,\ldots,n}$
Δ^n = the lattice generated by $\{\varepsilon_\alpha, \exp(\pi/3\tilde{J}_0)\,\varepsilon_\alpha\}_{\alpha=1,2,\ldots,n}$.

Theorem A shows that the compact quadratic s-manifolds are of two basic types.

THEOREM A. *Every compact quadratic s-manifold (M, s, ϕ) is one of the following two types.*

(i) *The angular parameter $\phi \neq 2\pi/3$, the canonical almost complex structure Φ is integrable, and, for any s-invariant metric g, (M, g) is a Hermitian symmetric space with respect to Φ. Moreover, if (M, g) has non-trivial Euclidean factor $\mathbf{C}^n/\Gamma$ in its symmetric space decomposition, then the lattice Γ is invariant by $\exp(\phi\tilde{J}_0)$ and necessarily $\phi = \pi/3$ or $\pi/2$.*

(ii) *The angular parameter $\phi = 2\pi/3$ and (M, s, ϕ) is a metrisable 3-symmetric space.*

The next two theorems (B and C) classify the compact quadratic s-manifolds of types (i) and (ii), thus affording a converse of Theorem A.

DEFINITION 2.5. Let (M, g) be a Hermitian symmetric space with complex structure J. We say a quadratic s-manifold structure (M, s, ϕ) is *associated* with the given Hermitian symmetric space if J is the canonical almost complex structure of (M, s, ϕ) and g is an s-invariant metric.

Theorem B describes the quadratic s-manifold structures associated with compact Hermitian symmetric spaces, and gives the classification (up to equivalence) of the s-manifolds of type (i) in Theorem A.

THEOREM B. (i) *Let (M_1, g_1) be a Hermitian symmetric space of compact type with complex structure J_1. Then, for each $\phi \in]0, \pi[$, there is a unique associated (M_1, s_1, ϕ).*

(ii) *Let (M_2, s_2, ϕ) be associated with a compact Hermitian symmetric space (M_2, g_2) of Euclidean type. Then $\phi = \pi/3$, $\pi/2$ or $2\pi/3$, M_2 is complex analytically diffeomorphic to $\mathbf{C}^n/\Sigma^n$ or $\mathbf{C}^n/\Delta^n$, and (M_2, s_2, ϕ) is equivalent to $(\mathbf{C}^n/\Lambda_\phi, \sigma_\phi, \phi)$ where*

$\Lambda_\phi = \Sigma^n$ *for* $\phi = \pi/2$, and

$\Lambda_\phi=\Delta^n$ for $\phi=\pi/3$ *or* $2\pi/3$;

in each case the symmetries are determined by

$$(\sigma_\phi)_{\alpha(0)}\circ\alpha=\alpha\circ\exp(\phi\tilde{J}_0),$$

where $\alpha:\mathbf{C}^n\to\mathbf{C}^n/\Lambda_\phi$ *is the natural projection.*

(iii) *Let* (M, s, ϕ) *be associated with a compact Hermitian symmetric space* $(M, g)=(M_1, g_1)\times(M_2, g_2)$ *with complex structure* J, *where the factors* (M_1, g_1) *of compact type and* (M_2, g_2) *of Euclidean type are each non-trivial. Then* $\phi=\pi/3, \pi/2$ *or* $2\pi/3$, *and* (M, s, ϕ) *is equivalent to* $(M_1\times\mathbf{C}^n/\Lambda_\phi, s_1\times\sigma_\phi, \phi)$ *for some* (M_1, s_1, ϕ), $(\mathbf{C}^n/\Lambda_\phi, \sigma_\phi, \phi)$ *defined in* (i), (ii) *above.*

(iv) *The compact quadratic s-manifolds with angular parameter* $\neq 2\pi/3$ *are* (*up to equivalence*) *precisely the folowing*

the (M_1, s_1, ϕ) *of* (i) *with* $\phi\neq 2\pi/3$,

the $(\mathbf{C}^n/\Delta^n, \sigma_{\pi/3}, \pi/3)$ *and* $(\mathbf{C}^n/\Sigma^n, \sigma_{\pi/2}, \pi/2)$ *of* (ii), *and the products* $(M_1\times\mathbf{C}^n/\Lambda_\phi, s_1\times\sigma_\phi, \phi)$ *of* (iii) *with* $\phi=\pi/3$ *or* $\pi/2$.

We now turn to case (ii) of Theorem A. Each coset space G/H given in Tables 1, 2 and 3 in §6 of [7] is defined by an automorphism $\theta\in\operatorname{Aut}G$ of order 3. In each case (G, H, θ) is a symmetric triple, and so determines (cf. Remarks 1.19) a related metrisable 3-symmetric space (M, s). We refer to these particular symmetric triples (G, H, θ) (and to the related (M, s)) as primitive. It follows from Proposition 1.18 and §6 of [7] that the simply-connected compact metrisable 3-symmetric spaces are (up to equivalence) precisely the products $(M_1\times M_2\times\cdots\times M_r, s_1\times s_2\times\cdots\times s_r)$ where each (M_i, s_i) is a primitive 3-symmetric space. The next theorem shows how all compact metrisable 3-symmetric spaces are constructed.

THEOREM C. *Let* (G_i, K_i, θ_i) *be primitive symmetric triples for* $i=1, 2, \dots, r$. *Let* $G_0(=\mathbf{R}^{2n})$ *be the translation group of a Euclidean vector space* $\mathbf{C}^n$ *with complex structure* $\tilde{J}_0$, *and write* $\theta_0=\exp(2\pi/3\tilde{J}_0)$. *Define*

$$\tilde{G}=G_0\times G_1\times\cdots\times G_r$$
$$\tilde{K}=\{0\}\times K_1\times\cdots\times K_r$$
$$Z=G_0\times Z_1\times\cdots\times Z_r$$
$$\tilde{\theta}=\theta_0\times\theta_1\times\cdots\times\theta_r,$$

where Z_i *denotes the centre of* G_i. *Let* Γ *be a discrete subgroup of* Z *such that* $\tilde{\theta}(\Gamma)=\Gamma$ *and* $\Gamma\cap G_0$ *is a* $2n$-*lattice. Define* $G=\tilde{G}/\Gamma$, *and let* $K=\pi(\tilde{K})$ *where* $\pi:\tilde{G}\to G$ *is the covering homomorphism. Define* $\theta\in\operatorname{Aut}G$ *by* $\theta\circ\pi=\pi\circ\tilde{\theta}$. *Then* (G, K, θ) *is a symmetric triple and the related* (M, s) *is a compact metrisable* 3-*symmetric space.*

Conversely, each compact metrisable 3-*symmetric space is equivalent to some* 3-*symmetric space constructed as above. Thus the compact quadratic s-manifolds of type* (ii) *in Theorem* A *are, up to equivalence, precisely the above* 3-*symmetric spaces.*

Remark 2.6. Observe that (M, s) is covered by the simply-connected (not necessarily compact) 3-symmetric space $(\tilde{M}, \tilde{s})$ related to the symmetric triple $(\tilde{G}, \tilde{K}, \tilde{\theta})$. For the covering $\alpha: \tilde{G}/\tilde{K} \to G/K$, defined by $\alpha(\tilde{x}\tilde{K}) = \pi(\tilde{x})K$, satisfies $\alpha \circ \tilde{s}_{\tilde{p}} = s_{\alpha(\tilde{p})} \circ \alpha$ for all $\tilde{p} \in \tilde{M} = \tilde{G}/\tilde{K}$.

Remark 2.7. Keep the notation of Theorem C. Let Λ be any discrete subgroup of Z. Then a typical element of Λ is of the form $(\lambda_0, \lambda_1, \ldots, \lambda_r)$, and for $\alpha = 0, 1, \ldots, r$ one has the subgroup Λ_α of Z_α consisting of those elements of Z_α appearing in the αth place for some element of Λ. (We call Λ_α the αth slot group of Λ.) Clearly $\bar{\theta}(\Lambda) = \Lambda$ implies $\theta_\alpha(\Lambda_\alpha) = \Lambda_\alpha$ for each α. The subgroups $\Lambda_0 \subset G_0$ invariant by θ_0 are precisely the conjugates of the triangle lattice Δ^n by non-singular complex linear transformations of $\mathbf{C}^n$; see the proof of Theorem B for details, particularly the identities (2) and (3). The next theorem treats explicitly the covering space problem for any primitive 3-symmetric space, and so gives for $\alpha > 0$ the possible θ_α-invariant subgroups $\Lambda_\alpha \subset Z_\alpha$.

THEOREM D. *Let $(\tilde{M}, \tilde{s})$ be the simply-connected compact metrisable* 3-*symmetric space related to a primitive symmetric triple $(\tilde{G}, \tilde{K}, \tilde{\theta})$. Then the (compact) metrisable* 3-*symmetric spaces (M, s) covered by $(\tilde{M}, \tilde{s})$ are, up to equivalence, precisely the* 3-*symmetric spaces related to the symmetric triples (G, K, θ) constructed as in Theorem* C *in terms of the $\tilde{\theta}$-invariant central subgroups Γ of $\tilde{G}$. Explicitly, the possibilities for such Γ are as follows:*

(i) *Suppose $\tilde{K}$ is a maximal rank subgroup of $\tilde{G}$; that is, $(\tilde{G}, \tilde{K}, \tilde{\theta})$ occurs in Table* 1 *or* 2 *in* §6 *of* [7]. *Then there is no non-trivial $\tilde{\theta}$-invariant Γ.*

(ii) *Suppose $\tilde{G} = \mathrm{Spin}\,8$, and $\tilde{K} = G_2$ or $SU(3)/\mathbf{Z}_3$ with the corresponding $\tilde{\theta}$ in each of the two cases determined by Theorem* 5.5 *of* [7]. *Then, in each case, $Z(\mathrm{Spin}\,8) = \mathbf{Z}_2 \times \mathbf{Z}_2$ is the only non-trivial $\tilde{\theta}$-invariant Γ.*

(iii) *Suppose $\tilde{G} = L \times L \times L/Z^*$, $\tilde{K} = L^*/Z^*$, where L is a simply-connected compact simple Lie group with centre Z (the symbol "*" denotes diagonal embedding); here $\tilde{\theta}$ is the automorphism induced on $\tilde{G}$ by cyclic permutation of the simple factors in $L \times L \times L$. The $\tilde{\theta}$-invariant central subgroups Γ of L^3/Z^* are given in* §A2 *of Appendix* A. *Then,* 3-*symmetric spaces related to symmetric triples (G, K, θ) constructed in terms of distinct Γ are inequivalent except when $L = D_{2k}$ $(k \geqslant 2)$. For $L = D_{2k}$ $(k \geqslant 2)$ there are precisely five non-trivial, proper, $\tilde{\theta}$-invariant subgroups Γ, denoted Γ_i $(i = 1, 2, \ldots, 5)$ in* §A2 *of Appendix* A. *In the case $L = D_{2k}$ $(k > 2)$ the* 3-*symmetric spaces determined by Γ_1 and Γ_2 are equivalent, likewise those determined by Γ_4 and Γ_5. In the case $L = D_4$ the* 3-*symmetric spaces determined by Γ_1, Γ_2 and Γ_3 are equivalent, likewise those determined by Γ_4 and Γ_5.*

§3. Proofs of Theorems

Proof of Theorem A. (i) Let (M, s, ϕ) be a compact quadratic s-manifold with $\phi \neq 2\pi/3$. Let g be any s-invariant metric on M, and let ∇ be the corresponding Riemannian connection. Since ∇S is s-invariant we have $S(\nabla_X S)(Y) = (\nabla_{SX} S)(SY)$ for all $X, Y \in \mathscr{X}(M)$. Consequently, for any eigenvalues ϱ_1, ϱ_2 of S (not necessarily distinct) and corresponding complex eigenvectors U, V at any point of M, $S(\nabla_U S)(V) = \varrho_1 \varrho_2 (\nabla_U S)(V)$. Since $\phi \neq 2\pi/3$, and ± 1 are not eigenvalues of S, then $\varrho_1 \varrho_2$ is not an eigenvalue of S, and it follows that $\nabla S = 0$. Consequently, the canonical almost complex structure $\Phi = (\sin\phi)^{-1}(S - (\cos\phi)I)$ is parallel with respect to ∇, and so Φ is integrable. Moreover, for each $p \in M$, S_p is orthogonal, and hence Φ_p is orthogonal. Thus (M, g) is a Kähler manifold with respect to Φ. Since S is parallel, (M, g) is locally Riemannian symmetric (cf. [4]); since Φ is parallel, then (M, g) is locally Hermitian symmetric with respect to Φ.

