Eigenvalue Inequalities for the Dirichlet Problem on Spheres and the Growth of Subharmonic Functions

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Eigenvalue Inequalities for the Dirichlet Problem on Spheres and the Growth of Subharmonic Functions

S. FRIEDLAND AND W. K. HAYMAN

1. Introduction

Let u be a subharmonic (s.h.) function in R^m, i.e.

(i) $-\infty \leq u(x) < +\infty$

(ii) $u(x)$ is upper-semicontinuous (u.s.c.) and

(iii) for every $x_0 \in R^m$, $u(x_0)$ does not exceed the average with respect to spherical measure of $u(x)$ on the hypersphere $|x - x_0| = r$.

Consider the set

$$E_R(K) = \{x \mid u(x) \geq K, |x| \leq R\}.$$

This set is compact, since $u(x)$ is u.s.c. and so can be divided into a number of components, $C_R(K)$ which are continua or points. In a recent paper Talpur [13] has proved the following 3 facts about these components.

(a) *Each $C_R(K)$ meets the boundary $|x| = R$.*

We only consider those components which contain at least one point x_0 in $|x_0| < R$, where $u(x_0) > K$. Such components will be called thick. Talpur [13] has shown that thick components necessarily have positive m-dimensional measure and thus their total number is finite or countable. He has also shown that there can be non-countably many thin components on which $u(x) \equiv K$, but that if $u(x)$ has such a thin component, with a point x_0 in $|x_0| < R$, then either $u(x) \equiv K$ in $|x| < R$, or $u(x)$ has infinitely many thick components.

(b) *If $v(x)$ is defined as follows*

$$v(x) = u(x) \text{ in a component } C_R(K)$$

$$v(x) = K \text{ elsewhere,}$$

then $v(x)$ is s.h. in $|x| < R$.

Let x_0 be a point of R^m, such that $u(x_0) > K$, and let $C_R(x_0, K)$ be the component of $E_R(K)$ containing x_0. Then evidently $C_R(x_0, K)$ expands with increasing R. We define

$$C(K)=C(x_0, K)=\bigcup_{R>|x_0|} C_R(x_0, K)$$

to be a limit component. Evidently two points x_1, x_2 for which $u(x_j)>K$ belong to the same limit component if they belong to the same component $C_R(K)$ for sufficiently large R. Two limit components are either identical or disjoint. We denote by $N(K)$ the total number of limit components, which may be either finite or $+\infty$. Since each component $C_R(x_0, K)$ and so also $C(x_0, K)$ has positive measure, it follows that the number of limit components is at most countably infinite.

We now define

$$B(r, u)=\sup_{|x|=r} u(x). \tag{1.1}$$

The quantity

$$\mu=\liminf_{r\to\infty}\frac{\log B(r, u)}{\log r}$$

is called the lower order of $u(x)$. Then Talpur [13] has proved

(c) *If $u(x)$ is bounded above with least upper bound K_0 then for $K<K_0$ there exists exactly one limit component $C(K)$. If $u(x)$ is not bounded above, but has finite lower order μ, then $u(x)$ is unbounded above on every limit component $C(x_0, K)$.*

We suppose henceforth that $u(x)$ is not bounded above but has finite lower order μ. If $\mu=+\infty$ all our results will be trivially true. It then follows from (c) that if $K_1<K_2$ each limit component $C(K_1)$ contains at least one limit component $C(K_2)$. Thus $N(K)$ is a monotonic increasing function of K. We define

$$N=\lim_{K\to+\infty} N(K)$$

to be the number of tracts of $u(x)$. In fact the limit N also exists if $u(x)$ is bounded above by K_2 say on some component $C(K_1)$. For in this case it turns out that $u(x)$ is unbounded above on every $C(K)$ for $K>K_2$ [13]. Thus the limit N still exists.

Let $l(N, m)$ be the greatest lower bound of all lower orders μ of subharmonic function in R^m having at least N tracts. Our aim in this paper is to obtain good lower bounds for $l(N, m)$. It will turn out that $l(2, m)=1$, so that unbounded functions of order $\mu<1$ always have exactly one tract. Thus it is possible to obtain a non-trivial lower bound only when $N\geq 2$. We shall also see that $l(N, m)\to\infty$ as $N\to\infty$ uniformly in m, so

$$l(\infty, m)=+\infty.$$

Thus we may assume that $2\leq N<\infty$, $2\leq m<\infty$.

2. Known Results

Suppose first that $m=2$. In this case the problem was solved by Heins [8], who proved

THEOREM A. *We have* $l(N, 2)=\frac{1}{2}N$.

This result of Heins represents a generalisation of Ahlfors', theorem on asymptotic values. In fact if $f(z)$ is an entire function with N distinct asymptotic values, where $N\geq 2$, then it is easy to see that $u(z)=\log|f(z)|$ is s.h. in R^2 and has at least N tracts, so that by Theorem A the lower order μ of $f(z)$ is at least $\frac{1}{2}N$. For $m=3$ Talpur [13] proved

THEOREM B. $l(N, 3)\geq\frac{1}{2}j_0\sqrt{(N-1)}-\frac{1}{2}$, *where* $j_0=2.4048\cdots$ *is the first zero of Bessel's function of order zero.*

For $m\geq 4$, the best result is due to Dinghas [5]

THEOREM C.

$$l(N, m)\geq\left\{C_m^2\Phi_m\left(\frac{1}{N}\right)^2+(m-2)^2\right\}^{1/2},\qquad N\geq 2,\qquad m\geq 3.$$

Here

$$C_m=\{\tfrac{1}{2}(m-1)\}^{(m-2)/(m-1)}\left\{\frac{\sigma_m}{\sigma_{m-1}}\right\}^{(m-3)/(m-1)},$$

$$\sigma_m=2\pi^{(1/2)m}/\Gamma(\tfrac{1}{2}m) \tag{2.1}$$

is the $(m-1)$*-dimensional measure of the surface of the unit-sphere* S_m *in*

R^m, *and*

$$\Phi_m(x) = x^{-1/(m-1)}(1-x)^{(m-2)/(m-1)}.$$

It should be said that Dinghas did not state his results in the above generality, but only for certain harmonic functions. However his proof can be extended to the general case, using a technique of Huber [9].

All the above results, as well as our own use a differental inequality technique, which in the case $m=2$ is due to Carleman [3] and was extended to the case $m=3$ by Keller [10] and the general case by Dinghas [4] and Huber [9]. It was Huber who removed the hypothesis on smoothness, which other authors needed, by an approximation technique. We now proceed to describe Huber's key result on which all subsequent work including our own has been based.

In order to do this we define, following Huber, the characteristic constant $\alpha(E)$ of a set E on the unit sphere

$$S_m : x_1^2 + x_2^2 + \cdots x_m^2 = 1$$

in R^m. Suppose that E is an open set. Let $\mathfrak{F}(E)$ be the class of functions which are Lipschitzian, nonnegative and not identically zero on S_m and which vanish outside E. Let

$$\lambda(E) = \inf_{f\in\mathfrak{F}(E)} \frac{\int |\text{grad}\, f|^2 \, d\sigma}{\int |f|^2 \, d\sigma}, \tag{2.2}$$

where $d\sigma$ denotes $(m-1)$ dimensional measure on S_m and grad f is the gradient, i.e. maximum directional derivative on the surface of the sphere. Huber notes without proof that if E is suitably smooth, this infimum is obtained for the solution of the Laplace-Beltrami equation

$$\nabla f + \lambda f = 0$$

on E, where $\lambda = \lambda(E)$ is the lowest eigenvalue of this equation. However we shall use this result only in a very special case, when we can give a simple proof (Lemma 1).

