Atoms of a Group Valued Measures

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Atoms of Group Valued Measures

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Any real valued measure may be written (uniquely) as a sum of an atomic and of an atomless measure. This result was extended first time by J. Hoffmann-Jørgensen ([1] Theorem 6) to absolutely continuous measures with values in locally convex spaces and then by K. Musiał ([2] Theorem 1) to the more general case of group valued measures satisfying *ccc*. The aim of this paper is to extend this result once more for group valued measures satisfying *asc* (Proposition 3.2 and Theorem 3.6). Any measure satisfying *ccc* (even locally) satisfies *asc* (Proposition 3.4). If the group G is complete any G-valued measure satisfies *asc* (Propositions 3.5 and 2.1). The nature of atoms for measures not satisfying *ccc* becomes by far more complicated and so a great part of this paper is dedicated to their study.

Throughout this paper we shall denote by $\mathfrak{R}$ *a* δ*-ring* (i.e. $\mathfrak{R} \neq \emptyset$ and for any sequence $(A_n)_{n\in\mathbf{N}}$ in $\mathfrak{R}$ we have $\bigcap_{n\in\mathbf{N}} A_n \in \mathfrak{R}$ and $A_0 \Delta A_1 \in \mathfrak{R}$) *by* $\mathfrak{K}$ *a subset of* $\mathfrak{R}$ *such that the union of any finite family in* $\mathfrak{K}$ *belongs to* $\mathfrak{K}$, *and by* G *a Hausdorff topological commutative group. We consider* $\mathfrak{R}$ *ordered by the inclusion relation and denote by* Λ *the set of lower directed nonempty subsets of* $\mathfrak{R}\backslash\{\emptyset\}$. For any $\mathfrak{A} \in \Lambda$ we denote by $\mathfrak{F}(\mathfrak{A})$ the filter on $\mathfrak{R}$ generated by the filter base

$$\{\{B \in \mathfrak{A} \mid B \subset A\} \mid A \in \mathfrak{A}\}.$$

A system of null sets of $\mathfrak{R}$ is a nonempty subset $\mathfrak{N}$ of $\mathfrak{R}$ such that: (a) any set of $\mathfrak{R}$ belongs to $\mathfrak{N}$ if it is contained in a set of $\mathfrak{N}$; (b) the union of any countable family in $\mathfrak{N}$ belongs to $\mathfrak{N}$ if it belongs to $\mathfrak{R}$.

1. Atoms

Throughout this section we shall denote by $\mathfrak{N}$ *a system of null sets of* $\mathfrak{R}$. We denote by $\Lambda(\mathfrak{N})$ the set of subsets $\mathfrak{A} \neq \emptyset$ of $\mathfrak{R}\backslash\mathfrak{N}$ such that the intersection of any countable family in $\mathfrak{A}$ belongs to $\mathfrak{A}$. It is obvious that $\Lambda(\mathfrak{N}) \subset \Lambda$. The maximal elements of $\Lambda(\mathfrak{N})$ (for the inclusion relation) will be called *atoms* (with respect to

$\mathfrak{N}$). A *set-atom* (with respect to $\mathfrak{N}$) is a set $A \in \mathfrak{R} \setminus \mathfrak{N}$ such that for any subset B of A belonging to $\mathfrak{R}$ we have either $B \in \mathfrak{N}$ or $A \setminus B \in \mathfrak{N}$. We say that $\mathfrak{N}$ satisfies the *countable chain condition* (abreviated *ccc*) if any disjoint family $(A_\iota)_{\iota \in I}$ in $\mathfrak{R} \setminus \mathfrak{N}$ (i.e. $A_\iota \cap A_{\iota'} = \emptyset$ for different elements ι, ι' of I) is countable. We say that $\mathfrak{N}$ satisfies *locally ccc* if any disjoint family $(A_\iota)_{\iota \in I}$ in $\mathfrak{R} \setminus \mathfrak{N}$ is countable if $\bigcup_{\iota \in I} A_\iota$ is contained in a set of $\mathfrak{R}$.