We now show that (M, g) is (globally) Hermitian symmetric with respect to Φ. Consider the simply-connected covering space $(\tilde{M}, \tilde{g})$ with the complex structure $\tilde{\Phi}$, where $\tilde{g}$ and $\tilde{\Phi}$ are the lifts of g and Φ. Then $(\tilde{M}, \tilde{g})$ is a Hermitian symmetric space with respect to $\tilde{\Phi}$. We have the decomposition $(\tilde{M}, \tilde{g}) = (\mathbf{C}^n, \tilde{g}_0) \times (\tilde{M}_1, \tilde{g}_1) \times \cdots \times (\tilde{M}_r, \tilde{g}_r)$, where $(\mathbf{C}^n, \tilde{g}_0)$ is a complex Euclidean vector space with complex structure $\tilde{J}_0$, and for $i = 1, 2, \ldots, r$, $(\tilde{M}_i, \tilde{g}_i)$ is an irreducible Hermitian symmetric space of compact type with complex structure $\tilde{J}_i$; moreover $\tilde{\Phi} = \tilde{J}_0 \times \tilde{J}_1 \times \cdots \times \tilde{J}_r$ (cf. the proof of Proposition 5.5 in Chapter VIII of [3]).

For the Riemannian covering $(\tilde{M}, \tilde{g}) \to (M, g)$ the group Γ of deck transformations is a group of Clifford translations of $(\tilde{M}, \tilde{g})$ because (M, g) is Riemannian homogeneous (cf. [6], Theorem 2.7.5). Furthermore, any $\gamma \in \Gamma$ is decomposable as $\gamma = \gamma_0 \times \gamma_1 \times \cdots \times \gamma_r$ where γ_0, γ_i are Clifford translations of $(\mathbf{C}^n, \tilde{g}_0)$, $(\tilde{M}_i, \tilde{g}_i)$ resp. (Corollary 3.1.4 of [5]). Let Γ_α be the αth slot group of Γ for $\alpha = 0, 1, \ldots, r$ (cf. Remark 2.7). Since $\tilde{\phi}$ is the lift of Φ, then $\tilde{\Phi}$ is Γ-invariant and from the above decomposition of $\tilde{\Phi}$ it follows that each $\tilde{J}_\alpha$ is Γ_α-invariant. Define $M_0 = \mathbf{C}^n/\Gamma_0$ and $M_i = \tilde{M}_i/\Gamma_i$ for $i = 1, 2, \ldots, r$. Let g_0, g_i and J_0, J_i be the metrics and parallel complex structures induced on the M_0, M_i respectively. Since Γ_0 is a group of translations on the real Euclidean space underlying $\mathbf{C}^n_1$ and Γ_0 preserves $\tilde{J}_0$, then (M_0, g_0) is a compact Hermitian symmetric space of Euclidean type. We claim that for $i = 1, 2, \ldots, r$ the group Γ_i is trivial. Consider the following two possibilities.

(a) Suppose $(\tilde{M}_i, \tilde{g}_i)$ is not a complex projective space $P_{2m\pm 1}(\mathbf{C})$ of odd complex dimension $2m+1 \geqslant 3$, nor a space $SO(4m+2)/U(2m+1)$ with $m > 0$. Then Theorem 5.5.1 of [5] implies that Γ_i is finite and centralises $I_0(\tilde{M}_i, \tilde{g}_i)$, and that (M_i, g_i) is Riemannian symmetric. Because J_i is a parallel complex structure on (M_i, g_i), then (M_i, g_i) is Hermitian symmetric with respect to J_i; it is of compact type, whence M_i is simply-connected. Thus Γ_i is trivial.

(b) Suppose $(\tilde{M}_i, \tilde{g}_i)$ is either $P_{2m\pm1}(\mathbf{C})$ or $SO(4m+2)/U(2m+1)$ for $m>0$. If (M_i, g_i) is Riemannian symmetric then, as in (a), Γ_i is trivial. If, for some non-trivial Γ_i, (M_i, g_i) is not Riemannian symmetric, then by Chapter 9 of [6] or 5.5.5 and 5.5.6 of [5], $\Gamma_i=\{1, \delta_i\}$ where δ_i is anti-holomorphic. This contradicts the fact that Γ_i preserves J_i. So again Γ_i must be trivial.

Since all the Γ_i are trivial, we have $\Gamma=\Gamma_0$. Consequently $(M, g)=(M_0, g_0)\times\times(\tilde{M}_1, \tilde{g}_1)\times\cdots\times(\tilde{M}_r, \tilde{g}_r)$ and $\Phi=J_0\times\tilde{J}_1\times\cdots\tilde{J}_r$. Thus (M, g) is a (globally) Hermitian symmetric space with respect to Φ.

Suppose now that the factor (M_0, g_0) is non-trivial (i.e., $\dim M_0=n>0$). From the above decomposition of Φ, it follows that the symmetry tensor field has a similar decomposition; consequently, $s=s_0\times s_1\times\cdots\times s_r$ such that for $\alpha=0, 1, \ldots, r$, the map $s_\alpha: M_\alpha\to \operatorname{Diff} M_\alpha$ defines a quadratic s_α-manifold $(M_\alpha, s_\alpha, \phi)$ with angular parameter ϕ. Consider now the simply-connected covering space $(\mathbf{C}^n, \tilde{s}_0)$ of (M_0, s_0) (cf. Remark 1.5). By Proposition 1.6, the symmetry $(\tilde{s}_0)_0$ at the origin 0 of $\mathbf{C}^n$ normalises Γ. Make the standard identification of $\mathbf{C}^n$ with the tangent space to $\mathbf{C}^n$ at 0, consider Γ as a lattice in $\mathbf{C}^n$, and write $A=(\tilde{s}_0)_0$; then Γ is invariant by the transformation A. Since (M_0, s_0, ϕ) is quadratic, then $A^2-2(\cos\phi)A+I=0$. Let $\{\tau_i\}_{i=1,2,\ldots,2n}$ be a set of generators of the lattice Γ, hence also a basis of the $\mathbf{R}^{2n}$ underlying $\mathbf{C}^n$. Define the matrix W by $A(\tau_i)=\sum_{j=1}^{2n} W_i^j\tau_j$; because A leaves Γ invariant, W is an integer matrix. We have det $W=1$, and

$$W^2-2(\cos\phi)W+I=0;$$

therefore, $(2\cos\phi)^{2n}=\det(W^2+I)\in\mathbf{Z}$. Since $\xi^2-2(\cos\phi)\xi+1$ is the minimal polynomial of W, then $\det(W-\xi I)=(\xi^2-2(\cos\phi)\xi+1)^n$. The term linear in ξ shows that $2n\cos\phi\in\mathbf{Z}$, that is, $\cos\phi=m/2n$ for some $m\in\mathbf{Z}$. Since $(2\cos\phi)^{2n}\in\mathbf{Z}$, it follows that $(m/n)^{2n}\in\mathbf{Z}$ and so $m/n\in\mathbf{Z}$ which implies that $\cos\phi=0, \pm\frac{1}{2}$, or ±1. By assumption, $\phi\in]0, \pi[$ and $\phi\neq 2\pi/3$; thus, $\phi=\pi/3$ or $\pi/2$. This completes the proof for case (i).

(ii) Let $\phi=2\pi/3$. Then (M, s, ϕ) is a metrisable 3-symmetric space by Remark 2.4. This completes the proof of Theorem A.

Proof of Theorem B. (i) Given a Hermitian symmetric space (M_1, g_1) of compact type with complex structure J_1, then the Lie group $G=I_0(M_1, g_1)$ is a compact semi-simple group of Hermitian isometries acting transitively on M_1. Thus, $M_1=G/H$ where H denotes the isotropy subgroup of G at some point $p\in M_1$. Now H may be identified with the linear isotropy group at $p=\nu(H)$ where $\nu: G\to M_1$ is the natural projection, and then $(J_1)_p$ may be considered as an element of the Lie algebra of H (cf. Chapter VIII in [3]). Given $\phi\in]0, \pi[$, consider $\exp(\phi(J_1)_p)\in H$, and define $\theta=\operatorname{ad}(\exp(\phi(J_1)_p))\in\operatorname{Aut} G$. Then (G, H, θ) is a symmetric triple, and determines a unique related metrisable regular s_1-manifold (M_1, s_1, ϕ) satisfying (a) and (b) in Proposition 1.18. From (a) it follows that $(s_1)_p=\exp(\phi(J_1)_p)$. By (b), G is a group of

s_1-preserving diffeomorphisms. Consequently, since G is a transitive group of Hermitian isometries of (M_1, g_1) with respect to J_1, the metric g_1 is s-invariant and $(s_1)_q = \exp(\phi(J_1)_q)$ for all $q \in M_1$. It follows that the symmetry tensor field $S_1 = \exp\phi J_1 = (\cos\phi)I + (\sin\phi)J_1$, and so J_1 is the canonical almost complex structure on (M_1, s_1). Moreover, $(S_1)^2 - 2(\cos\phi)\, S_1 + I = 0$, whence (M_1, s_1) is quadratic with angular parameter ϕ; thus, (M_1, s_1, ϕ) is associated with the given Hermitian symmetric space. Suppose (M_1, s, ϕ) is also associated with the given Hermitian symmetric space; then, $s_p = (s_1)_p$, for their differentials have the common value $\exp(\phi(J_1)_p)$ at p and each is an isometry of (M_1, g_1). Since the symmetry s_p determines (M_1, s) (cf. Corollary 1.12), this proves uniqueness of the associated (M_1, s_1, ϕ).