The characteristic constant $\alpha(E)$ is defined to be the positive root of the equation

$$\alpha(\alpha+m-2)=\lambda. \tag{2.3}$$

If E is a general compact set on S_m, such that E and the complement of E are not empty we define $\alpha(E)$ to be the upper bound of $\alpha(D)$ over the class of all open sets D containing E. Finally if E is a general set, then $\alpha(E)$ is defined to be the lower bound of $\alpha(F)$ over compact sets F contained in E. Thus $\alpha(E)$ is now defined for all measurable sets E on S_m and $\alpha(E)$ is a decreasing function of E, i.e. if $E_1\subset E_2$ then $\alpha(E_1)\geq\alpha(E_2)$.

Finally if E is a set on the sphere $|x|=R$, let $\hat{E}$ be the projection of E on $|x|=1$. In other words $\hat{E}$ consists of all points $x/|x|$, where $x\in E$. Then we define $\alpha(E)=\alpha(\hat{E})$.

Suppose now that $u(x)$ is s.h. and non-negative in $|x|<R$. Let $D(r)$ be the intersection of the set $u>0$ with $|x|=r$ and let $\alpha(r)$ be the characteristic constant of $D(r)$. If $D(r)$ has $(m-1)$-dimensional measure zero, it follows from Poisson's inequality that $u(x)\equiv 0$ for $|x|<r$. We suppose that $u(x)>0$ for some x in $|x|<R$, so that $D(r)$ has positive measure for $r_0\leq r<R$ say. We now set

$$m^2(r)=m^2(r,u)=\frac{1}{\sigma_m r^{m-1}}\int_{D(r)}u^2(x)\,d\omega(x), \tag{2.4}$$

where $\omega(x)$ denotes surface area on $|x|=r$, and σ_m is given by (2.1). Then Huber [9, p. 112] proved the following fundamental convexity theorem.

THEOREM D. *We have with the above hypotheses*

$$r^{m-2}m^2(r)-r_0^{m-2}m^2(r_0)\geq C_0\int_{r_0}^{r}\exp\left\{2\int_{r_0}^{\rho}\alpha(t)\frac{dt}{t}\right\}\rho^{m-3}\,d\rho, \qquad r_0<r<R$$

where

$$C_0=r_0^{2-m}\left\{r\frac{d}{dr}\,m^2(r)r^{m-2}\right\}_{r=r_0}\geq(m-2)m^2(r_0).$$

COROLLARY. *If $m\geq 3$ and $2r_0<r<R$, we have*

$$m(r)\geq\frac{1}{\sqrt{2}}\,m(r_0)\exp\left\{\int_{r_0}^{(1/2)r}\alpha(t)\frac{dt}{t}\right\}$$

The corollary follows at once from the main result. In fact we have in this case for $\frac{1}{2}r \le \rho \le r$

$$\exp\left\{2\int_{r_0}^{\rho} \alpha(t)\frac{dt}{t}\right\} \ge \exp\left\{2\int_{r_0}^{(1/2)r} \alpha(t)\frac{dt}{t}\right\} = B \text{ say.}$$

Thus

$$r^{m-2}m^2(r) \ge C_0\int_{(1/2)r}^{r} B\rho^{m-3}\,d\rho = \frac{BC_0 r^{m-2}}{(m-2)}(1-2^{2-m}) \ge \frac{BC_0 r^{m-2}}{2(m-2)}$$

Thus

$$m^2(r) \ge \frac{BC_0}{2(m-2)} \ge \frac{Bm^2(r_0)}{2},$$

and this proves the Corollary. We shall in general assume $m \ge 3$ in the sequel since Theorem A solves our main problem for $m = 2$.

3. Statement of our Results

We can now return to our problem. Suppose that $u(x)$ is s.h. in R^m and has at least N tracts there. Then for some positive K the set $u(x) \ge K$ has N limit components C_1 to C_N, each containing a point x_j such that $u(x_j) > K$.

We now choose

$$\tfrac{1}{2}R \ge r_0 > \max_{j=1 \text{ to } N} |x_j|$$

and define the functions $v_j(x, R)$ as follows. Let $C_j(K, R)$ be that component of $u(x) \ge K$ in $|x| \le R$, which contains x_j. We define

$$v_j(x, R) = u(x) - K, \qquad x \in C_j(x, R)$$

$$v_j(x, R) = 0, \qquad \text{elsewhere.}$$

By (b) of section 1, $v_j(x, R)$ is s.h. in $|x| < R$. Let

$$B_j(r, R) = \sup_{|x|=r} v_j(x, R), \qquad r_0 < r < R.$$

Let $\alpha_j(r, R)$ be the characteristic constant of $C_j(K, R) \cap (|x| = r)$. It follows from Theorem D, Corollary that

$$B_j(r, R) \geq \tfrac{1}{2} m_j(r_0, R) \exp \int_{r_0}^{(1/2)r} \alpha_j(t, R) \frac{dt}{t}, \tag{3.1}$$

where

$$m_j^2(r_0, R) = \frac{1}{\sigma_m r_0^{m-1}} \int_{C_j(K, R) \cap |x| = r_0} (u(x) - K)^2 \, d\sigma(x).$$

Since $C_j(K, R)$ expands with increasing R it follows that we can replace R by any smaller number without destroying the inequality (3.1). In particular we may replace R by $2r_0$. We deduce that

$$B_j(r, R) \geq C_j \exp \left\{ \int_{r_0}^{(1/2)r} \alpha_j(t, R) \frac{dt}{t} \right\}, \qquad R > r > 2r_0,$$

where the constants C_j are independent of R and positive. Thus

$$\left\{ \prod_{j=1}^{N} B_j(r, R) \right\}^{1/N} \geq C_0 \exp \left\{ \int_{r_0}^{(1/2)r} \frac{1}{N} \sum_{j=1}^{N} \alpha_j(t, R) \frac{dt}{t} \right\}, \qquad 2r_0 < r < R,$$

where

$$C_0 = \left(\prod_{j=1}^{N} C_j \right)^{1/N}$$

is again independent of R. Thus if $B(r, u)$ is defined by (1.1), we deduce

$$B(r, u) \geq K + C_0 \exp \left\{ \int_{r_0}^{(1/2)r} \frac{1}{N} \sum_{j=1}^{N} \alpha_j(t, R) \frac{dt}{t} \right\}, \qquad r_0 < \tfrac{1}{2} r < R. \tag{3.2}$$

We note that the $\alpha_j(t, R)$ are characteristic constants of mutually disjoint sets. In order to achieve good lower bounds for $B(r, u)$ we proceed to obtain lower bounds for such characteristic constants. A fundamental result was recently published by Sperner [11].

THEOREM E. *Among all sets E with given $(m-1)$-dimensional surface area $\sigma_m S$ on the unit sphere in R^m, a spherical cap, i.e. a set of the form*

$c < x_1 \le 1$, *has the smallest characteristic constant* $\alpha(S, m)$. *Here* $0 < S < 1$ *and* c, S *are related by the equations*

$$S = \frac{\sigma_{m-1}}{\sigma_m} \int_0^{\phi} (\sin t)^{m-2}\, dt, \qquad c = \cos \phi, \qquad 0 < \phi < \pi. \tag{3.3}$$

We deduce

THEOREM 1. *Let* $\alpha(S, m)$ *be the function defined in Theorem E, and suppose that*

$$\alpha(S, m) \ge \phi(S), \qquad 0 < S < 1,$$

where $\phi(S)$ *is a convex decreasing function of* S. *Then*

$$l(N, m) \ge \phi\left(\frac{1}{N}\right).$$

The idea of this result was used by Dinghas [5] and Talpur [13] for their proof of Theorems B and C. It might be desirable to prove that $\alpha(S, m)$ is itself a convex function of S, but this seems likely to be difficult. We can however obtain various inequalities. We note

THEOREM 2. *The quantities* $\alpha(S, m)$ *for fixed* S *and* $l(N, m)$ *for fixed* N *are monotonic decreasing functions of* m.