PROPOSITION 1.1. *Assume $\mathfrak{N}$ satisfies locally ccc and let $\mathfrak{A} \in \Lambda$. Then there exists a decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in $\mathfrak{A}$ such that we have for any $A \in \mathfrak{A}$*

$$\bigcap_{n \in \mathbf{N}} A_n \setminus A \in \mathfrak{N}.$$

Assume that for any decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in $\mathfrak{A}$ there exists $A \in \mathfrak{A}$ such that

$$\bigcap_{n \in \mathbf{N}} A_n \setminus A \notin \mathfrak{N}.$$

Let ω_1 be the first uncountable ordinal number. We shall construct inductively a family $(B_\xi)_{\xi < \omega_1}$ in $\mathfrak{A}$ such that

$$\bigcap_{\xi < \mu} B_\xi \setminus B_\mu \notin \mathfrak{N}$$

for any $\mu < \omega_1$. Let $\mu < \omega_1$ and assume the family $(B_\xi)_{\xi < \mu}$ with the required property is constructed. Since this family is countable and since $\mathfrak{A}$ is lower directed there exists a decreasing sequence $(A_n)_{n \in \mathbf{N}}$ in $\mathfrak{A}$ such that

$$\bigcap_{n \in \mathbf{N}} A_n \subset \bigcap_{\xi < \mu} B_\xi.$$

By the hypothesis there exists $B_\mu \in \mathfrak{A}$ with

$$\bigcap_{n \in \mathbf{N}} A_n \setminus B_\mu \notin \mathfrak{N}.$$

Hence

$$\bigcap_{\xi < \mu} B_\xi \setminus B_\mu \notin \mathfrak{N}.$$

The existence of the family $(\bigcap_{\xi < \mu} B_\xi \setminus B_\mu)_{\mu < \omega_1}$ contradicts the hypothesis that $\mathfrak{N}$ satisfies locally *ccc*. ∎

PROPOSITION 1.2. *For any* $A \in \mathfrak{R}\backslash\mathfrak{N}$ *we set*

$$\mathfrak{A}_A := \{B \in \mathfrak{R} \mid A\backslash B \in \mathfrak{N}\}$$

Then:

(a) $\mathfrak{A}_A \in \Lambda(\mathfrak{N})$;

(b) $\mathfrak{A}_A$ *is an atom* $\Leftrightarrow$ A *is a set-atom;*

(c) *if* $\mathfrak{N}$ *satisfies locally ccc then for any atom* $\mathfrak{A}$ *there exists a set-atom* $A \in \mathfrak{R}\backslash\mathfrak{N}$ *with* $\mathfrak{A}_A = \mathfrak{A}$; *in particular for any* $B \in \mathfrak{R}$ *the set of atoms* $\mathfrak{A}$ *such that* $B \in \mathfrak{A}$ *is countable.*

(a) The intersection of any sequence in $\mathfrak{A}_A$ belongs to $\mathfrak{A}_A$.

(b) Let A be a set-atom. Let $\mathfrak{A} \in \Lambda(\mathfrak{N})$ with $\mathfrak{A}_A \subset \mathfrak{A}$ and let $B \in \mathfrak{A}$. Then $A \cap B \in \mathfrak{A}$ and therefore $A \cap B \notin \mathfrak{N}$. We get $A\backslash B \in \mathfrak{N}$ and therefore $B \in \mathfrak{A}_A$. Hence $\mathfrak{A}_A = \mathfrak{A}$. Thus $\mathfrak{A}_A$ is an atom.

Assume now $\mathfrak{A}_A$ is an atom. If A is not a set-atom then there exists $B \in \mathfrak{R}\backslash\mathfrak{N}$ such that $B \subset A$ and $A\backslash B \notin \mathfrak{N}$. We get $\mathfrak{A}_B \in \Lambda(\mathfrak{N})$, $\mathfrak{A}_A \subset \mathfrak{A}_B$, and $B \in \mathfrak{A}_B\backslash\mathfrak{A}_A$. Hence $\mathfrak{A}_A$ is not a maximal element of $\Lambda(\mathfrak{N})$ and this is a contradiction.

(c) Assume $\mathfrak{N}$ satisfies locally *ccc* and let $\mathfrak{A}$ be an atom. By Proposition 1.1 there exists $A \in \mathfrak{A}$ such that $\mathfrak{A} \subset \mathfrak{A}_A$. Since $\mathfrak{A}$ is maximal in $\Lambda(\mathfrak{N})$ we deduce by (a) $\mathfrak{A} = \mathfrak{A}_A$. By (b) A is a set-atom.