(ii) Given a compact Hermitian symmetric space (M_2, g_2) of Euclidean type, then $M_2 = \mathbf{C}^n/\Gamma$ where Γ is a $2n$-dimensional real lattice in $\mathbf{C}^n$ and g_2 is the flat metric induced by the Euclidean metric on $\mathbf{C}^n$. The complex structure J_2 on M_2 is that induced by the complex structure $\tilde{J}_0$ on $\mathbf{C}^n$. Consider (M_2, s_2, ϕ) associated with the given Hermitian symmetric space, and the simply-connected covering space $(\mathbf{C}^n, \tilde{s}, \phi)$ of (M_2, s_2, ϕ) where for all $\tilde{p}, \tilde{q} \in \mathbf{C}^n$

$$\tilde{s}_{\tilde{p}}(\tilde{q}) = \tilde{p} + \exp(\phi\tilde{J}_0)\,(\tilde{q} - \tilde{p}). \tag{1}$$

Observe that, by Proposition 1.6, the lattice Γ must be invariant by the orthogonal transformation $A = \exp(\phi\tilde{J}_0)$ of the Euclidean vector space $\mathbf{R}^{2n}$ underlying $\mathbf{C}^n$; cf. the last paragraph in the proof for case (i) of Theorem A.

Let $e_1 \neq 0$ be a lattice point of Γ nearest to the origin 0 of $\mathbf{R}^{2n}$. Since the eigenvalues of A are $e^{\pm i\phi}$ with $\phi \in]0, \pi[$, then $\{e_1, Ae_1\}$ spans a 2-plane π_1, which is A-invariant because $A^2 = 2(\cos\phi)\, A - I$. Consequently, as in Lemma 3.5.2 of [6], every lattice point in π_1 is an integer linear combination of e_1 and Ae_1; moreover, $\phi = \pi/3$, $\pi/2$ or $2\pi/3$. Consider now a lattice point $e_2 \notin \pi_1$ at minimal distance from π_1. Then $Ae_2 \notin \pi_1$ because π_1 is A-invariant; it follows that $\{e_1, Ae_1, e_2, Ae_2\}$ spans an A-invariant 4-plane π_2. We now show that every lattice point γ lying in π_2 is an integer linear combination e_1, Ae_1, e_2 and Ae_2. Let N be the normal vector from e_2 to π_1; then $N' = A(N)$ is the normal vector from Ae_2 to π_1. By subtracting from γ an appropriate integer linear combination of e_1, Ae_1, e_2, Ae_2, one obtains $\gamma' \in \pi_2 \cap \Gamma$ satisfying $\gamma' = ae_1 + a'Ae_1 + be_2 + b'Ae_2$, where a, a', b, b' each have absolute value $\leqslant \frac{1}{2}$. Now the distance from γ' to π_1 is

$$\begin{aligned} d(\gamma', \pi_1) &= \|bN + b'N'\| \\ &< (|b| + |b'|)\|N\| \\ &\leqslant \|N\|, \end{aligned}$$

the first inequality being strict because $N' = A(N)$ cannot be parallel to N (by the eigenvalues of A). Since $\|N\|$ is the distance $d(e_2, \pi_1)$ which is minimal by hypothesis, it follows that $b=b'=0$. Thus $\gamma'\in\pi_1$, and so a and a' are integers. Hence, γ is an integer linear combination of e_1, Ae_1, e_2, Ae_2 as claimed. Continuing by induction, one obtains $\{e_i, Ae_i\}_{i=1,2,\dots,n}$ which generates Γ and forms a basis of $\mathbf{R}^{2n}$.

As just shown in the preceding paragraph, $\phi=\pi/3$, $\pi/2$ or $2\pi/3$. Consider now the standard basis $\{\varepsilon_i, \tilde{J}_0(\varepsilon_i)\}_{i=1,2,\dots,n}$ of $\mathbf{R}^{2n}$, and observe that the lattice Λ_ϕ generated by $\{\varepsilon_i, A\varepsilon_i\}_{i=1,2,\dots,n}$ is Σ^n for $\phi=\pi/2$ and Δ^n for $\phi=\pi/3$ or $2\pi/3$. (Δ^n and Σ^n are defined in §2 immediately before the statement of Theorem A.) Define the non-singular real linear transformation $\tilde{f}:\mathbf{R}^{2n}\to\mathbf{R}^{2n}$ by $\tilde{f}(e_i)=\varepsilon_i$, $\tilde{f}(Ae_i)=A\varepsilon_i$. Then

$$\tilde{f}\Gamma\tilde{f}^{-1}=\Lambda_\phi \tag{2}$$

$$\tilde{f}A\tilde{f}^{-1}=A. \tag{3}$$

Equation (3) is equivalent to

$$\tilde{f}\circ\tilde{J}_0=\tilde{J}_0\circ\tilde{f} \tag{3'}$$

which is the condition that $\tilde{f}$ be complex linear. Consequently, we have a complex analytic diffeomorphism $f:\mathbf{C}^n/\Gamma\to\mathbf{C}^n/\Lambda_\phi$ defined by $f\circ\alpha'=\alpha\circ\tilde{f}$ where $\alpha':\mathbf{C}^n\to\mathbf{C}^n/\Gamma$ and $\alpha:\mathbf{C}^n\to\mathbf{C}^n/\Lambda_\phi$ denote the natural projections.

Observe that $A=\exp(\phi\tilde{J}_0)$ leaves the lattice Λ_ϕ invariant. Consequently, Λ_ϕ is a group of $\tilde{s}$-preserving translations of $\mathbf{C}^n$. Moreover, by (1) and (3′), each symmetry $\tilde{s}_{\tilde{p}}$ normalises Λ_ϕ. Therefore (cf. Proposition 1.6) there exists a quadratic σ_ϕ-manifold $(\mathbf{C}^n/\Lambda_\phi, \sigma_\phi, \phi)$ covered by $(\mathbf{C}^n, \tilde{s}, \phi)$ with symmetries satisfying

$$(\sigma_\phi)_{\alpha(\tilde{p})}\circ\alpha=\alpha\circ\tilde{s}_{\tilde{p}} \quad \text{for } \tilde{p}\in\mathbf{C}^n.$$

Now $\tilde{f}(0)=0$, and from (3′) we have that $\tilde{f}$ is $\tilde{s}$-preserving; hence, equation (2) and Proposition 1.13 imply that (M_2, s_2, ϕ) is equivalent to $(\mathbf{C}^n/\Lambda_\phi, \sigma_\phi, \phi)$.

(iii) Consider now a compact Hermitian symmetric space $(M,g)=(M_1,g_1)\times\times(M_2,g_2)$ with complex structure $J=J_1\times J_2$, where (M_1,g_1) (resp. (M_2,g_2)) is the factor of compact (resp. Euclidean) type. We suppose $\dim M_1>0$ and $\dim M_2>0$, for otherwise the situation reduces to (i) or (ii). As in (i) and (ii) we can write $M_1=G/K$ and $M_2=\mathbf{C}^n/\Gamma$, and we have the natural projections $\nu:G\to M_1$ and $\alpha':\mathbf{C}^n\to\mathbf{C}^n/\Gamma$. We write $p=\nu(K)\in M_1$ and $p'=\alpha'(0)\in M_2$. Suppose now that (M,s,ϕ) is associated with the Hermitian symmetric space (M,g) with complex structure J. Then the symmetry at $(p,p')\in M=M_1\times M_2$ has differential $\exp(\phi J_{(p,p')})=\exp(\phi(J_1)_p)\times\exp(\phi(J_2)_{p'})$ at the point (p,p'), and consequently

$$s_{(p,p')}=(s_1)_p\times(s_2)_{p'}. \tag{4}$$

Now an element of $G\times\mathbf{C}^n$ acts as a Hermitian isometry on $(M_1\, g)$, and, since (M,s,ϕ) is associated with the given Hermitian symmetric space, we deduce that such an element preserves the symmetry tensor field S and hence preserves s (cf. the proof of Proposition 1.9). As a consequence of (4), $s=s_1\times s_2$, and hence $(M, s, \phi)=(M_1, s_1, \phi)\times(M_2, s_2, \phi)$. Then (iii) follows from (i) and (ii).

(iv) Consider a compact quadratic s-manifold (M, s, ϕ) with angular parameter $\phi\neq 2\pi/3$. By Theorem A the canonical almost complex structure Φ is integrable and, for any s-invariant metric g, (M, g) is a compact Hermitian symmetric space with respect to Φ. Then (iv) follows immediately from Definition 2.5 and (i), (ii), (iii) above. This completes the proof of Theorem B.

Proof of Theorem C. From Proposition 1.18 and the fact that $\bar{\theta}^3=\mathrm{id}$, it follows that the construction described in Theorem C yields a metrisable 3-symmetric space (M, s). Since $\Gamma\cap G_0$ is a $2n$-lattice, $G=\bar{G}/\Gamma$ is compact, so $M=G/K$ is compact.

Now consider any compact metrisable 3-symmetric space (M, s), and write $G=\Sigma_0(M, s)$. Thus, G is a compact connected Lie group acting effectively and transitively on M. Choose a point $p\in M$, and let K denote the isotropy subgroup of G at p. By Proposition 1.18 there exists $\theta\in\mathrm{Aut}\,G$ such that (G, K, θ) is a symmetric triple related to (M, s), and $\theta^3=\mathrm{id}$.

Let $\tilde{G}$ be the simply-connected covering group of G, denote the covering homomorphism by $\tilde{\pi}:\tilde{G}\to G$, and let $D=\ker\tilde{\pi}$. Define $K^*=(\tilde{\pi}^{-1}(K))_0$; then $\tilde{G}/K^*$ is the simply-connected covering space of G/K with the projection induced by $\tilde{\pi}$. Let N be the kernel of the natural action of $\tilde{G}$ on $\tilde{G}/K^*$; thus, N is a closed normal subgroup of $\tilde{G}$ and $N\subset K^*$. Define $\bar{G}=\tilde{G}/N$ and $\bar{K}=K^*/N$; then $\bar{G}/\bar{K}$ is an effective coset space diffeomorphic to $\tilde{G}/K^*$. The group $\tilde{\pi}(N)$ is normal in G and $\tilde{\pi}(N)\subset K$, hence $\tilde{\pi}(N)=\{e\}$ because G/K is effective, and thus $N\subset D$. Thus, the kernel Γ of the covering $\pi:\bar{G}\to G$ is a discrete central subgroup of $\bar{G}$ isomorphic to D/N. The automorphism $\theta\in\mathrm{Aut}\,G$ is covered by a unique $\tilde{\theta}\in\mathrm{Aut}\,\tilde{G}$. Because (G, K, θ) is a symmetric triple, θ fixes K pointwise; consequently, since K^* is connected, $\tilde{\theta}$ fixes K^* pointwise. Since $N\subset K^*$ one thus has $\tilde{\theta}(N)=N$, and so $\tilde{\theta}$ covers a unique $\bar{\theta}\in\mathrm{Aut}\,\bar{G}$. It follows that $\bar{\theta}(\Gamma)=\Gamma$ and that $\bar{\theta}$ covers θ.