In view of Theorem 2 we may consider the function

$$\alpha(S, \infty) = \lim_{m \to \infty} \alpha(S, m)$$

and try to obtain a convex lower bound for this. We will then, in view of Theorem 1, deduce a lower bound for $l(N, m)$ which is independent of m. Our result is contained in

THEOREM 3. *We have* $\alpha(S, \infty) \ge \phi_\infty(S)$, *where*

$$\phi_\infty(S) = \begin{cases} \frac{1}{2} \log\left(\frac{1}{4S}\right) + \frac{3}{2}, & 0 < S \le \frac{1}{4} \\ 2(1 - S), & \frac{1}{4} \le S < 1. \end{cases} \tag{3.4}$$

Hence we have, for all m, $l(2, m)=1$, $l(3, m)\geq\frac{4}{3}$,

$$l(N, m)\geq\tfrac{1}{2}\log\left(\frac{N}{4}\right)+\tfrac{3}{2},\qquad N\geq 4.$$

COROLLARY. *If* $u(x)$ *is s.h. in* R^m *then either all the sets* $u(x)\geq K$ *are connected in* R^m *or else*

$$\liminf_{r\to\infty}\frac{B(r, u)}{r}>0.$$

The function $|x_1|$ *shows that this limit need not be infinite.*

The result of Theorem 3 gives the right order of magnitude for $l(N, m)$ when the order of $\log N$ is not greater than that of m. In fact if we choose $u=|x_1 x_2\cdots x_m|$, we see that for $K>0$ the set $u>K$ has $N=2^m$ components, while the order μ of u is just m. Thus

$$l(N, m)\leq m=\frac{\log N}{\log 2}$$

while Theorem 3 gives for all m

$$l(N, m)\geq\tfrac{1}{2}\log N.$$

When $\log N$ is large compared with m we can also obtain the right order of magnitude.

THEOREM 4. *We have* $\alpha(S, m)\geq\phi_m(S)$ *and hence*

$$l(N, m)\geq\phi_m\left(\frac{1}{N}\right),$$

where $\phi_m(S)=2(1-S)$, *for* $\frac{1}{2}\leq S<1$ *and for* $0<S<\frac{1}{2}$, *we have*

$$\phi_m(S)=\max\left\{\phi_\infty(S),\ j_k\left(\frac{d_m}{S}\right)^{1/(m-1)}-\tfrac{2}{5}j_k-\tfrac{1}{2}(m-2)\right\},\qquad 4\leq m<\infty,\tag{3.5}$$

$$\phi_3(S)=\max\left\{\phi_\infty(S),\ \tfrac{1}{2}j_0\left(\frac{1}{S}-\frac{1}{2}\right)^{1/2}-\frac{1}{2}\right\}.\tag{3.6}$$

Also

$$\phi_2(S) = \frac{1}{2S}, \qquad 0 < S < 1. \tag{3.7}$$

Here j_k *is the first zero of* Bessel's *function of order* $k = \frac{1}{2}(m-3)$, *and in particular* $j_0 = 2.4048 \cdots$. *Also*

$$d_m = \frac{\Gamma(\frac{1}{2}m)}{2\pi^{1/2}\Gamma\{\frac{1}{2}(m+1)\}}.$$

For $m = 2$ we obtain the classical result of Heins [8]. The results for $m \geq 3$ sharpen the previous bounds of Talpur [13] and Dinghas [5], particularly when N is small. We shall also in Theorem 5 obtain a more precise bound for $l(N, m)$ when $m = 4$. It is shown elsewhere that [2, 6]

$$l(N, m) \leq (m-1)(\tfrac{1}{2}N)^{1/(m-1)}, \qquad N \geq 2^m,$$

and in fact a little more. We note that [1, p. 371]

$$j_k \sim k \sim \tfrac{1}{2}m, \qquad d_m^{1/(m-1)} \to 1,$$

as $m \to \infty$. Thus Theorem 4 gives

$$l(N, m) \geq (\tfrac{1}{2} - \varepsilon_m) m \{N^{1/(m-1)} - \tfrac{3}{2}\}, \qquad N > 2^m$$

where $\varepsilon_m \to 0$ as $m \to \infty$, and so we still obtain the right order of magnitude for $l(N, m)$ when $N > 2^m$.

In fact these remarks probably underrate the accuracy of the above results. Thus one of us has sketched elsewhere [6] an example to show that

$$l(N, 3) < 1.013 \frac{j_0}{2} N^{1/2}$$

when N is large. This compares with Theorem 4 which yields

$$l(N, 3) > \tfrac{1}{2} j_0 (N - \tfrac{1}{2})^{1/2} - \tfrac{1}{2}, \qquad N \geq 2.$$

4. Proof of Theorem 1

In order to prove Theorem 1 we use the inequality (3.2). Let $\sigma_m r^{m-1} S_j(r, R)$ be the $(m-1)$-dimensional area of $C_j(K, R) \cap (|x| = r)$. Since the $C_j(K, R)$ are disjoint for different j, we deduce that

$$\sum_{j=1}^{N} S_j(r, R) \leq 1.$$

Let $\alpha_j(t)$ be the characteristic constant of $C_j(K, R) \cap (|x| = t)$. Then we deduce from Theorem E, that

$$\alpha_j(t) \geq \alpha(S_j, m),$$

where $S_j = S_j(t, R)$ and $\alpha(S, m)$ is the function of Theorem 1. Thus

$$\sum_{j=1}^{N} \alpha_j(t) \geq \sum_{j=1}^{N} \alpha(S_j, m) \geq \sum_{j=1}^{N} \phi(S_j).$$

Now since $\phi(S)$ is a convex decreasing function of S, we deduce from Jensen's inequality that

$$\phi\left(\frac{1}{N}\right) \leq \phi\left(\sum_{j=1}^{N} \frac{S_j}{N}\right) \leq \frac{1}{N} \sum_{j=1}^{N} \phi(S_j).$$

Thus

$$\sum_{j=1}^{N} \alpha_j(t) \geq N\phi\left(\frac{1}{N}\right).$$

Writing $\alpha_j(t, R)$ instead of $\alpha_j(t)$ and substituting in (3.2) we deduce that

$$B(r, u) \geq K + C_0 \exp\left\{\int_{r_0}^{(1/2)r} \phi\left(\frac{1}{N}\right)\frac{dt}{t}\right\}$$

$$= K + C_0 \exp\left\{\phi\left(\frac{1}{N}\right)\log\left(\frac{r}{2r_0}\right)\right\} = K + C_0\left(\frac{r}{2r_0}\right)^{\alpha}, \tag{4.1}$$

where $\alpha = \phi(1/N)$. Since C_0 is a positive constant and this inequality holds for all large r, we deduce that u has lower order at least α. This result is true for any s.h. function u having at least N tracts and Theorem 1 is proved. In fact (4.1) shows a little more namely that $B(r, u)$ has at least positive lower type of order $\phi(1/N)$ in this case.