Let $B \in \mathfrak{R}$ and assume that the set of atoms $\mathfrak{A}$ such that $B \in \mathfrak{A}$ is uncountable. Let ω_1 be the first uncountable ordinal number. There exists a family $(\mathfrak{A}_\xi)_{\xi<\omega_1}$ of atoms such that $B \in \mathfrak{A}_\xi$ for any $\xi < \omega_1$ and such that $\mathfrak{A}_\xi \neq \mathfrak{A}_\eta$ for any $\xi < \eta < \omega_1$. For any $\xi < \omega_1$ let A_ξ be a set-atom such that $\mathfrak{A}_\xi = \mathfrak{A}_{A_\xi}$. Let $\xi < \eta$. If $A_\xi \cap A_\eta \notin \mathfrak{N}$ then $A_\eta\backslash A_\xi \in \mathfrak{N}$ and we get for any $C \in \mathfrak{A}_\xi$

$$A_\xi\backslash C \in \mathfrak{N}, \qquad A_\eta\backslash C \in \mathfrak{N}, \qquad C \in \mathfrak{A}_\eta.$$

Hence $\mathfrak{A}_\xi \subset \mathfrak{A}_\eta$ and this leads to the contradiction $\mathfrak{A}_\xi = \mathfrak{A}_\eta$. Hence $A_\xi \cap A_\eta \in \mathfrak{N}$. We get for any $\eta < \omega_1$

$$A_\eta \cap B \backslash \bigcup_{\xi<\eta} A_\xi \in \mathfrak{A}_\eta$$

Hence $(A_\eta \cap B\backslash \bigcup_{\xi<\eta} A_\xi)_{\eta<\omega_1}$ is an uncountable disjoint family of subsets of B in $\mathfrak{R}\backslash\mathfrak{N}$ and this contradicts the hypothesis that $\mathfrak{N}$ satisfies locally *ccc*. ■

Remark. From (c) it follows that if $\mathfrak{N}$ satisfies locally *ccc* the atoms and the set-atoms may be identified.

PROPOSITION 1.3. *Let* $\mathfrak{A}$ *be an atom. Then*

(a) *If* $B\in\mathfrak{R}$ *and if there exists* $A\in\mathfrak{A}$ *such that* $A\backslash B\in\mathfrak{N}$ *then* $B\in\mathfrak{A}$.

(b) *If* $B\in\mathfrak{R}$ *and if* $A\cap B\notin\mathfrak{N}$ *for any* $A\in\mathfrak{A}$ *then* $B\in\mathfrak{A}$.

(c) *If* $(A_\iota)_{\iota\in I}$ *is a countable family in* $\mathfrak{R}$ *whose union belongs to* $\mathfrak{A}$ *then there exists* $\iota\in I$ *with* $A_\iota\in\mathfrak{A}$.

(d) $\mathfrak{A}$ *is a maximal element of* Λ.

(a) The set $\{C\in\mathfrak{R}\mid \exists A\in\mathfrak{A}, A\backslash C\in\mathfrak{N}\}$ belongs to $\Lambda(\mathfrak{N})$ and contains $\mathfrak{A}$. Since $\mathfrak{A}$ is a maximal element of $\Lambda(\mathfrak{N})$ this set coincides with $\mathfrak{A}$. Hence $B\in\mathfrak{A}$.

(b) The set $\{C\in\mathfrak{R}\mid \exists A\in\mathfrak{A}, C\supset A\cap B\}$ belongs to $\Lambda(\mathfrak{N})$ and contains $\mathfrak{A}$. Since $\mathfrak{A}$ is a maximal element of $\Lambda(\mathfrak{N})$ this set coincides with $\mathfrak{A}$. Hence $B\in\mathfrak{A}$.