From §6 of [7] and Remarks 1.20 it follows that, up to equivalence of the 3-symmetric spaces, we can assume that $\bar{G}$, $\bar{K}$, and $\bar{\theta}$ admit the following decompositions:

$$\bar{G}=G_0\times G_1\times\cdots\times G_r,$$
$$\bar{K}=\{0\}\times K_1\times\cdots\times K_r,$$
$$\bar{\theta}=\theta_0\times\theta_1\times\cdots\times\theta_r,$$

for some set of primitive (G_i, K_i, θ_i), $i=1, 2, \ldots, r$, and G_0 the translation group of some Euclidean vector space $\mathbf{C}^n$ with complex structure $\mathcal{J}_0$ and $\theta_0=\exp(\phi\mathcal{J}_0)$. As noted above, Γ is a discrete central subgroup of $\bar{G}$ and $\bar{\theta}(\Gamma)=\Gamma$. Since G is compact, then $\Gamma\cap G_0$ is a $2n$-lattice. Finally, from Theorem 6.4 of [7] it follows that $K=\pi(\bar{K})$. This completes the proof of Theorem C.

Proof of Theorem D. We first prove two lemmas.

LEMMA 3.1. *Let $(\tilde{M}, \tilde{g}, \tilde{s})$ be the simply-connected covering space of a Riemannian regular s-manifold (M, g, s). Let $\tilde{G}'$ be the group of $\tilde{s}$-preserving isometries of $(\tilde{M}, \tilde{g})$ which preserve the fibres of the covering $\alpha: \tilde{M} \to M$ with the Lie group structure induced by inclusion in $I(\tilde{M}, \tilde{g}, \tilde{s})$, and let $G' = I(M, g, s)$. Then there exists a smooth covering homomorphism $\pi': \tilde{G}' \to G'$ with kernel Δ = the group of deck transformations of the covering α. Moreover, let $\tilde{G} = (\tilde{G}')_0$ and $G = I_0(M, g, s)$. Then $\pi = \pi'|_{\tilde{G}}: \tilde{G} \to G$ is a smooth covering homomorphism with kernel $\Gamma = \Delta \cap \tilde{G}$(central in $\tilde{G}$).*

Proof. For $\tilde{x} \in \tilde{G}'$ define the diffeomorphism $x: M \to M$ by $x \circ \alpha = \alpha \circ \tilde{x}$. If $\alpha(\tilde{p}) = p$, then $s_p \circ \alpha = \alpha \circ \tilde{s}_{\tilde{p}}$ and so $x \circ s_p \circ \alpha = x \circ \alpha \circ \tilde{s}_{\tilde{p}} = \alpha \circ \tilde{x} \circ \tilde{s}_{\tilde{p}} = \alpha \circ \tilde{s}_{\tilde{x}(\tilde{p})} \circ \tilde{x} = s_{x(p)} \circ \alpha \circ \tilde{x} = s_{x(p)} \circ x \circ \alpha$. Thus $x \in G'$ and so a map $\pi': \tilde{G}' \to G'$ is defined by $\pi'(\tilde{x}) \circ \alpha = \alpha \circ \tilde{x}$. Since $\pi'(\tilde{x}.\tilde{y}) \circ \alpha = \alpha \circ (\tilde{x}.\tilde{y}) = \pi'(\tilde{x}) \circ \pi'(\tilde{y}) \circ \alpha$, then π' is a homomorphism. Moreover, π' is surjective. For let $x \in G'$, and let $\tilde{x}: \tilde{M} \to \tilde{M}$ be a lift of x. It is easily seen that $\tilde{x}$ is an $\tilde{S}$-preserving isometry of $(\tilde{M}, \tilde{g}, \tilde{s})$ where $\tilde{S}$ is the symmetry tensor field. Hence (cf. Proof of Proposition 1.9) $\tilde{x}$ is $\tilde{s}$-preserving. Thus $\tilde{x} \in \tilde{G}'$, and $\pi'(\tilde{x}) = x$; this proves π' surjective. We next show that π' is a smooth map; for this purpose it suffices to prove continuity since π' is a homomorphism of Lie groups $\tilde{G}'$, G'. Let $(\tilde{x}_n)$ be a sequence in $\tilde{G}'$ such that $\tilde{x}_n \to \tilde{x}$. Then for each $\tilde{p} \in \tilde{M}$, $\tilde{x}_n(\tilde{p}) \to \tilde{x}(\tilde{p})$. Hence $(\pi'(\tilde{x}_n))(\alpha(\tilde{p})) = \alpha(\tilde{x}_n(\tilde{p})) \to \alpha(\tilde{x}(\tilde{p})) = (\pi'(\tilde{x}))(\alpha(\tilde{p}))$. Since G' is a closed topological subgroup of $I(M, g)$ with the compact open topology, it follows that $\pi'(\tilde{x}_n) \to \pi'(\tilde{x})$, so π' is continuous, and hence smooth. The final statement of the lemma is immediate; thus the proof of Lemma 3.1 is complete.

LEMMA 3.2. *Let $(\tilde{M}, \tilde{s})$ be the simply-connected 3-symmetric space related to a primitive symmetric triple $(\tilde{G}, \tilde{K}, \tilde{\theta})$, and let $\tilde{g}$ be any $\tilde{s}$-invariant metric on $\tilde{M}$. Then $\tilde{G} = \Sigma_0(\tilde{M}, \tilde{s}) = I_0(\tilde{M}, \tilde{g}, \tilde{s})$. If, moreover, $(\tilde{M}, \tilde{g}, \tilde{s})$ is the simply-connected covering space of (M, g, s) with covering map $\alpha: \tilde{M} \to M$, then each element of $\tilde{G}$ preserves the fibres of the covering α.*

Proof. Let $\tilde{v}: \tilde{G} \to \tilde{G}/\tilde{K}$ be the natural projection. Because $(\tilde{M}, \tilde{s})$ and $(\tilde{G}, \tilde{K}, \tilde{\theta})$ are related, $\tilde{M} = \tilde{G}/\tilde{K}$, $\tilde{s}_{\tilde{K}} \circ \tilde{v} = \tilde{v} \circ \tilde{\theta}$, and $\tilde{s}_{\tilde{x}\tilde{K}} = \tilde{x} \circ \tilde{s}_{\tilde{K}} \circ \tilde{x}^{-1}$ for all $\tilde{x}\tilde{K} \in \tilde{G}/\tilde{K}$. It follows that the group Ψ generated by the symmetries is contained in $\tilde{G} \cup \tilde{G}.\tilde{s}_{\tilde{K}} \cup \tilde{G}.(\tilde{s}_{\tilde{K}})^2$. By Proposition 1.18, there exists an $\tilde{s}$-invariant metric $\tilde{g}$ on $\tilde{M}$ such that $\tilde{G} \subset I_0(\tilde{M}, \tilde{g}, \tilde{s})$. Consequently, since the Lie group $\tilde{G}$ acts transitively on $\tilde{M}$, it follows that $\tilde{G}$ is a closed topological Lie subgroup of $I_0(\tilde{M}, \tilde{g}, \tilde{s})$ (cf. Remark 2 on page 176 of [3]). Since $\Sigma_0(\tilde{M}, \tilde{s})$ is the identity component of the closure of Ψ in $I(\tilde{M}, \tilde{g}, \tilde{s})$, then $\Sigma_0(\tilde{M}, \tilde{s}) \subset \tilde{G}$. Moreover, since $\tilde{G}$ is a group of $\tilde{s}$-preserving isometries of $(\tilde{M}, \tilde{g})$, then $\Sigma_0(\tilde{M}, \tilde{s})$ is a normal Lie subgroup of $\tilde{G}$. In all cases except $\tilde{G} = L^3/Z^*$, the group $\tilde{G}$ is simple in Tables 1, 2 and 3 of [7], and so in these cases we have $\tilde{G} = \Sigma_0(\tilde{M}, \tilde{s})$. For $\tilde{G} = L^3/Z^*$, one notes that $\Sigma_0(\tilde{M}, \tilde{s})$ must be $\tilde{\theta}$-invariant and since $\tilde{\theta}$ here is induced by cyclic permutation of the three simple factors in $L \times L \times L$, again $\tilde{G} = \Sigma_0(\tilde{M}, \tilde{s})$.

Now, for any $\tilde{s}$-invariant metric $\tilde{g}$ on $\tilde{M}$, $\tilde{G} = \Sigma_0(\tilde{M}, \tilde{s})$ is a normal closed topological Lie subgroup if $I_0(\tilde{M}, \tilde{g}, \tilde{s})$. Moreover, $\bar{\theta}$ is the restriction to $\bar{G}$ of the automorphism $\mathrm{ad}(\tilde{s}_{\bar{K}}) \in \mathrm{Aut}\, I(\tilde{M}, \tilde{g}, \tilde{s})$. Decomposing the compact connected Lie group $I_0(\tilde{M}, \tilde{g}, \tilde{s})$ (which acts transitively and effectively on the simply-connected $\tilde{M}$) as in the first paragraph of §6 in [7], one sees that $\bar{G} = I_0(\tilde{M}, \tilde{g}, \tilde{s})$. This completes the proof of the first statement in the lemma.

Suppose now that $(\tilde{M}, \tilde{g}, \tilde{s})$ covers (M, g, s) with covering map $\alpha: \tilde{M} \to M$. Then $\alpha \circ \tilde{s}_{\tilde{p}} = s_{\alpha(\tilde{p})} \circ \alpha$ for each $\tilde{p} \in \tilde{M}$, whence each symmetry $\tilde{s}_{\tilde{p}}$ is fibre preserving. So the group Ψ generated by the symmetries (and hence its closure $\Sigma(\tilde{M}, \tilde{s})$ in $I(\tilde{M}, \tilde{g}, \tilde{s})$) is fibre preserving. Thus, $\bar{G} = \Sigma_0(\tilde{M}, \tilde{s})$ is fibre preserving, and this completes the proof of Lemma 3.2.

We now return to the proof of Theorem D. Consider then a primitive symmetric triple $(\bar{G}, \bar{K}, \bar{\theta})$ and the related simply-connected compact metrisable 3-symmetric space $(\tilde{M}, \tilde{s})$.

Let Γ be a $\bar{\theta}$-invariant central subgroup of $\bar{G}$; for the covering $\pi: \bar{G} \to G = \bar{G}/\Gamma$, define $K = \pi(\bar{K})$ and define $\theta \in \mathrm{Aut}\, G$ by $\theta \circ \pi = \pi \circ \bar{\theta}$. By Theorem C, (G, K, θ) is a symmetric triple and determines a related metrisable 3-symmetric space (M, s). By Remark 2.6, (M, s) is covered by $(\tilde{M}, \tilde{s})$.