5. Proof of Theorem 2

It is fairly clear that $l(N, m)$ is a decreasing function of m for fixed N. In fact suppose that $u(x_1, x_2, \ldots, x_m)$ is a s.h. function in R^m having N tracts and lower order λ. We then define

$$v(x_1, x_2, \ldots, x_{m+1}) = u(x_1, x_2, \ldots, x_m).$$

Then $v(x)$ also has N tracts, since any continuum in R^{m+1} on which $v \geq K$ projects onto a continuum in R^m, where $u \geq K$. If there exist N points $\xi^{(j)} = (x_1^{(j)}, \ldots, x_m^{(j)})$, such that $u(\xi^j) > K$, but no pair $\xi^{(i)}$, $\xi^{(j)}$ for $i \neq j$ can be joined by a continuum on which $u \geq K$, then two points $(x_1^{(j)}, \ldots, x_m^{(j)}, 0)$ for different j cannot be joined by a continuum on which $v \geq K$.

Further v has the same order as u. For

$$B(R, v) = \sup_{x_1^2 + \cdots + x_{m+1}^2 = R^2} v(x) = \sup_{x_1^2 + \cdots + x_m^2 \leq R^2} u(x) = B(R, u),$$

since $B(R, u)$ increases with R. Thus v is a competing function for the class defining $l(N, m+1)$ and so

$$l(N, m+1) \leq \lambda.$$

Since we may choose λ as close as we wish to $l(N, m)$, we deduce

$$l(N, m+1) \leq l(N, m).$$

Before we can continue with the proof of Theorem 2, we need to consider more closely the extremal functions in Theorem E. In fact Sperner proves a little more than is stated in that theorem. He shows that given any function $f \in \mathfrak{F}(E)$, where $\mathfrak{F}(E)$ is the class defined in connection with (2.2), then there exists a symmetrized function f^* on S_m, depending only on x_1 and belonging to $\mathfrak{F}(E^*)$, where E^* is the spherical cap $c < x_1 \leq 1$, having the same area as E. Furthermore

$$\int |\text{grad } f^*|^2 \, d\sigma \leq \int |\text{grad } f|^2 \, d\sigma$$

and

$$\int |f^*|^2 \, d\sigma = \int |f|^2 \, d\sigma.$$

Thus in order to find the minimum value of $\lambda(E)$ for sets E of given area it is enough to consider spherical caps and functions on them which depend on x_1 only.

It is convenient to write $x_1 = \cos\theta$, and assume that $E = E(\theta_0)$ takes the form $0 \leq \theta < \theta_0$. We then write $f = f(\theta)$. Since $f \in \mathfrak{F}(E)$, it follows that $f(\theta)$ is Lipschitzian and in particular absolutely continuous in $[0, \pi]$, $f(\theta) = 0$, $\theta_0 \leq \theta \leq \pi$. It is clear that the spherical area $\sigma(\theta_0)$ of E is given by

$$\sigma(\theta_0) = \sigma_{m-1} \int_0^{\theta_0} (\sin t)^{m-2} \, dt,$$

so that the proportionate area is

$$S(\theta_0) = \frac{\sigma_{m-1}}{\sigma_m} \int_0^{\theta_0} (\sin t)^{m-2} \, dt, \tag{5.1}$$

where σ_m is given by (2.1). Thus

$$J(f) = \frac{\int (\text{grad } f)^2 \, d\sigma}{\int f^2 \, d\sigma} = \frac{\int_0^{\theta_0} f'(\theta)^2 (\sin\theta)^{m-2} \, d\theta}{\int_0^{\theta_0} f(\theta)^2 (\sin\theta)^{m-2} \, d\theta}. \tag{5.2}$$

We thus need to find the minimum value of $J(f)$ over the class of functions described. The technique for this is classical. We state the result in

LEMMA 1. *For all functions $f(\theta)$ which are Lip in $[0, \pi]$, not identically zero, but such that $f(\theta) = 0$, $\theta_0 \leq \theta \leq \pi$, where $0 < \theta_0 < \pi$, we have $J(f) \geq J(F) = \lambda$, where $u = (\sin\theta)^{(1/2)(m-2)} F(\theta)$ is a solution of the differential equation*

$$\frac{d^2u}{d\theta^2} + \left\{ \lambda + \tfrac{1}{4}(m-2)^2 + \frac{(m-2)(4-m)}{4\sin^2\theta} \right\} u = 0, \tag{5.3}$$

and the positive number λ *is so chosen that* F *remains analytic at* $\theta=0$, $F'(0)=0$, $F(\theta_0)=0$ *and* $F(\theta)>0$ *for* $0<\theta<\theta_0$.

We note that (5.3) can also be written as

$$\frac{d}{d\theta}(\sin\theta)^{m-2}F' = -\lambda(\sin\theta)^{m-2}F \tag{5.4}$$

or

$$F''+(m-2)\cot\theta F'+\lambda F=0. \tag{5.5}$$

Thus F is Legendre's function.

It follows from standard results on Legendre-functions, that for given positive λ a solution $F(\theta)$ exists which is positive at $\theta=0$ and has at least one zero in $(0, \pi)$. Then $\theta_0=\theta_0(\lambda)$ is the smallest zero of this function. Also if $m\geq 3$, θ_0 decreases continuously from π to 0 as λ increases from 0 to ∞. Thus for each θ_0 there exists exactly one corresponding λ, and λ is a continuous function of θ_0. Thus the function $F(\theta)$ exists.

Next we write $p(\theta)=(\sin\theta)^{m-2}$ and obtain from (5.4)

$$\int_0^{\theta_0} p(\theta)F'(\theta)^2\,d\theta = [F(\theta)p(\theta)F'(\theta)]_0^{\theta_0} - \int_0^{\theta_0} F(\theta)\frac{d}{d\theta}\{p(\theta)F'(\theta)\}\,d\theta$$

$$= \lambda\int_0^{\theta_0} p(\theta)F(\theta)^2\,d\theta,$$

so that $J(F)=\lambda$.

To prove that $J(f)\geq J(F)$, suppose first that f vanishes for $\theta\geq\theta_1$, where $\theta_1<\theta_0$. We have, using (5.4),

$$p(f'^2-\lambda f^2) = pf'^2 + \frac{f^2\left(\frac{d}{d\theta}pF'\right)}{F} = pf'^2 + p\frac{f^2F''}{F} + p'f^2\frac{F'}{F}$$

$$= p\left(f' - \frac{F'}{F}f\right)^2 + \frac{d}{d\theta}\left(pf^2\frac{F'}{F}\right) \geq \frac{d}{d\theta}\left(pf^2\frac{F'}{F}\right).$$

This holds whenever f' exists, i.e. almost everywhere in $[0, \theta_1]$. Also since f has a bounded derivative, the right hand side is the derivative of an absolutely continuous function, which vanishes for $\theta=0$, and $\theta>\theta_1$. Thus we may integrate both sides from 0 to θ_0 and obtain

$$\int_0^{\theta_0} p(\theta)\{f'(\theta)^2-\lambda f(\theta)^2\}\, d\theta \geq \left\{p(\theta)f^2(\theta)\frac{F'(\theta)}{F(\theta)}\right\}_0^{\theta_1}=0.$$

This proves the Lemma if $f(\theta)=0$ for $\theta>\theta_0-\varepsilon$, for some positive ε. In the general case we apply this conclusion to $f(t\theta)$, where $t>1$ and allow t to tend to one. This proves Lemma 1.