(c) Assume $A_\iota\notin\mathfrak{A}$ for any $\iota\in I$. By (b) there exists for any $\iota\in I$ a $B_\iota\in\mathfrak{A}$ such that $A_\iota\cap B_\iota\in\mathfrak{N}$. From

$$\left(\bigcup_{\iota\in I} A_\iota\right)\cap\left(\bigcap_{\iota\in I} B_\iota\right)\subset\bigcup_{\iota\in I}(A_\iota\cap B_\iota)$$

it follows $(\bigcup_{\iota\in I}A_\iota)\cap(\bigcap_{\iota\in I}B_\iota)\in\mathfrak{N}\cap\mathfrak{A}$ and this is a contradiction.

(d) Let $\mathfrak{B}\in\Lambda$ with $\mathfrak{A}\subset\mathfrak{B}$. Let $B\in\mathfrak{B}\backslash\mathfrak{A}$. By (b) there exists $A\in\mathfrak{A}$ with $A\cap B\in\mathfrak{N}$. By (a) we get $A\backslash B\in\mathfrak{A}$ and therefore

$$\emptyset=(A\backslash B)\cap B\in\mathfrak{B}$$

which is a contradiction. Hence $\mathfrak{B}=\mathfrak{A}$ and $\mathfrak{A}$ is a maximal element of Λ. ■

COROLLARY 1.4. *Let* $\mathfrak{N}'$, $\mathfrak{N}''$ *be two systems of null sets on* $\mathfrak{R}$ *and let* $\mathfrak{A}$ *be an atom with respect to* $\mathfrak{N}'$. *Then either* $\mathfrak{A}$ *is an atom with respect to* $\mathfrak{N}''$ *or* $\mathfrak{A}\cap\mathfrak{N}''\neq\emptyset$.

If $\mathfrak{A}\cap\mathfrak{N}''=\emptyset$ then $\mathfrak{A}\in\Lambda(\mathfrak{N}'')$. By the Proposition 1.3(d) $\mathfrak{A}$ is a maximal element of Λ and therefore a fortiori it is a maximal element of $\Lambda(\mathfrak{N}'')$. ■

PROPOSITION 1.5. *Let* Φ *be a countable set of atoms. Then there exists a disjoint family* $(A_\mathfrak{A})_{\mathfrak{A}\in\Phi}$ *in* $\mathfrak{R}$ *such that* $A_\mathfrak{A}\in\mathfrak{A}$ *for any* $\mathfrak{A}\in\Phi$.

Let $\mathfrak{A}$, $\mathfrak{B}$ be two different atoms of Φ and let $A_{\mathfrak{A},\mathfrak{B}}\in\mathfrak{A}\backslash\mathfrak{B}$. By Proposition 1.3(b) there exists $B_{\mathfrak{A},\mathfrak{B}}\in\mathfrak{B}$ such that

$$A_{\mathfrak{A},\mathfrak{B}}\cap B_{\mathfrak{A},\mathfrak{B}}\in\mathfrak{N}.$$

By Proposition 1.3(a)

$$A_{\mathfrak{A},\mathfrak{B}} \setminus B_{\mathfrak{A},\mathfrak{B}} \in \mathfrak{A}, \qquad B_{\mathfrak{A},\mathfrak{B}} \setminus A_{\mathfrak{A},\mathfrak{B}} \in \mathfrak{B}.$$

We set for any $\mathfrak{A} \in \Phi$

$$A_{\mathfrak{A}} := \bigcap_{\substack{\mathfrak{B} \in \Phi \\ \mathfrak{B} = \mathfrak{A}}} ((A_{\mathfrak{A},\mathfrak{B}} \setminus B_{\mathfrak{A},\mathfrak{B}}) \cap (B_{\mathfrak{B},\mathfrak{A}} \setminus A_{\mathfrak{B},\mathfrak{A}})).$$

It is obvious that $(\mathfrak{A}_A)_{\mathfrak{A} \in \Phi}$ possesses the required properties. ■

PROPOSITION 1.6. *Let Φ be a set of atoms such that for any $A \in \mathfrak{R}$ the set $\{\mathfrak{A} \in \Phi \mid A \in \mathfrak{A}\}$ is countable. Then there exists a family $(A_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ in $\mathfrak{R}$ such that $A_{\mathfrak{A}} \in \mathfrak{A} \setminus \mathfrak{A}'$ for any $\mathfrak{A}, \mathfrak{A}' \in \Phi$, $\mathfrak{A} \neq \mathfrak{A}'$.*