Conversely, suppose (M, s) is a compact metrisable 3-symmetric space covered by $(\tilde{M}, \tilde{s})$, and let Δ be the group of deck transformations of the covering $\alpha: \tilde{M} \to M$. Let g be any s-invariant metric on M, and let $\tilde{g}$ be the corresponding $\tilde{s}$-invariant metric $\alpha^* g$ on $\tilde{M}$. Set $G = I_0(M, g, s)$ and let K be the isotropy subgroup of G at the point $\alpha(\bar{K})$ in M. Now, by Lemma 3.2, $\bar{G} = I_0(\tilde{M}, \tilde{g}, \tilde{s})$ and each element of $\bar{G}$ preserves the fibres of the covering α. Consequently, by Lemma 3.1, there is a smooth covering homomorphism $\pi: \bar{G} \to G$ defined by $\pi(\bar{x}) \circ \alpha = \alpha \circ \bar{x}$ for $\bar{x} \in \bar{G}$, and so $\Gamma = \ker \pi = \Delta \cap \bar{G}$ is a discrete central subgroup of $\bar{G}$. By Lemma 3.2, $\bar{G} = \Sigma_0(\tilde{M}, \tilde{s})$, so the map $\bar{\phi}: \bar{G} \to \bar{G}$, defined by $\bar{\phi}(\bar{x}) = \tilde{s}_{\bar{K}} \circ \bar{x} \circ \tilde{s}_{\bar{K}}^{-1}$ for $\bar{x} \in \bar{G}$, is an automorphism of $\bar{G}$. For $\bar{x}, \bar{y} \in \bar{G}$, $\bar{\phi}(\bar{x}) \bar{y} \bar{K} = \bar{\theta}(\bar{x}) \bar{y} \bar{K}$ because $\tilde{s}_{\bar{K}}(\bar{y}\bar{K}) = \bar{\theta}(\bar{y})\bar{K}$, and hence $\bar{\phi} = \bar{\theta}$ because $\bar{G}/\bar{K}$ is effective. Define $\theta \in \mathrm{Aut}\, G$ by $\theta(x) = s_K \circ x \circ s_K^{-1}$ for $x \in G$. Since $\alpha \circ \tilde{s}_{\bar{K}} = s_K \circ \alpha$, then $(\theta \circ \pi(\bar{x})) \circ \alpha = \alpha \circ (\bar{\theta}(\bar{x}))$ for $\bar{x} \in \bar{G}$, and so $\pi \circ \bar{\theta} = \theta \circ \pi$ by the defining property of π. Consequently $\bar{\theta}(\Gamma) = \Gamma$.

Let $\bar{\pi}: \tilde{G} \to \bar{G}$ be the simply-connected covering group of $\bar{G}$, and define the homomorphism $\tilde{\pi} = \pi \circ \bar{\pi}$. Since $\bar{K} = (\bar{G}^{\bar{\theta}})_0$ and, by Proposition 1.18, $(G^\theta)_0 \subset K \subset G^\theta$, then $(\bar{\pi}^{-1}(\bar{K}))_0 = (\tilde{\pi}^{-1}(K))_0$, since their Lie algebras coincide. Now $\tilde{G}/\bar{\pi}^{-1}(\bar{K})$ is diffeomorphic to $\bar{G}/\bar{K} = \tilde{M}$, because $\bar{\pi}^{-1}(\bar{K})$ is the isotropy subgroup of $\tilde{G}$ acting naturally on $\bar{G}/\bar{K}$; since $\bar{G}/\bar{K}$ is simply-connected, then $\bar{\pi}^{-1}(\bar{K})$ is connected. The kernel of the action of $\tilde{G}$ on $\tilde{G}/\tilde{\pi}^{-1}(K))_0$ equals the kernel of the homomorphism $\bar{\pi}$, and it follows from the proof of Theorem 6.4 of [7] that $K = \pi(\bar{K})$. To summarise, $G = \bar{G}/\Gamma$, $K = \pi(\bar{K})$ and $\theta \circ \pi = \pi \circ \theta$ where $\pi: \bar{G} \to G$ is the natural projection and Γ is a $\bar{\theta}$-invariant central subgroup of $\bar{G}$.

Thus we have shown that the compact metrisable 3-symmetric spaces covered by $(\tilde{M}, \tilde{s})$ are precisely those (M, s) constructed from the symmetric triple $(\bar{G}, \bar{K}, \bar{\theta})$ in terms of $\bar{\theta}$-invariant central subgroups Γ of $\bar{G}$ as described in Theorem C.

With the above notation, the covering $\alpha:\tilde{G}/\tilde{K}\to G/K$ is given by $\alpha(\tilde{x}\tilde{K})=\pi(\tilde{x})K$ for $\tilde{x}\tilde{K}\in\tilde{G}/\tilde{K}$. Since $\tilde{G}/\tilde{K}$ is effective, then $\Gamma\cap\tilde{K}$ is the identity element. It follows that the group of deck transformations of the covering α is precisely Γ, acting naturally on $\tilde{G}/\tilde{K}$ as a subgroup of $\tilde{G}$. Given two $\tilde{\theta}$-invariant central subgroups Γ_1, Γ_2 of $\tilde{G}$, construct, as above, symmetric triples (G_1, K_1, θ_1), (G_2, K_2, θ_2) and the related 3-symmetric spaces (M_1, s_1), (M_2, s_2). Now, by Proposition 1.13, (M_1, s_1) and (M_2, s_2) are equivalent if and only if there exists an $\tilde{s}$-preserving diffeomorphism $\tilde{f}$ of $\tilde{M}=\tilde{G}/\tilde{K}$ such that $\tilde{f}(\tilde{K})=\tilde{K}$ and $\tilde{f}\Gamma_1\tilde{f}^{-1}=\Gamma_2$. Since $\tilde{G}=\Sigma_0(\tilde{M}, \tilde{s})$, such an $\tilde{f}$ exists if and only if there exists $\tilde{\phi}\in\operatorname{Aut}\tilde{G}$ such that $\tilde{\phi}(\tilde{K})=\tilde{K}$, $\tilde{\phi}\tilde{\theta}=\tilde{\theta}\tilde{\phi}$ and $\tilde{\phi}(\Gamma_1)=\Gamma_2$.

We now describe the $\tilde{\theta}$-invariant central subgroups of $\tilde{G}$ for the various primitive symmetric triples $(\tilde{G}, \tilde{K}, \tilde{\theta})$, and check for equivalence of the resultant 3-symmetric spaces.

Case (i): if $(\tilde{G}, \tilde{K}, \tilde{\theta})$ occurs in Table 1 or 2 of §6 in [7], then $\tilde{G}$ has trivial centre, so there are no non-trival $\tilde{\theta}$-invariant Γ.

Case (ii): $\tilde{G}=\mathrm{Spin}\,8$, $\tilde{K}=G_2$ or $SU(3)/\mathbf{Z}_3$.
In the case $\mathrm{Spin}\,8/G_2$ the automorphism $\tilde{\theta}$ is the triality automorphism η of Spin 8; the centre Z (Spin 8)$=\mathbf{Z}_2\times\mathbf{Z}_2$. Since η fixes the identity and cyclically permutes the other three elements of Z (Spin 8), then the only non-trivial η-invariant subgroup of Z (Spin 8) is Z (Spin 8) itself. In the case $\mathrm{Spin}\,8/(SU(3)/\mathbf{Z}_3)$ the automorphism $\tilde{\theta}$ is $\eta\circ\mathrm{ad}(u)$ for some $u\in\mathrm{Spin}\,8$. Since the inner automorphism $\mathrm{ad}(u)$ leaves Z (Spin 8) pointwise fixed, it follows that (as in the G_2 case) the only non-trivial $\eta\circ\mathrm{ad}(u)$-invariant subgroup of Z (Spin 8) is Z (Spin 8) itself.

Case (iii): $\tilde{G}=L^3/Z^*$, $\tilde{K}=L^*/Z^*$, where L is a compact simply-connected simple Lie group with centre Z and "*" denotes diagonal embedding into $L^3=L\times L\times L$; the automorphism $\tilde{\theta}\in\operatorname{Aut}L^3/Z^*$ is induced by cyclic permutation of the simple factors in L^3. For each of the simple groups L, all $\tilde{\theta}$-invariant central subgroups Γ of L^3/Z^* are listed in §A2 of Appendix A. Each such Γ yields a symmetric triple (G, K, θ) and so defines, as described above, a related 3-symmetric space. By §A3 of Appendix A, the equivalence of the resultant 3-symmetric spaces is as stated in Theorem D.

§A1. Automorphisms of L^3/Z^*

Let L be a simply-connected compact simple Lie group with centre Z. Consider the product group $L^3=L\times L\times L$. Let $p_{ijk}\in\operatorname{Aut}L^3$ be defined by

$$p_{ijk}(x_1, x_2, x_3)=(x_i, x_j, x_k),$$

where (i, j, k) is some permutation of $(1, 2, 3)$. Then if $\theta_1, \theta_2, \theta_3\in\operatorname{Aut}L$, it follows that $(\theta_1\times\theta_2\times\theta_3)\circ p_{ijk}\in\operatorname{Aut}L^3$. Conversely, if $\theta\in\operatorname{Aut}L^3$, then, because L is simple, θ permutes the factors of L^3 and so $\theta=(\theta_1\times\theta_2\times\theta_3)\circ p_{ijk}$ for some $\theta_1, \theta_2, \theta_3\in\operatorname{Aut}L$ and some p_{ijk}.

Consider the quotient group L^3/Z^* where $Z^*\{(z, z, z)\in L^3 : z\in Z\}$. Define Aut $(L^3, Z^*)=\{\alpha\in \operatorname{Aut} L^3 : \alpha(Z^*)=Z^*\}$, and let $\pi : L^3 \to L^3/Z^*$ be the natural projection. There is a unique isomorphism β: Aut $(L^3, Z^*)\to \operatorname{Aut} L^3/Z^*$ satisfying $\beta(\phi)\circ\pi=\pi\circ\phi$ for all $\phi\in \operatorname{Aut}(L^3, Z^*)$. Now the elements of $\operatorname{Aut} L^3/Z^*$ are precisely the automorphisms $\beta((\theta_1\times\theta_2\times\theta_3)\circ p_{ijk})$ with θ_1, θ_2 and θ_3 in the same ad$L$-coset of Aut$L$. For $(\theta_1\times\theta_2\times\theta_3)\circ p_{ijk}\in \operatorname{Aut} L^3$ preserves Z^* if and only if $\theta_1(z)=\theta_2(z)=\theta_3(z)$ for all $z\in Z$, and automorphisms in distinct ad$L$-cosets of Aut$L$ have distinct actions on Z.

§ A2. $\bar{\theta}$-invariant Central Subgroups Γ

With the notation of § A1, consider $p_{312}\in \operatorname{Aut} L^3$. Then $p_{312}(Z^*)=Z^*$, and we define $\bar{\theta}=\beta(p_{312})\in \operatorname{Aut} L^3/Z^*$. We now find the $\bar{\theta}$-invariant central subgroups of L^3/Z^* as required for Theorem D.

The centre of L^3/Z^* is Z^3/Z^*, which we identify with $Z\times Z$ via the isomorphism $\delta : Z^3/Z^*\to Z\times Z$ defined by $\delta((z_1, z_2, 1)Z^*)=(z_1, z_2)$ for $z_1, z_2\in Z$. Now $\bar{\theta}((z_1, z_2, 1)Z^*)=(1, z_1, z_2)Z^*=(z_2^{-1}, z_1z_2^{-1}, 1)Z^*$, so $\bar{\theta}$ acts on $Z\times Z$ by $\bar{\theta}(z_1, z_2)=(z_2^{-1}, z_1z_2^{-1})$ for $(z_1, z_2)\in Z\times Z$. We list below the non-trivial $\bar{\theta}$-invariant subgroups $\Gamma\subset Z\times Z$ for the various simple groups L. For cases (a), (b), (c) and (d), the results follow either directly or from Appendix B.