We also note

LEMMA 2. *Suppose that* $(x_1, x_2, \ldots, x_m)$ *is a point in* R^m, *and* r, θ *are defined by* $|x|=r$, $r\cos\theta=x_1$, *where* $0\leq\theta\leq\pi$. *Then if* $F(\theta)$ *is the function of Lemma* 1, $v=r^\alpha F(\theta)$ *is harmonic and positive in the cone* C *given by* $0\leq\theta<\theta_0$, *and zero on the boundary of* C, *where* α, λ *are related by* (2.3).

In fact the harmonicity of v is equivalent to the equation (5.5). The other properties are obvious. We deduce

LEMMA 3. *Suppose that* $0<S<1$ *and* $\mathfrak{F}_1(S)$ *is the class of functions* v, *s.h. in* R^m *and such that the intersection of the set* $v>0$ *with the sphere* $|x|=r$ *always has area and at least* $\sigma_m Sr^m$. *Then the minimum of the lower orders of* $v\in\mathfrak{F}_1(S)$ *is precisely* α, *where* λ *is given in terms of* θ_0 *by Lemma* 1, S, θ_0 *are related by* (5.1), *and* α, λ *are related by* (2.3).

It follows from Theorem E and Lemma 1, that if α, λ are related by (2.3), then the characteristic constant of the set $v>0$ on $|x|=r$ is at least α. Thus in (3.1) we may take $j=1$ and $\alpha_j(t, R)\geq\alpha$ and this gives

$$B(r, R)\geq Cr^\alpha.$$

This shows that the lower order of v is at least α. On the other hand Lemma 2 shows that this lower bound is attained by the function $r^\alpha F(\theta)$.

We can now prove that

$$\alpha(S, m+1)\leq\alpha(S, m). \tag{5.6}$$

Let v be the function of Lemma 2 and set $v=0$ outside the cone C. Then evidently v is s.h. in R^m. We now define

$$V(x_1, x_2, \ldots, x_{m+1})=v(x_1, x_2, \ldots, x_m)$$

for all points in R^{m+1}. Clearly V is s.h. in R^{m+1} and has order $\alpha=\alpha(S, m)$. Also to obtain the m-dimensional area of the intersection of $V>0$ with $|x|=r$, we consider the $(m-1)$-dimensional area for fixed x_{m+1} and integrate. This $(m-1)$-dimensional area is for every x_{m+1}, precisely S times the area of the whole $(m-1)$ sphere given by

$$x_1^2+x_2^2+\cdots x_m^2=r^2-x_{m+1}^2.$$

We deduce that $V>0$ meets the m-sphere $|x|=r$ in a set of m-dimensional area precisely $S\sigma_{m+1}r^m$, while V has lower order α. Thus by Lemma 3

$$\alpha(S, m)=\alpha\geq\alpha(S, m+1).$$

This proves (5.6) and completes the proof of Theorem 2.

6. Proof of Theorem 3

Our proof of Theorem 3 depends on [7].

THEOREM F. *Let $H_\alpha(x)$ denote Hermite's function, so that*

$$F(x)=e^{-(1/4)x^2}H_\alpha(x/\surd 2)$$

satisfies the differential equation

$$\frac{d^2F}{dx^2}+(\alpha+\tfrac{1}{2}-\tfrac{1}{4}x^2)F=0 \tag{6.1}$$

and $F'(0)/F(0)=-2^{1/2}\Gamma\{(1-\alpha)/2\}/\Gamma(-\alpha/2)$. Let $h=h(\alpha)$ be the largest real zero of F. Then if

$$S=\frac{1}{\surd(2\pi)}\int_h^\infty e^{-(1/2)t^2}\,dt, \tag{6.2}$$

we have

$$\alpha\geq\phi_\infty(S), \tag{6.3}$$

where $\phi_\infty(S)$ is given by (3.4).

We note that if μ is a positive integer, then $H_\mu(t)$ is a polynomial. In particular $H_1(t)=t$, $H_2(t)=1-2t^2$, $H_3(t)=t-2t^3/3$, etc. Thus $h(1)=0$, $h(2)=1$, $h(3)=\sqrt{3}$.

We have not been able to give an analytical proof of (6.3) and in certain ranges it has been necessary to verify the result by computer. We proceed to show that if α, S are related as in Theorem F, then $\alpha(S,\infty)=\alpha(S)$.

To see this we set

$$\phi=\frac{\pi}{2}-\frac{h}{\sqrt{m}}, \qquad t=\frac{\pi}{2}-\frac{x}{\sqrt{m}},$$

and we obtain from (3.3)

$$S=\tfrac{1}{2}-\frac{\sigma_{m-1}}{\sigma_m}\int_{(\pi/2)-(h/\sqrt{m})}^{\pi/2}(\sin t)^{m-2}\,dt=\tfrac{1}{2}-f_m\int_0^h\left(\cos\frac{x}{\sqrt{m}}\right)^{m-2}dx.$$

where

$$f_m=\frac{\sigma_{m-1}}{\sigma_m\sqrt{m}}=\frac{1}{\pi^{1/2}\sqrt{m}}\frac{\Gamma\{\frac{1}{2}(m)\}}{\Gamma\{\frac{1}{2}(m-1)\}}\sim(2\pi)^{-(1/2)}, \quad \text{as } m\to\infty.$$

Also for bounded x we have

$$\cos\left(\frac{x}{\sqrt{m}}\right)^{m-2}=\exp\left\{(m-2)\log\left(\cos\frac{x}{\sqrt{m}}\right)\right\}$$

$$=\exp(m-2)\log\left(1-\frac{x^2}{2m}+\frac{O(1)}{m^2}\right)=\exp\left\{-\frac{x^2}{2}+\frac{O(1)}{m}\right\}.$$

Thus as $m\to\infty$, while h remains fixed we see that

$$S\to\tfrac{1}{2}-\frac{1}{(2\pi)^{1/2}}\int_0^h e^{-(1/2)x^2}\,dx=\frac{1}{(2\pi)^{1/2}}\int_h^\infty e^{-(1/2)t^2}\,dt.$$

which is (6.2). Again if

$$u=u_m=(\sin t)^{(1/2)(m-2)}F_m(t)$$

is the function of Lemma 1 we see that as $m \to \infty$

$$u = \left(\cos \frac{x}{\sqrt{m}}\right)^{(1/2)(m-2)} F\left(\frac{\pi}{2} - \frac{x}{\sqrt{m}}\right) \sim \left(1 - \frac{x^2}{2m}\right)^{(1/2)m} F\left(\frac{\pi}{2} - \frac{x}{\sqrt{m}}\right)$$

$$\sim e^{-(1/4)x^2} F\left(\frac{\pi}{2} - \frac{x}{\sqrt{m}}\right).$$

The differential equation for u in terms of x becomes, in view of (5.3) with x instead of θ

$$\frac{d^2u}{dx^2} + \frac{1}{m}\left\{\lambda + \tfrac{1}{4}(m-2)^2 - \frac{(m-2)(m-4)}{4\cos^2(x/\sqrt{m})}\right\} u = 0.$$

Writing $\lambda = \alpha(\alpha + m - 2)$ and neglecting terms of order $1/m$ we obtain

$$\frac{d^2u}{dx^2} + \frac{1}{m(\cos^2 x/\sqrt{m})}\left\{[\alpha + \tfrac{1}{2}(m-2)]^2\left(1 - \frac{x^2}{m} + \cdots\right) - \tfrac{1}{4}(m-2)(m-4)\right\} u = 0$$

$$\frac{d^2u}{dx^2} + \left(\alpha + \frac{1}{2} - \frac{x^2}{4} + \frac{O(1)}{m}\right) u = 0.$$

Thus if we multiply u by a suitable constant c_m, the limiting function will certainly satisfy (6.1), where we have written x instead of t, provided that the limiting solution is chosen so that F and its derivative remain bounded and do not both vanish at $x = 0$. To see this we require a Lemma which describes the behaviour of $F_m(t)$ near $t = 0$.