Let $(B_{\mathfrak{A}})_{\mathfrak{A}}$ be a family in $\mathfrak{R}$ such that $B_{\mathfrak{A}} \in \mathfrak{A}$ for any $\mathfrak{A} \in \Phi$. Let $\mathfrak{A} \in \Phi$ and let

$$\Psi(\mathfrak{A}) := \{\mathfrak{A}' \in \Phi \mid B_{\mathfrak{A}} \in \mathfrak{A}'\}.$$

By the hypothesis $\Psi(\mathfrak{A})$ is countable. By Proposition 1.5 there exists a disjoint family $(C_{\mathfrak{A}'})_{\mathfrak{A}' \in \Psi(\mathfrak{A})}$ such that $C_{\mathfrak{A}'} \in \mathfrak{A}'$ for any $\mathfrak{A}' \in \Psi(\mathfrak{A})$. We set

$$A_{\mathfrak{A}} := B_{\mathfrak{A}} \cap C_{\mathfrak{A}}.$$

Let $\mathfrak{A}, \mathfrak{A}'$ be two different atoms of Φ. Then $A_{\mathfrak{A}} \in \mathfrak{A}$. If $\mathfrak{A}' \notin \Psi(\mathfrak{A})$ then $B_{\mathfrak{A}} \notin \mathfrak{A}'$ and therefore $A_{\mathfrak{A}} \notin \mathfrak{A}'$ (Proposition 1.3(a)). If $\mathfrak{A}' \in \Psi(\mathfrak{A})$ then

$$A_{\mathfrak{A}} \cap A_{\mathfrak{A}'} \subset C_{\mathfrak{A}} \cap C_{\mathfrak{A}'} = \varnothing$$

and this shows that $A_{\mathfrak{A}} \notin \mathfrak{A}'$. ■

2. Measures

A *measure on* $\mathfrak{R}$ is a map μ of $\mathfrak{R}$ into a Hausdorff topological commutative group such that for any countable disjoint family $(A_\iota)_{\iota \in I}$ in $\mathfrak{R}$ whose union belongs to $\mathfrak{R}$ the family $(\mu(A_\iota))_{\iota \in I}$ is summable and its sum is $\mu(\bigcup_{\iota \in I} A_\iota)$. A measure μ on $\mathfrak{R}$ is called *$\mathfrak{K}$-regular* if for any $A \in \mathfrak{R}$ and for any neighbourhood V of $\mu(A)$ there exists $K \in \mathfrak{K}$ contained in A such that

$$\{\mu(B) \mid B \in \mathfrak{R}, K \subset B \subset A\} \subset V.$$

For any measure μ on $\mathfrak{R}$ we set

$$\mathfrak{N}(\mu) := \{A \in \mathfrak{R} \mid \forall B \in \mathfrak{R}, B \subset A \Rightarrow \mu(B) = 0\}.$$

$\mathfrak{N}(\mu)$ is a system of null sets of $\mathfrak{R}$. We say that μ *satisfies ccc* (resp. that μ *satisfies locally ccc*) if $\mathfrak{N}(\mu)$ satisfies *ccc* (resp. satisfies locally *ccc*). It is obvious that the set of G-valued measures on $\mathfrak{R}$, the set of $\mathfrak{K}$-regular G-valued measures on $\mathfrak{R}$, and the set of G-valued measures on $\mathfrak{R}$ satisfying *ccc* or satisfying locally *ccc* are subgroups of $G^{\mathfrak{R}}$. For any measure μ on $\mathfrak{R}$ and for any $\mathfrak{A} \in \Lambda$ we denote by $\mu(\mathfrak{F}(\mathfrak{A}))$ the image of the filter $\mathfrak{F}(\mathfrak{A})$ through μ (i.e. the filter generated by the filter base $\{\mu(\mathfrak{B}) \mid \mathfrak{B} \in \mathfrak{F}(\mathfrak{A})\}$); if this filter converges we denote by $\mu_{\mathfrak{A}}$ its limit. By Proposition 1.1 $\mu(\mathfrak{F}(\mathfrak{A}))$ converges for any measure μ satisfying locally *ccc* and for any $\mathfrak{A} \in \Lambda$. We call *atom of* μ any atom with respect to $\mathfrak{N}(\mu)$. An atom $\mathfrak{A}$ of μ is called *improper* if $\mu(\mathfrak{F}(\mathfrak{A}))$ converges to 0; otherwise we call it *proper.* Let $\mathfrak{A}$ be an improper atom of μ. If for any $A \in \mathfrak{A}$ there exists a proper atom $\mathfrak{A}'$ of μ such that $A \in \mathfrak{A}'$ we say that $\mathfrak{A}$ is *of the first kind.* An improper atom which is not of the first kind will be called *of the second kind.* We call *set-atom of* μ any set-atom with respect to $\mathfrak{N}(\mu)$.