(a) $L=G_2$, F_4 or E_8; Z is trivial. There are no non-trivial $\bar{\theta}$-invariant subgroups $\Gamma\subset Z\times Z$.

(b) $L=B_l$ $(l\geqslant 2)$, C_l $(l\geqslant 3)$ or E_7; $Z=\mathbf{Z}_2$. Here $\Gamma=Z\times Z$ is the only possibility.

(c) $L=E_6$; $Z=\mathbf{Z}_3$. $\Gamma=\{(0, 0), (1, 2), (2, 1)\}$ or $Z\times Z$.

(d) $L=D_{2k+1}$ $(k\geqslant 2)$; $Z=\mathbf{Z}_4$. $\Gamma=\mathbf{Z}_2\times\mathbf{Z}_2$ or $Z\times Z$.

(e) $L=D_{2k}$ $(k\geqslant 2)$; $Z=\mathbf{Z}_2\times\mathbf{Z}_2$. Explicitly let A and B be the generators of the two copies of $\mathbf{Z}_2$; thus $Z=\{A, B : A^2=B^2=1, AB=BA\}$. Directly one finds that the non-trivial $\bar{\theta}$-invariant subgroups of $Z\times Z$ are:

$$\Gamma_1=\{(1, 1), (1, A), (A, A), (A, 1)\},$$
$$\Gamma_2=\{(1, 1), (1, B), (B, B), (B, 1)\},$$
$$\Gamma_3=\{(1, 1), (1, AB), (AB, AB), (AB, 1)\},$$
$$\Gamma_4=\{(1, 1), (AB, A), (A, B), (B, AB)\},$$
$$\Gamma_5=\{(1, 1), (AB, B), (B, A), (A, AB)\},$$

and

$$\Gamma_6=Z\times Z.$$

(f) $L=A_l$ $(l\geqslant 1)$; $Z=\mathbf{Z}_{l+1}$. The $\bar{\theta}$-invariant subgroups of $Z\times Z$ are the groups listed in § B6 of Appendix B (with $n=l+1$).

§ A3. Equivalence

The $\bar{\theta}$-invariant central subgroups Γ of L^3/Z^* described in § A2 are used in Theorem D to construct 3-symmetric spaces. For given L, two $\bar{\theta}$-invariant central subgroups Γ, $\Gamma' \subset L^3/Z^*$ yield equivalent 3-symmetric spaces if and only if there exists $\bar{\Phi} = \operatorname{Aut} L^3/Z^*$ such that $\bar{\Phi}(L^*/Z^*) = L^*/Z^*$, $\bar{\theta}\bar{\Phi} = \bar{\Phi}\bar{\theta}$ and $\bar{\Phi}(\Gamma) = \Gamma'$, where $L^* = \{(a, a, a) \in L^3 : a \in L\}$; for details see proof of Theorem D.

Now the elements of $\operatorname{Aut} L^3/Z^*$ which preserve L^*/Z^* and commute with $\bar{\theta}$ are precisely the automorphisms of the form $\beta((\psi \times \psi \times \psi) \circ (p_{312})^i) = \beta(\psi \times \psi \times \psi) \circ \bar{\theta}^i$ where $\psi \in \operatorname{Aut} L$ and $i = 0$, 1 or 2. For, using the results of § A1, one sees that the elements of $\operatorname{Aut} L^3/Z^*$ which preserve L^*/Z^* are precisely those of the form $\beta((\psi \times \psi \times \psi) \circ p_{ijk})$ for $\psi \in \operatorname{Aut} L$; moreover, such an element commutes with $\bar{\theta} = \beta(p_{312})$ if and only if $p_{ijk} = \mathrm{id}$, p_{312} or p_{231}, because β is an isomorphism and p_{312} commutes with $\psi \times \psi \times \psi$.

Observe that $(\beta(\psi \times \psi \times \psi))\ ((z_1, z_2, 1)Z^*) = (\psi(z_1), \psi(z_2), 1)Z^*$ for $\psi \in \operatorname{Aut} L$, $z_1, z_2 \in Z$. Hence, identifying the centre of L^3/Z^* with $Z \times Z$ as in § A2, two $\bar{\theta}$-invariant central subgroups Γ, $\Gamma' \subset Z \times Z$ yield equivalent 3-symmetric spaces if and only if there exists $\psi \in \operatorname{Aut} L$ such that $(\psi \times \psi)\ (\Gamma) = \Gamma'$.

For $L \neq D_{2k}$ $(k \geqslant 2)$, then each $\psi \in \operatorname{Aut} L$ either fixes Z pointwise or maps each element of Z to its inverse, and so $\psi \times \psi$ preserves any $\bar{\theta}$-invariant subgroup $\Gamma \subset Z \times Z$. Consequently, for $L \neq D_{2k}$ $(k \geqslant 2)$, distinct Γ yield inequivalent 3-symmetric spaces.

Consider now $L = D_{2k}$ $(k \geqslant 2)$ and the $\bar{\theta}$-invariant subgroups $\Gamma_i \subset Z \times Z$, $i = 1, 2, \ldots, 5$, defined in § A2. For $L = D_{2k}$ $(k > 2)$, it follows from § A1 that, for any outer automorphism $\psi \in \operatorname{Aut} L$, $\psi \times \psi$ interchanges Γ_1 and Γ_2, likewise Γ_4 and Γ_5, whilst preserving Γ_3. Consequently, Γ_1 and Γ_2 define equivalent 3-symmetric spaces, likewise Γ_4 and Γ_5; there are no other equivalences. For $L = D_4 = \operatorname{Spin} 8$, we have two basic outer automorphisms; namely, α which fixes AB and interchanges A and B, and the triality automorphism η which cyclically permutes $A \to B \to AB \to A$. Then $E(D_4)$ is the dihedral group generated by α and η. For $\psi \in E(D_4)$, the action of $\psi \times \psi$ on the $\Gamma_i \subset Z \times Z$ is as follows:

$\alpha \times \alpha$ interchanges Γ_1, Γ_2, interchanges Γ_4, Γ_5, and preserves Γ_3;
$(\eta \circ \alpha) \times (\eta \circ \alpha)$ interchanges Γ_1, Γ_3, interchanges Γ_4, Γ_5, and preserves Γ_2;
$(\eta^2 \circ \alpha) \times (\eta^2 \circ \alpha)$ interchanges Γ_2, Γ_3, interchanges Γ_4, Γ_5, and preserves Γ_1;
$\eta \times \eta$ permutes $\Gamma_1 \to \Gamma_2 \to \Gamma_3 \to \Gamma_1$, and preserves Γ_4, Γ_5;
$\eta^2 \times \eta^2$ permutes $\Gamma_1 \to \Gamma_3 \to \Gamma_2 \to \Gamma_1$, and preserves Γ_4, Γ_5.

Consequently, Γ_1, Γ_2 and Γ_3 yield equivalent 3-symmetric spaces, likewise Γ_4 and Γ_5; there are no other equivalences.

Appendix B

Let $\mathbf{Z}_n$ denote the additive group of integers modulo n. If $a, b \in \mathbf{Z}$ and $a(\text{mod}\,n)$, $b(\text{mod}\,n)$ denote their respective equivalence classes modulo n, then we denote $(a(\text{mod}\,n), b(\text{mod}\,n)) \in \mathbf{Z}_n \oplus \mathbf{Z}_n$ by (a, b). We denote the greatest common divisor of a and b by $[a, b]$.

The purpose of this appendix is to find all the subgroups $\Gamma \subset \mathbf{Z}_n \oplus \mathbf{Z}_n$ invariant by the automorphism θ_n defined by $\theta_n((a, b)) = (-b, a-b)$ for $(a, b) \in \mathbf{Z}_n \oplus \mathbf{Z}_n$. These groups are needed in the proof of Theorem D; cf. also §A2 of Appendix A.

§B1

PROPOSITION. *Let $n = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_r^{\lambda_r}$ be the prime power decomposition of n with $p_1 < p_2 < \dots < p_r$, and write $n_i = p^{\lambda_i}$. The isomorphism $\mathbf{Z}_n = \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \dots \oplus \mathbf{Z}_{n_r}$ yields the isomorphism $\mathbf{Z}_n \oplus \mathbf{Z}_n = (\mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_1}) \oplus (\mathbf{Z}_{n_2} \oplus \mathbf{Z}_{n_2}) \oplus \dots \oplus (\mathbf{Z}_{n_r} \oplus \mathbf{Z}_{n_r})$; moreover, $\theta_n = \theta_{n_1} \times \theta_{n_2} \times \dots \times \theta_{n_r}$. Hence a subgroup $\Gamma \subset \mathbf{Z}_n \oplus \mathbf{Z}_n$ is θ_n-invariant if and only if $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \dots \oplus \Gamma_r$ for some θ_{n_i}-invariant subgroups $\Gamma_i \subset \mathbf{Z}_{n_i} \oplus \mathbf{Z}_{n_i}$, $i = 1, 2, \dots, r$.*

Proof. Given θ_{n_i}-invariant subgroups $\Gamma_i \subset \mathbf{Z}_{n_i} \oplus \mathbf{Z}_{n_i}$, one sees immediately that $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \dots \oplus \Gamma_r$ is invariant by θ_n.

Conversely, suppose $\Gamma \subset \mathbf{Z}_n \oplus \mathbf{Z}_n$ is θ_n-invariant. For $i = 1, 2, \dots, r$ let π_i be the projection of $\mathbf{Z}_n \oplus \mathbf{Z}_n$ onto the factor $\mathbf{Z}_{n_i} \oplus \mathbf{Z}_{n_i}$, and define $\Gamma_i = \pi_i(\Gamma)$. Because $\theta_n(\Gamma) = \Gamma$, $\theta_{n_i}(\Gamma_i) = \Gamma_i$ for $i = 1, 2, \dots, r$. We claim that $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \dots \oplus \Gamma_r$. Since it is immediate that $\Gamma \subset \Gamma_1 \oplus \Gamma_2 \oplus \dots \oplus \Gamma_r$, it remains only to show that $\Gamma_1 \oplus \Gamma_2 \oplus \dots \oplus \Gamma_r \subset \Gamma$. For a fixed i, consider any $(a_i, b_i) \in \Gamma_i$. By definition of Γ_i, there exists $z \in \Gamma$ such that $\pi_i(z) = (a_i, b_i)$. Define $m = n^2 \cdot (n_i)^{-2}$. Observing that $(n_j)^2$ is the order of $\mathbf{Z}_{n_j} \oplus \mathbf{Z}_{n_j}$, one sees that $\pi_j(m \cdot z) = (0, 0)$ for $j \neq i$. Because $[m, (n_i)^2] = 1$, there exist integers s, t such that $sm = 1 - t(n_i)^2$, and so $sm \cdot (a_i, b_i) = (a_i, b_i)$. Consequently $((0, 0), \dots, (a_i, b_i), \dots, (0, 0)) = sm \cdot z \in \Gamma$; thus $\{(0, 0)\} \oplus \dots \oplus \Gamma_i \oplus \dots \oplus \{(0, 0)\} \subset \Gamma$ for $i = 1, 2, \dots, r$, and so $\Gamma_1 \oplus \Gamma_2 \oplus \dots \oplus \Gamma_r \subset \Gamma$. This completes the proof of the proposition.