LEMMA 4. *Let $F(\theta) = F_m(\theta, \lambda)$ be the function described in Lemma 1. Then at any rate for $m \geq 6$ we have*

$$\frac{F'_m(\pi/2)}{F_m(\pi/2)} = 2\frac{\Gamma(-\tfrac{1}{2}\alpha + \tfrac{1}{2})\Gamma(\tfrac{1}{2}\alpha + (m-1)/2)}{\Gamma(-\tfrac{1}{2}\alpha)\Gamma(\tfrac{1}{2}\alpha + (m-2)/2)} \tag{6.4}$$

(In fact the result is true for all m, but we do not require this.)

We write $c = \cos\theta$ and express the equation (5.5) in terms of c. We obtain

$$(1 - c^2)\frac{d^2F}{dc^2} - (m-1)c\frac{dF}{dc} + \lambda F = 0. \tag{6.5}$$

The solution is analytic at $c=0$ as a function of c and so we obtain a series expansion for F. We write $F(\theta)=g(c)$, where $c=\cos\theta$ and obtain

$$g=\sum_0^\infty a_n c^n$$

The equation (6.5) leads to

$$a_{n+2}=\frac{n(n+m-2)-\lambda}{(n+1)(n+2)}a_n=\frac{(n-\alpha)(n+\alpha+m-2)}{(n+1)(n+2)}a_n, \tag{6.6}$$

since $\lambda=\alpha(\alpha+m-2)$. We write

$$b_n=a_n\frac{\Gamma(\frac{1}{2}n+\frac{1}{2})\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2}n-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}n+\frac{1}{2}(\alpha+m-2))}$$

Then the equation (6.6) takes the simple form

$$b_{n+2}=b_n.$$

Thus $b_n=b_0$ if n is even, $b_n=b_1$ if n is odd and our expansion takes the form

$$g(c)=b_0\sum_{\nu=0}^\infty d_{2\nu}c^{2\nu}+b_1\sum_{\nu=0}^\infty d_{2\nu+1}c^{2\nu+1}$$

where

$$d_n=\frac{\Gamma(\frac{1}{2}n-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}n+\frac{1}{2}(\alpha+m-2))}{\Gamma(\frac{1}{2}n+\frac{1}{2})\Gamma(\frac{1}{2}n+1)}\sim(\tfrac{1}{2}n)^{(1/2)(m-5)},\quad \text{as } n\to\infty.$$

Suppose now that $b_0+b_1\neq 0$. Then, since $m\geq 6$ by hypothesis,

$$d_{2\nu+1}\sim d_{2\nu}\to+\infty,\qquad \text{as } \nu\to\infty.$$

Thus as $c\to 1-$

$$g(c)=\sum_{\nu=0}^\infty(b_0d_{2\nu}+b_1cd_{2\nu+1})c^{2\nu}\sim(b_0+b_1)\sum_{\nu=0}^\infty d_{2\nu}c^{2\nu}\to\infty.$$

However we know from Lemma 1 that F remains bounded as $\theta \to 0$, i.e. as $c \to 1-$. Thus $b_0 + b_1 = 0$, and

$$\frac{F'_m(\pi/2)}{F_m(\pi/2)} = -\frac{g'(0)}{g(0)} = \frac{d_1}{d_0} = \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\alpha+\frac{1}{2}(m-1))\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\alpha+\frac{1}{2}(m-2))\Gamma(\frac{3}{2})\Gamma(2)}.$$

This yields Lemma 4.

We now let $m \to \infty$ for fixed α and express F as a function of $x = \sqrt{m}(\pi/2-\theta) \sim \sqrt{m}\sin(\pi/2-\theta) = c\sqrt{m}$, where $c = \cos\theta$ as in the previous Lemma. Then if $F_m(t) = h(x)$, we deduce that

$$\frac{h'(0)}{h(0)} = \frac{m^{-(1/2)}g'(0)}{g(0)} \sim -2m^{-(1/2)}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha)\Gamma\{m/2+[(\alpha-1)/2]\}}{\Gamma(-\frac{1}{2}\alpha)\Gamma\{m/2+[(\alpha-2)/2]\}},$$

$$\frac{h'(0)}{h(0)} \sim -2m^{-(1/2)}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha)}{\Gamma(-\frac{1}{2}\alpha)}\left(\frac{m}{2}\right)^{1/2} = \frac{-2^{1/2}\Gamma(\frac{1}{2}-\frac{1}{2}\alpha)}{\Gamma(-\frac{1}{2}\alpha)}.$$

Thus $F_\infty(x) = \lim_{m\to\infty} F_m(\pi/2 - x/\sqrt{m}, \alpha(\alpha+m-2))$ has the behaviour described in Theorem F. In particular $F(x)$ satisfies (6.1) and has the right behaviour at $x=0$. These conditions determine $F(x)$ uniquely, and so $F(x)$ is the function given in Theorem F, apart from a multiplicative constant. Also for given α we see that $F \neq 0$, provided that $x > h(\alpha)$ and in particular for $S = S(x)$ not satisfying (6,3).

We deduce that

$$\alpha_\infty(S) = \lim_{m\to\infty} \alpha(S, m)$$

satisfies the inequality (6.3) for α and this proves the first part of Theorem 3. Evidently $\phi_\infty(S)$ is convex for $0 \le S < 1$ and, in view of Theorem 2,

$$\alpha(S, m) \ge \phi_\infty(S), \qquad m = 2, 3, \ldots.$$

Hence Theorem 1 shows that

$$l(N, m) \ge \phi_\infty\left(\frac{1}{N}\right), \qquad N \ge 2, \qquad m \ge 2,$$

and this completes the proof of Theorem 3.

We note in particular that $\phi_\infty(\frac{1}{2})=1$ and so $l(2, m)\geq 1$ for all m. Since the function $|x_1|$ shows that $l(2, m)\leq 1$, we deduce that $l(2, m)=1$ for all m.

In fact (4.1) shows that with the hypotheses of Theorem 1, we have

$$\liminf_{r\to\infty} \frac{B(r)}{r^\alpha}>0 \quad \text{where } \alpha=\phi\left(\frac{1}{N}\right).$$

Now the Corollary to Theorem 3 follows with $N=2$, $\alpha=1$.

7. Further Estimates for $\alpha(S, m)$, and $l(N, m)$

Theorem 3 gives quite good results when S is not much smaller than $1/m$, or $\log N$ is not much bigger than m and m is large. In other cases we can obtain more precise bounds.

In order to obtain such bounds for $\alpha(S, m)$ we employ Lemma 1. We note first that $\alpha(S, m)$ is given by (2.3), i.e.

$$\alpha(\alpha+m-2)=\lambda, \tag{7.1}$$

where λ is so chosen that the smallest positive zero θ_0 of the function u satisfying (5.3) and $u(0)=0$ is such that the spherical cap, $\theta\leq\theta_0$, just satisfies

$$S=\frac{\sigma_{m-1}}{\sigma_m}\int_0^{\theta_0}(\sin t)^{m-2}\, dt. \tag{7.2}$$

The result remains true for $m=2$ if we write $u'(0)=0$, instead of $u(0)=0$. The equation (5.3) reduces in this case to

$$\frac{d^2u}{d\theta^2}+\alpha^2u=0, \tag{7.3}$$

so that $u=\cos(\alpha\theta)$, and $\theta_0=\pi/(2\alpha)$. Also (7.2) gives $S=\theta_0/\pi$. Thus in this case $\alpha=1/(2S)$, which is (3.7). Since $1/(2S)$ is itself a convex function of S we obtain from Theorem 2 that

$$l(N, 2)\geq\frac{N}{2}$$

which is Theorem A. The functions $u=r^{(1/2)N}|\sin(\frac{1}{2}N\theta)|$, which have N tracts and order $\frac{1}{2}N$ show that this result is sharp, i.e. $l(N,2)=\frac{1}{2}N$.