A *preorder relation* on a set I is a binary relation $\leqslant$ on I such that:

(a) $\iota \in I \Rightarrow \iota \leqslant \iota$;

(b) $\iota, \iota', \iota'' \in I, \iota \leqslant \iota', \iota' \leqslant \iota'' \Rightarrow \iota \leqslant \iota''$.

An *upper directed preordered set* is a set I endowed with a preorder relation $\leqslant$ such that for any $\iota', \iota'' \in I$ there exists $\iota \in I$ with $\iota' \leqslant \iota, \iota'' \leqslant \iota$. The section filter of an upper directed nonempty set $(I, \leqslant)$ is the filter on I generated by the filter base

$$\{\{\iota \in I \mid \iota \geqslant \lambda\} \mid \lambda \in I\}.$$

A *net* in a set X is a pair (I, f) such that I is an upper directed preordered set and f is a map of I into X.

Let X be a topological space. An ω-*net in* X is a net (I, f) in X such that for any increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in X the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ is convergent. An ω-*filter on* X is a filter $\mathfrak{F}$ on X such that there exists an ω-net (I, f) in X such that $f(\mathfrak{F}) \subset \mathfrak{F}$, where $\mathfrak{G}$ denotes the section filter of I. An ω *space* is a topological space for which any ω-filter converges.

PROPOSITION 2.1. *Any ω-filter on a uniform space is a Cauchy filter. Hence any complete uniform space is an ω-space.*

Let X be a uniform space, let (I, f) be an ω-net in X, and let $\mathfrak{F}$ be the section filter of I. Let further U be an arbitrary entourage (=vicinity) of X and let V be an entourage of X such that $V \circ V^{-1} \subset U$. Assume that for any $\iota \in I$ there exists $\lambda \in I$ such that $\lambda \geqslant \iota$ and $(f(\iota), f(\lambda)) \notin V$. Then we may construct inductively an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $(f(\iota_n), f(\iota_{n+1})) \notin V$ for any $n \in N$. The sequence $(f(\iota_n))_{n\in\mathbf{N}}$ being convergent this is a contradiction. Hence there exists $\iota \in I$ with $(f(\iota), f(\lambda)) \in V$ for any $\lambda \in I$, $\lambda \geqslant \iota$. We get $(f(\iota'), f(\iota'')) \in U$ for any $\iota', \iota'' \in I$ with $\iota' \geqslant \iota$, $\iota'' \geqslant \iota$. Hence $f(\mathfrak{F})$ is a Cauchy filter. ■

PROPOSITION 2.2. *For any measure μ on $\mathfrak{R}$ and for any $\mathfrak{A} \in \Lambda$, $\mu(\mathfrak{F}(\mathfrak{A}))$ is an ω-filter and therefore a Cauchy filter.*

Let us order $\mathfrak{A}$ by the converse inclusion relation, let $\mathfrak{G}$ be the section filter of $\mathfrak{A}$, and let $\mu \mid \mathfrak{A}$ be the restriction of μ to $\mathfrak{A}$. Then $(\mathfrak{A}, \mu \mid \mathfrak{A})$ is an ω-net and

$$\mu(\mathfrak{F}(\mathfrak{A})) = \mu \mid \mathfrak{A}(\mathfrak{G}).$$

Hence $\mu(\mathfrak{F}(\mathfrak{A}))$ is an ω-filter. By Proposition 2.1 it is a Cauchy filter. ■

PROPOSITION 2.3. *Let μ be a measure on $\mathfrak{R}$ and let $\mathfrak{A}$ be a maximal element of Λ. Then either $\mathfrak{A}$ is an atom of μ or $\mu(\mathfrak{F}(\mathfrak{A}))$ converges to* 0.