§B2

The above proposition reduces the problem to the case of a prime power. Consider then in this section a subgroup $\Gamma \subset \mathbf{Z}_n \oplus \mathbf{Z}_n$ satisfying $\theta_n(\Gamma) = \Gamma$, where $n = p^\lambda$. Define Λ_1 (resp. Λ_2) as the subgroup of $\mathbf{Z}_n$ obtained by projecting Γ on the first (resp. second) factor in $\mathbf{Z}_n \oplus \mathbf{Z}_n$. Suppose $a \in \Lambda_1$, and that $(a, x) \in \Gamma$. Then $\theta_n^2((a, x)) = \theta_n(-x, a-x) = (x-a, -a)$, whence $a \in \Lambda_2$. Hence, $\Lambda_1 \subset \Lambda_2$; likewise $\Lambda_2 \subset \Lambda_1$, so $\Gamma \subset \Lambda \oplus \Lambda$ where $\Lambda = \Lambda_1 = \Lambda_2$. We call Λ the slot group of Γ. Now Λ is a subgroup of $\mathbf{Z}_n$, so $\Lambda \cong \mathbf{Z}_{p^\sigma}$ for some $0 \leqslant \sigma \leqslant \lambda$. The order of Γ is $\geqslant p^\sigma$, and since $\Gamma \subset \Lambda \oplus \Lambda$, the order of Γ is $p^{\sigma+\mu}$ for some $0 \leqslant \mu \leqslant \sigma$. We denote by $\Gamma_\alpha(p^\lambda, \sigma, \mu)$ the θ_n-invariant subgroups of $\mathbf{Z}_n \oplus \mathbf{Z}_n$

($n=p^\lambda$) which have slot group $\Lambda=\mathbf{Z}_{p^\sigma}$ and order $p^{\sigma+\mu}$; here α runs over some indexing set necessarily finite and possibly void for certain values of p, λ, σ, μ. Observe that the subgroup $\Gamma_\alpha(p^\lambda, \sigma, \mu)$ of $\mathbf{Z}_{p^\lambda}\oplus\mathbf{Z}_{p^\lambda}$ are isomorphic to the subgroups $\Gamma_\alpha(p^\sigma, \sigma, \mu)$ of $\mathbf{Z}_{p^\sigma}\oplus\mathbf{Z}_{p^\sigma}$.

§B3

By the last observation, our problem is finally reduced to the following: for $n=p^\sigma$ and $0\leqslant\mu\leqslant\sigma$ find all subgroups $\Gamma\subset\mathbf{Z}_n\oplus\mathbf{Z}_n$ satisfying

(i) $\theta_n(\Gamma)=\Gamma$

(ii) order of $\Gamma=p^{\sigma+\mu}$

and

(iii) the associated slot group $\Lambda\cong\mathbf{Z}_n$.

In the next three sections we solve this problem explicitly. For $\mu=\sigma$ there is exactly one subgroup of $\mathbf{Z}_n\oplus\mathbf{Z}_n$ satisfying (i), (ii) and (iii); namely, $\mathbf{Z}_n\oplus\mathbf{Z}_n$ itself, denoted $\Gamma(p^\sigma, \sigma, \sigma)$.

We now consider the case $0\leqslant\mu<\sigma$. Then there must be at least one element $x\in\mathbf{Z}_n$ such that there are at least p^μ distinct elements of the form $(x, *)$ in Γ, for otherwise order of $\Gamma<p^{\sigma+\mu}$. Moreover, for any $y\in\mathbf{Z}_n$ there must be an $a_y\in\mathbf{Z}_n$ such that $(-x+y, a_y)\in\Gamma$ by property (iii); consequently there are at least p^μ distinct elements of the form $(y, *)\in\Gamma$. Since order $\Gamma=p^{\sigma+\mu}=n\cdot p^\mu$, there must be precisely p^μ elements $(y, *)$ in Γ for each $y\in\mathbf{Z}_n$.

Consider an integer x satisfying $0\leqslant x<n$ and such that $(1, x)\in\Gamma$. Then $(x, x^2)=x\cdot(1, x)\in\Gamma$, and since $\theta_n(\Gamma)=\Gamma$, we also have $(-x, 1-x)\in\Gamma$, whence $(0, x^2-x+1)\in\Gamma$. Now if $[x^2-x+1, n]=1$, then $(x^2-x+1) \pmod n$ generates $\mathbf{Z}_n$, so that $\{0\}\oplus\mathbf{Z}_n\subset\Gamma$; $\theta_n(\Gamma)=\Gamma$ then implies $\mathbf{Z}_n\oplus\{0\}\subset\Gamma$, whence $\Gamma=\mathbf{Z}_n\oplus\mathbf{Z}_n$ which contradicts order $\Gamma=p^{\sigma+\mu}<n^2$. Consequently,

$$x^2-x+1\equiv0 \pmod p). \tag{B.1}$$

For $p=2$ the congruence (1) is not soluble; for $p=3$ it has the solution $x\equiv2 \pmod 3$. For a prime $p>3$, Euler's criterion asserts that (B.1) has a solution if and only if $(-3)^{(p-1)/2}\equiv1 \pmod p$, and for such p there are precisely two solutions mod$p$.

We thus observe that, if $p=2$ or if the prime p is >3 and $(-3)^{(p-1)/2}\not\equiv1 \pmod p$, then for $0\leqslant\mu<\sigma$ there is no subgroup $\Gamma\subset\mathbf{Z}_n\oplus\mathbf{Z}_n$ ($n-p^\sigma$) satisfying (i), (ii) and (iii) above.

§B4

Leaving the case $p=3$ for the moment, we now investigate the case when $0\leqslant\mu<\sigma$, $p>3$ and $(-3)^{(p-1)/2}\equiv1 \pmod p$. The congruence (B1) then has two distinct solutions mod$p$, which we write $x_\pm$ (mod$p$) where the integers $x_\pm$ satisfy $0 \leqslant x_- < x_+ < p$. If, for $0 \leqslant x < n = p^\sigma$, we have $(1, x) \in \Gamma$, then by §B3, $x^2 - x + 1 \equiv 0 \pmod{p}$ and so $x = x_\pm + kp$ for some $0 \leqslant k < p^{\sigma-1}$.

Consider now all the elements in Γ of the form $(1, *)$. Firstly, either all such elements are of the form $(1, x_+ + kp)$ or all are of the form $(1, x_- + kp)$. For suppose $(1, x_+ + kp)$ and $(1, x_- + k'p) \in \Gamma$ for some k, k'; then $(0, (x_+ - x_-) + (k - k')p) \in \Gamma$. Because p does not divide $x_+ - x_-$, the integer $((x_+ - x_-) + (k - k')p)$ (mod$n$) generates $\mathbf{Z}_n$, and so $\{0\} \oplus \mathbf{Z}_n \subset \Gamma$. Then $\mathbf{Z}_n \oplus \{0\} \subset \Gamma$ also, because $\theta_n(\Gamma) = \Gamma$, whence $\Gamma = \mathbf{Z}_n \oplus \mathbf{Z}_n$ which contradicts order $\Gamma < p^{2\sigma} = n^2$. Consequently, the p^μ distinct elements $(1, *)$ of Γ can be written either all in the form $(1, x_+ + k_i p)$ or all in the form $(1, x_- + k_i p)$ for appropriate $0 \leqslant k_1 < k_2 < \cdots < k_{p^\mu} < p^{\sigma-1}$. Secondly, we show that for each i, $(k_i - k_{i-1})p = p^{\sigma-\mu}$. Suppose that for some i, $[(k_i - k_{i-1})p, p^{\sigma-\mu}] < p^{\sigma-\mu}$. Then, for $\lambda = 1, 2, \ldots, p^\mu + 1$, we have the $p^\mu + 1$ distinct elements $(0, \lambda(k_i - k_{i-1})p) \in \Gamma$, contradicting the fact that there are only p^μ elements $(0, *)$ in Γ. Consequently, $[(k_i - k_{i-1})p, p^{\sigma-\mu}] = p^{\sigma-\mu}$, whence $k_i p = k_1 p + (i-1)p^{\sigma-\mu}$ foı $i = 1, 2, \ldots, p^\mu$. Thus, the elements in Γ of the form $(1, *)$ may be written as $(1, \bar{x} + ip^{\sigma-\mu})$ for $i = 1, 2, \ldots, p^\mu$, where $\bar{x} = x_\pm + k_1 p$ for some k_1 satisfying $0 \leqslant k_1 < p^{\sigma-\mu-1}$.

For each $i = 1, 2, \ldots, p^\mu$, let $\{(1, \bar{x} + ip^{\sigma-\mu})\}$ denote the cyclic subgroup of Γ generated by $(1, \bar{x} + ip^{\sigma-\mu})$. Now for any integers a, b, m, m', Γ contains the element

$$(a+b, (a+b)\bar{x} + (am + bm')p^{\sigma-\mu}) = a \cdot (1, \bar{x} + mp^{\sigma-\mu}) + b \cdot (1, \bar{x} + m'p^{\sigma-\mu}).$$

If p does not divide $a+b$, this element is simply $(a+b) \cdot (1, \bar{x} + m''p^{\sigma-\mu})$ for an integer m'' satisfying $m''(a+b)p^{\sigma-\mu} \equiv (am + bm')p^{\sigma-\mu} \pmod{p^\sigma}$. Considering integers a, b such that p divides $a+b$, one sees that Γ must contain all elements of the form $(rp, rp\bar{x} + mp^{\sigma-\mu})$, r and m arbitrary integers. Observe that if, for some $i \neq i'$, $a \cdot (1, \bar{x} + ip^{\sigma-\mu})$ belongs to $\{(1, \bar{x} + i'p^{\sigma-\mu})\}$, then $a = rp$ for some r. A count of elements now shows that the subgroup of Γ generated by the p^μ cyclic groups $\{(1, \bar{x} + ip^{\sigma-\mu})\}$ has order $p^{\sigma+\mu}$, and hence coincides with Γ.

Now a subgroup of $\mathbf{Z}_n \oplus \mathbf{Z}_n$ generated by such cyclic groups is θ_n-invariant if and only if $\theta_n((1, \bar{x} + ip^{\sigma-\mu})) \in \Gamma$ for $i = 1, 2, \ldots, p^\mu$; equivalently if and only if $\bar{x}^2 - \bar{x} + 1 \equiv 0$ (mod$p^{\sigma-\mu}$). We claim that this congruence has precisely two solutions (mod$n = p^\sigma$). First we prove the following

LEMMA. *Let $\nu > 0$. Then the congruence*

$$\bar{x}^2 - \bar{x} + 1 \equiv 0 \pmod{p^\nu} \tag{B.2}$$

has a unique solution $\bar{x}_+$ (modp^ν) congruent to x_+ (mod$p$), likewise for x_-.