Another case when we can evaluate $\alpha(S,m)$ exactly is the case $m=4$. In this case the equation (5.3) reduces to

$$\frac{d^2u}{d\theta^2}+(\alpha+1)^2u=0,$$

so that $u=\sin\{(\alpha+1)\theta\}$, $\alpha=\pi/\theta_0-1$. Also (7.2) reduces to

$$S=\frac{1}{\pi}\{\theta_0-\tfrac{1}{2}\sin(2\theta_0)\},$$

so that

$$S=\frac{1}{\alpha+1}-\frac{1}{2\pi}\sin\frac{2\pi}{\alpha+1}. \tag{7.4}$$

This equation gives $\alpha=\alpha(S,4)$ in terms of S. In this case we can again show that α is a convex function of S. We note that

$$\frac{dS}{d\alpha}=-\frac{1}{(\alpha+1)^2}\left(1-\cos\frac{2\pi}{\alpha+1}\right)<0, \qquad 0<\alpha<\infty.$$

Also

$$\frac{d^2\alpha}{dS^2}=\frac{d}{dS}\{S'(\alpha)\}^{-1}=\frac{d\alpha}{dS}\cdot\frac{d}{d\alpha}\{S'(\alpha)\}^{-1}=-\frac{S''(\alpha)}{S'(\alpha)^3}.$$

Thus $\alpha(S,4)$ is convex since

$$S''(\alpha)=\frac{d^2S}{d\alpha^2}=\frac{2}{(\alpha+1)^3}\left\{1-\cos\left(\frac{2\pi}{\alpha+1}\right)\right\}+\frac{2\pi}{(\alpha+1)^4}\sin\frac{2\pi}{\alpha+1}>0.$$

We deduce that we may take $\phi(S)=\alpha(S,4)$ in Theorem 1. Thus

$$l(N,4)\geq\alpha_N,$$

where α_N is the solution of the equation

$$\frac{1}{\alpha_N+1}-\frac{1}{2\pi}\sin\frac{2\pi}{\alpha_N+1}=\frac{1}{N}. \tag{7.5}$$

Since $\alpha(S, m)$ is monotonic, we also deduce

THEOREM 5. *The function $\alpha=\alpha(S,4)$ is given by the equation (7.4). Hence*

$$l(N,3)\geq l(N,4)\geq \alpha_N,$$

where α_N is given by (7.5).

We note that

$$\alpha_N=\left(\frac{2\pi^2N}{3}\right)^{1/3}-1+O(N^{-(1/3)}),\quad \text{as } N\to\infty. \tag{7.6}$$

To obtain (7.6) we write $t=2\pi/(\alpha_N+1)$ and deduce that

$$t-\sin t=2\pi/N.$$

Thus

$$2\pi/N=\frac{t^3}{6}+O(t^5),\qquad t=(12\pi/N)^{1/3}+O(N^{-1})$$

$$\alpha_N+1=\frac{2\pi}{t}=\left(\frac{2\pi^2N}{3}\right)^{1/3}+O(N^{-(1/3)}),$$

which is (7.6).

Prof. J. G. Wendel has kindly calculated the value of α_N for us, when $N\leq 16$. He obtains'

N	2	3	4	5	6	7	8	9
α_N	1	1.41167	1.72013	1.97339	2.19110	2.38336	2.55718	2.71583
N		10	11	12	13	14	15	16
α_N		2.86241	2.99898	3.12708	3.24791	3.36243	3.47158	3.57541

Table 1.

In the opposite direction we note that

$$l(N, m) \leq l(N, 2) = \tfrac{1}{2}N, \quad \text{for } N \geq 2, \qquad m \geq 2. \tag{7.7}$$

In particular $1.4116 \leq l(3, m) \leq 1.5$ for $m = 3$, 4 while Theorem 3 yields

$$\tfrac{4}{3} \leq l(3, m) \leq \tfrac{3}{2}, \qquad m \geq 5.$$

It seems quite possible that $l(3, m) = \frac{3}{2}$ for all m. We can divide the sphere in R^3 into four congruent equilateral spherical triangles and this configuration probably yields the extreme functions for $l(4, 3)$. For large values of N and $m > 2$ the inequality (7.7) is certainly not sharp. Thus the function $u_1 = |x_1 x_2 x_3|$ in 3 dimensions is s.h. and has 8 tracts so that

$$2.5571 < l(8, 3) \leq 3.$$

Again in 4 dimensions we may set $x_1 + ix_2 = r_1 e^{i\theta_1}$, $x_3 + ix_4 = r_2 e^{i\theta_2}$ and $u_2 = r_1^{3/2} r_2^{3/2} |\sin(\frac{3}{2}\theta_1)| \, |\sin(\frac{3}{2}\theta_2)|$. The function u_2 is certainly s.h. in R^4 since it is continuous and harmonic at all points where u_2 is positive. For at such points we have

$$\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} = \frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_4^2} = 0.$$

But u_2 has 9 tracts and order 3, so that

$$2.7158 < l(9, 4) \leq 3.$$

The function u_2, unlike u_1, is not the modulus of a harmonic polynomial and suggests that harmonic polynomials may not yield extremal examples for $l(N, m)$ in all cases. In fact it seems reasonable to conjecture that the harmonic polynomials referred to above [2] have the maximum possible number of tracts for given degree and for these polynomials the maximum number of tracts for degree N is certainly 2^N, if the number of variables is allowed to be arbitrary.

We note in particular that $u_3 = |x_1 x_2 x_3 x_4|$ has 16 tracts and order 4, so that

$$3.5754 < l(16, 4) \leq 4.$$

8. Proof of Theorem 4

When N is large we can obtain convenient bounds for $\alpha(S, m)$ from the Sturm-Liouville theory. We note that

$$\frac{1}{\sin^2\theta}=\sum_{n=-\infty}^{+\infty}\left(\frac{1}{\theta+n\pi}\right)^2$$

and so

$$\frac{d}{d\theta}\left(\frac{1}{\sin^2\theta}-\frac{1}{\theta^2}\right)=2\sum_{1}^{\infty}\left\{\left(\frac{1}{n\pi-\theta}\right)^3-\left(\frac{1}{n\pi+\theta}\right)^3\right\}>0,\qquad 0<\theta<\pi. \tag{8.1}$$

Thus, as θ increases from 0 to $\pi/2$, $(\sin\theta)^{-2}-\theta^{-2}$ increases from $\frac{1}{3}$ to $1-4/\pi^2$. Hence

$$\frac{1}{\theta^2}+\frac{1}{3}<\frac{1}{\sin^2\theta}\leq\frac{1}{\theta^2}+1-\frac{4}{\pi^2},\qquad 0<\theta\leq\frac{\pi}{2}.$$

Since $\theta=0$, θ_0 are successive zeros of a solution of the equation (5.3) it follows(1) that we obtain lower and upper bounds for θ_0 by replacing

$$\lambda+\tfrac{1}{4}(m-2)^2+\frac{(m-2)(4-m)}{4\sin^2\theta}$$

by a smaller or larger function. We shall assume $\theta_0\leq\pi/2$, which corresponds to $S\leq\frac{1}{2}$. Also the equation(2)

$$\frac{d^2v}{d\theta^2}+\left(c^2+\frac{\frac{1}{4}-k^2}{\theta^2}\right)v=0$$

has a solution $v=\theta^{1/2}J_k(c\theta)$, where $J_k(\theta)$ is Bessel's function of order k.

We recall (7.1) and define

$$k=\tfrac{1}{2}(m-3),\quad\text{so that}\quad\frac{(m-2)(4-m)}{4}=\tfrac{1}{4}-k^2.$$

(1) See e.g. [12, p. 19]

(2) Ibid. p. 17

Then if

$$c_1^2=[\alpha+\tfrac{1}{2}(m-2)]^2+\tfrac{1}{3}(\tfrac{1}{4}-k^2),\qquad c_2^2=[\alpha+\tfrac{1}{2}(m-2)]^2+\left(1-\frac{4}{\pi^2}\right)(\tfrac{1}{4}-k^2) \tag{8.2}$$

θ_0 lies between the first zero of $J_k(c_1\theta)$ and $J_k(c_2\theta)$, i.e. θ_0 lies between j_k/c_1 and j_k/c_2, where j_k is the smallest zero of $J_k(\theta)$. If $m=4$, $k=\tfrac{1}{2}$, $c_1=c_2=\alpha+1$ and $j_{1/2}=\pi$. If $m=3$, $k=0$ and

$$c_1^2=\alpha^2+\alpha+\tfrac{1}{3},\qquad c_2^2=\alpha^2+\alpha+\tfrac{1}{2}-\frac{1}{\pi^2},\qquad j_0=2.4048\cdots$$

Also from (7.2)

$$S=\tfrac{1}{2}(1-\cos\theta_0)=\sin^2\frac{\theta_0}{2},\qquad 0<\theta_0<\frac{\pi}{2}.$$

Thus

$$(\alpha+\tfrac{1}{2})^2+\tfrac{1}{4}-\frac{1}{\pi^2}>\frac{j_0^2}{\theta^2}>j_0^2\left[\frac{1}{4\sin^2(\tfrac{1}{2}\theta_0)}+\frac{4}{\pi^2}-\frac{1}{2}\right],$$

since $(\sin\tfrac{1}{2}\theta)^{-2}-(\tfrac{1}{2}\theta)^{-2}$ increases with θ for $0\le\theta\le\pi/2$. Thus

$$(\alpha+\tfrac{1}{2})^2>\frac{j_0^2}{4S}+\frac{1}{\pi^2}-\frac{1}{4}-j_0^2\left(\frac{1}{2}-\frac{4}{\pi^2}\right)>\frac{j_0^2}{4}\left(\frac{1}{S}-\frac{1}{2}\right),$$

and

$$\alpha=\alpha(S,3)>\frac{j_0}{2}\left(\frac{2-S}{2S}\right)^{1/2}-\tfrac{1}{2}.$$

This yields (3.6) and improves slightly Talpur's Theorem B. We may again apply Theorem 1, since $\phi_3(S)$ is convex and decreasing and deduce

$$l(N,3)\ge\frac{j_0}{2}(N-\tfrac{1}{2})^{1/2}-\tfrac{1}{2}=1.2024\cdots(N-\tfrac{1}{2})^{1/2}-\tfrac{1}{2}. \tag{8.3}$$

Finally we turn to the case $m \geq 4$. We note that by (7.2)

$$S = \frac{\sigma_{m-1}}{\sigma_m}\int_0^{\theta_0}(\sin t)^{m-2}\,dt \geq \frac{\sigma_{m-1}}{\sigma_m}\int_0^{\theta_0}(\sin t)^{m-2}\cos t\,dt$$

$$= \frac{\sigma_{m-1}}{(m-1)\sigma_m}(\sin \theta_0)^{m-1}.$$

Thus

$$\left(\frac{d_m}{S}\right)^{\frac{1}{m-1}} \leq \frac{1}{\sin \theta_0}, \quad \text{where} \quad d_m = \frac{\sigma_{m-1}}{(m-1)\sigma_m} = \frac{\Gamma(\frac{1}{2}m)}{2\pi^{1/2}\Gamma\{\frac{1}{2}(m+1)\}}.$$

We also recall that by (8.1)

$$\left(\frac{1}{\sin t}\right)^2 - \left(\frac{1}{t}\right)^2 \quad \text{and hence also} \quad \frac{1}{\sin t} - \frac{1}{t}$$

increases with increasing t for $0 < t \leq \pi/2$ and so

$$\frac{1}{\sin \theta_0} \leq \frac{1}{\theta_0} + 1 - \frac{2}{\pi} < \frac{1}{\theta_0} + \frac{2}{5}, \qquad 0 < \theta_0 < \frac{\pi}{2}.$$

Again we noted earlier that

$$\theta_0 \geq \frac{j_k}{c_1},$$

where c_1 is given by (8.2). Thus

$$c_1^2 = [\alpha + \tfrac{1}{2}(m-2)]^2 - \tfrac{1}{12}(m-3)^2 + \tfrac{1}{12} \geq j_k^2\frac{1}{\theta_0^2} \geq j_k^2\left\{\frac{1}{\sin\theta_0} - \frac{2}{5}\right\}^2$$

$$\geq j_k^2\left\{\left(\frac{d_m}{S}\right)^{1(m-1)} - \frac{2}{5}\right\}^2$$

Thus, since $m \geq 4$, we deduce that

$$\alpha \geq j_k\left(\frac{d_m}{S}\right)^{1/(m-1)} - \frac{2}{5}j_k - \tfrac{1}{2}(m-2).$$

This proves (3.5) and concludes the proof of Theorem 4, since the right hand side of (3.5) is clearly a convex decreasing function of S. For completeness we include a table of the quantity

$$\beta_N = 1.2024(N-\tfrac{1}{2})^{1/2}-\tfrac{1}{2},$$

which by (8.3) gives a lower bound for $l(N,3)$. We are indebted to Prof. H. L. Montgomery for the calculations.

N	2	3	4	5	6	7	8	9
β_N	0.97263	1.40116	1.74948	2.05068	2.31988	2.56553	2.79291	3.00557
N	10	11	12	13	14	15	16	
β_N	3.20605	3.39622	3.57754	3.75113	3.91790	4.07860	4.23385	

Table 2

We note that $\beta_N > \alpha_N$, for $N \geq 4$. The table also shows that a s.h. function of lower order 2 in R^3 and in particular the modulus of a harmonic polynomial of degree 2 in 3 variables can have at most 4 tracts, since $l(5,3)>2$. This bound is sharp as the polynomial $x_1 x_2$ shows. Similarly the modulus of a polynomial of degree 3 in 3 variables and more generally a s.h. function of order 3 in R^3 can have at most 8 tracts, since $l(9,3)>3$. This result is also sharp as the polynomial $x_1 x_2 x_3$ shows.

Finally we note that the lower bound for $l(N, m)$ can by its nature be sharp only if R^m can be exactly divided into N congruent right circular cones. This is the case only when $N=2$ or $m=2$ and that is why our results are sharp only in these 2 cases. However we have seen that our results always give the correct order of magnitude for $l(N, m)$.

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Institute of Advanced Study,
Princeton, N.J. and
Imperial College,
London, S.W.7.

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