Proof. To prove the lemma for the x_+ case, we will show, by induction on ν, that there is a unique solution to (B.2) of the form

$$\bar{x}=x_{+}+\alpha p \text{ where } 0\leqslant\alpha<p^{v-1}. \tag{B.3}$$

We write $u_{+}=2x_{+}-1$, and note that by definition of x_{+},

$$x_{+}^{2}-x_{+}+1\equiv 0(\mathrm{mod}\,p). \tag{B.4}$$

Step 1. If $v=1$, then necessarily $\alpha=0$ and $\bar{x}_{+}=x_{+}$ is the unique solution to (B.2) of the form (B.3).

Step 2. Suppose $v\geqslant 2$. Observe that (B.4) implies that $x_{+}^{2}-x_{+}+1\equiv r_{1}p$ for some integer r_{1}, and (B.2) holds if and only if

$$(r_{1}+u_{+}\alpha)\,p+\alpha^{2}p^{2}\equiv 0(\mathrm{mod}\,p^{v}). \tag{B.5}$$

Consequently, a necessary condition for $\bar{x}=x_{+}+\alpha p$ to solve (B.2) is

$$r_{1}+u_{+}\alpha\equiv 0(\mathrm{mod}\,p). \tag{B.6}$$

Now $[u_{+},p]=1$, so (B.6) has unique solution $\alpha\equiv a_{0}(\mathrm{mod}\,p)$ where we may assume $0\leqslant a_{0}<p$. Write $\alpha=a_{0}+\alpha_{1}p$; then $0\leqslant\alpha_{1}<p^{v-2}$ because $0\leqslant\alpha<p^{v-1}$. If $v=2$, then necessarily $\alpha_{1}=0$, and (B.6) is a sufficient condition for $\bar{x}=x_{+}+\alpha p$ to solve (B.2), so $\bar{x}_{+}=x_{+}+a_{0}p$ is the unique solution to (B.2) of the form (B.3).

Step 3. Suppose $v\geqslant 3$. Continue, and by induction, arrive at:

Step i. Suppose $v\geqslant i$. Observe that (B.2i) implies that

$$(r_{i-2}+u_{+}a_{i-3})\,p^{i-2}+2a_{2}(a_{0}+a_{1}p+\cdots+a_{i-2}p^{i-2})\,p^{i-1}=r_{i-1}p^{i-1}$$

for some integer r_{i-1}, and (B.2) holds if and only if

$$(r_{i-1}+u_{+}\alpha_{i-2})\,p^{i-1}+2\alpha_{i-2}(a_{0}+a_{1}p+\cdots a_{i-3}p^{i-3})\,p^{i}\equiv 0(\mathrm{mod}\,p^{v}). \tag{B.2i+1}$$

Consequently, a necessary condition for $\bar{x}=x_{+}+\alpha p$ to solve (B.2) is

$$r_{i-1}+u_{+}\alpha_{i-2}\equiv 0\quad(\mathrm{mod}\,p). \tag{B.2i+2}$$

Now $[u_{+},p]=1$, so (B.2i+2) has unique solution $\alpha_{i-2}\equiv\alpha_{i-2}(\mathrm{mod}\,p)$ where we may assume $0\leqslant a_{i-2}<p$. Write $\alpha_{i-2}=a_{i-2}+\alpha_{i-1}p$; then $0\leqslant\alpha_{i-1}<p^{v-i}$ because $0\leqslant\alpha_{i-2}<p^{v-(i-1)}$. If $v=i$, then necessarily $\alpha_{i-1}=0$, and (B.2i+2) is a sufficient condition for $\bar{x}=x_{+}+\alpha p$ to solve (B.2), so $\bar{x}_{+}=x_{+}+(a_{0}+a_{1}p+\cdots+a_{i-2}p^{i-2})\,p$ is the unique solution to (B.2), of the form (B.3).

This process clearly halts at Step v. This completes the proof of the Lemma.

Thus, by the Lemma, the congruence $\bar{x}^{2}-\bar{x}+1\equiv 0(\mathrm{mod}\,p^{\sigma-\mu})$ has a unique solution $\bar{x}_{+}(\mathrm{mod}\,p^{\sigma-\mu})$ such that $\bar{x}_{+}\equiv x_{+}(\mathrm{mod}\,p)$. Similarly, the given congruence has a unique solution $\bar{x}_-(\bmod p^{\sigma-\mu})$ such that $\bar{x} \equiv x_-\ (\bmod p)$. Clearly $\bar{x}_+ \not\equiv \bar{x}_-\ (\bmod p^{\sigma-\mu})$, for $x_+ \not\equiv x_-\ (\bmod p)$.

Define $\Gamma_\pm\ (p^\sigma, \sigma, \mu)$ to be the two subgroups of $\mathbf{Z}_n \oplus \mathbf{Z}_n$ generated by the p^μ cyclic groups $\{(1, \bar{x}_\pm + ip^{\sigma-\mu})\}$, $i=1, 2, \ldots, p^\mu$, respectively.

We summarise the results of this section. Given $n=p^\sigma$ where the prime $p>3$ satisfies

$$(-3)^{(p-1)/2} \equiv 1 \quad (\bmod p),$$

let $\bar{x}_\pm$ be the two solutions to the congruence $\bar{x}^2 - \bar{x} + 1 \equiv 0 (\bmod p^{\sigma-\mu})$ where $0 \leqslant \mu < \sigma$. Then the θ_n-invariant subgroups of $\mathbf{Z}_n \oplus \mathbf{Z}_n$ having order $p^{\sigma+\mu} (0 \leqslant \mu < \sigma)$ and slot group $\Lambda = \mathbf{Z}_n$ are the two groups $\Gamma_\pm (p^\sigma, \sigma, \mu)$ just defined.

§B5

We now treat the problem posed in §B3 for $n=p^\sigma$ in the case $p=3$. For $p=3$, the congruence (B.1) has the unique solution $2 (\bmod 3)$. Arguing as in §B4, one may show that the elements in Γ of the form $(1, *)$ may be written as $(1, \bar{x} + i3^{\sigma-\mu})$, $i=1, 2, \ldots, 3^{\sigma-\mu}$, where $\bar{x} = 2 + 3k$ for some $0 \leqslant k < 3^{\sigma-\mu-1}$; moreover, $\bar{x}$ must satisfy the congruence

$$\bar{x}^2 - \bar{x} + 1 \equiv 0 \quad (\bmod 3^{\sigma-\mu}),$$

that is,

$$9k^2 + 9k + 3 \equiv 0 \quad (\bmod 3^{\sigma-\mu}). \tag{B!}$$

Now (B!) has a solution for k only if $\mu = \sigma$ of $\sigma - 1$. So, if $0 \leqslant \mu \leqslant \sigma - 2$, there are no θ_n-invariant subgroups $\Gamma(3^\sigma, \sigma, \mu)$. As pointed out in §B3, if $\mu = \sigma$, there is exactly one θ_n-invariant subgroup with slot group $\mathbf{Z}_n$, namely $\Gamma(3^\sigma, \sigma, \sigma) = \mathbf{Z}_n \oplus \mathbf{Z}_n$ itself. Assume then that $\mu = \sigma - 1$. Then any integer k solves (B!). One observes that the subgroup of $\mathbf{Z}_n \oplus \mathbf{Z}_n$ generated by the cyclic groups $\{(1, 2+3i)\}$, $i=1, 2, \ldots, 3^{\sigma-1}$, is the θ_n-invariant subgroup $\Gamma(3^\sigma, \sigma, \sigma-1)$ of order $3^{2\sigma-1}$ with slot group equal to $\mathbf{Z}_n$.

§B6

In this section we give the complete solution to the problem posed in §B1.

Consider any integers λ and σ satisfying $0 \leqslant \sigma \leqslant \lambda$. Then:

(i) for any prime p we have the subgroup $\Gamma(p^\lambda, \sigma, \sigma) = \mathbf{Z}_{p^\sigma} \oplus \mathbf{Z}_{p^\sigma}$ in $\mathbf{Z}_{p^\lambda} \oplus \mathbf{Z}_{p^\lambda}$;

(ii) for $p=3$ we also have the subgroup $\Gamma(3^\lambda, \sigma, \sigma-1)$ in $\mathbf{Z}_{3^\lambda} \oplus \mathbf{Z}_{3^\lambda}$ generated by the cyclic groups $\{3^{\lambda-\sigma} \cdot (1, 2+3\mathrm{i})\}$, $i=1, 2, \ldots, p^{\sigma-1}$; and

(ii′) for prime $p>3$ such that $(-3)^{(p-1)/2} \equiv 1 (\bmod p)$ we also have, for each integer μ satisfying $0 \leqslant \mu < \sigma$, the two subgroups $\Gamma_\pm (p^\lambda, \sigma, \mu)$ in $\mathbf{Z}_{p^\lambda} \oplus \mathbf{Z}_{p^\lambda}$ generated by the p^μ

cyclic groups $\{p^{\lambda-\mu}\cdot(1, \bar{x}_{\pm}+ip^{\sigma-\mu})\}$, $i=1, 2, \dots, p^{\mu}$, respectively, where $\bar{x}_+$ and $\bar{x}_-$ are the two solutions $(\bmod p^{\sigma-\mu})$ to the congruence $\bar{x}^2-\bar{x}+1\equiv 0(\bmod p^{\sigma-\mu})$ as described in § B4.

Now consider any positive integer n and let $n=p_1^{\lambda_1}\, p_2^{\lambda_2}\dots\, p_r^{\lambda_r}$ be the unique decomposition of n into prime powers with

$$p_1=2<p_2=3<p_3<\cdots<p^r,$$

and such that $\lambda_1, \lambda_2\geqslant 0$, $\lambda_i>0$ for $i\geqslant 3$. Then by §§B1 to B5 the θ_n-invariant subgroups of $\mathbf{Z}_n\oplus\mathbf{Z}_n$ are precisely the direct sums $\Gamma_1\oplus\Gamma_2\oplus\cdots\oplus\Gamma_r$ where the Γ_i are as follows.

$\Gamma_1=\Gamma(2^{\lambda_1}, \sigma_1, \sigma_1)$ for some $0\leqslant\sigma_1\leqslant\lambda_1$.

$\Gamma_2=\Gamma(3^{\lambda_2}, \sigma_2, \mu_2)$ for some $0\leqslant\sigma_2\leqslant\lambda_2$ and $\mu_2=\sigma_2$ or σ_2-1.

For $i\geqslant 3$: if $(-3)^{(p_i-1)/2}\not\equiv 1\,(\bmod p_i)$, then

$$\Gamma_i=\Gamma(p_i^{\lambda_i}, \sigma_i, \sigma_i) \text{ for some } 0\leqslant\sigma_i\leqslant\lambda_i;$$

whereas, if $(-3)^{(p_i-1)/2}\equiv 1\,(\bmod p_i)$, then

$$\Gamma_i=\Gamma(p_i^{\lambda_i}, \sigma_i, \sigma_i) \text{ for some } 0\leqslant\sigma_i\leqslant\lambda_i,$$

or

$$\Gamma_i=\Gamma_{\pm}(p_i^{\lambda_i}, \sigma_i, \mu_i) \text{ for some } 0\leqslant\mu_i<\sigma_i\leqslant\lambda_i.$$

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