Assume that $\mu(\mathfrak{F}(\mathfrak{A}))$ does not converge to 0. By Proposition 2.2 there exist a 0-neighbourhood V and an $A \in \mathfrak{A}$ such that

$$\{\mu(B) \mid B \in \mathfrak{A}, B \subset A\} \cap V = \varnothing.$$

Let $(A_n)_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{A}$. If $\bigcap_{n\in\mathbf{N}} A_n = \varnothing$ then there exists a decreasing sequence $(B_n)_{n\in\mathbf{N}}$ in $\mathfrak{A}$ with empty intersection and such that $B_0 \subset A$. It follows that $(\mu(B_n))_{n\in\mathbf{N}}$ converges to 0 and this contradicts the above relation. Hence $\bigcap_{n\in\mathbf{N}} A_n \neq \varnothing$. The set

$$\left\{B \in \mathfrak{R} \mid \exists C \in \mathfrak{A}, \left(\bigcap_{n\in\mathbf{N}} A_n\right) \cap C \subset B\right\}$$

belongs to Λ and contains $\mathfrak{A}$. Since $\mathfrak{A}$ is maximal it coincides with $\mathfrak{A}$. We deduce $\bigcap_{n\in\mathbf{N}} A_n \in \mathfrak{A}$. Since $\mathfrak{A} \cap \mathfrak{N}(\mu) = \varnothing$ we deduce $\mathfrak{A} \in \Lambda(\mathfrak{N}(\mu))$. It is obvious that $\mathfrak{A}$ is a maximal element of $\Lambda(\mathfrak{N}(\mu))$. Hence $\mathfrak{A}$ is an atom of μ. ■

COROLLARY 2.4. *Let μ, ν be measures on $\mathfrak{R}$ and let $\mathfrak{A}$ be an atom of μ. Then either $\mathfrak{A}$ is an atom of ν or $\nu(\mathfrak{F}(\mathfrak{A}))$ converges to* 0.

By Proposition 1.3(d) $\mathfrak{A}$ is a maximal element of Λ and the assertion follows from the proposition. ■

PROPOSITION 2.5. *Let μ be a $\mathfrak{K}$-regular measure on $\mathfrak{R}$ and let $\mathfrak{A}$ be a proper atom of μ. Then for any $A \in \mathfrak{A}$ there exists $K \in \mathfrak{K} \cap \mathfrak{A}$ with $K \subset A$.*

By Proposition 2.3 $\mu(\mathfrak{F}(\mathfrak{A}))$ is a Cauchy filter. Since $\mathfrak{A}$ is a proper atom of μ, 0 is not an adherent point of this filter. Hence there exist a neighbourhood V of 0 and a set $B \in \mathfrak{A}$ such that

$$\{\mu(C) \mid C \in \mathfrak{A}, C \subset B\} \cap V = \varnothing.$$

Since μ is $\mathfrak{K}$-regular there exists $K \in \mathfrak{K}$ such that $K \subset A \cap B$ and

$$\{\mu(C) \mid C \in \mathfrak{R}, C \subset A \cap B \setminus K\} \subset V.$$

Let $C \in \mathfrak{A}$. If $C \cap K \in \mathfrak{N}(\mu)$ then by Proposition 1.3(a) $C \setminus K \in \mathfrak{A}$ and therefore

$$A \cap B \cap (C \setminus K) \in \mathfrak{A}, \qquad \mu(A \cap B \cap (C \setminus K)) \in V$$

which is a contradiction. Hence $C \cap K \notin \mathfrak{N}(\mu)$ for any $C \in \mathfrak{A}$. By Proposition 1.3(b) we get $K \in \mathfrak{A}$. ■

3. Atomic Measures

A measure possessing no proper atom is called *atomless*. If it possesses no atoms at all it is called *strictly atomless*. Any improper atom of an atomless measure is of the second kind.

PROPOSITION 3.1. *The set of atomless (resp. strictly atomless) G-valued measures on $\mathfrak{R}$ is a subgroup of $G^{\mathfrak{R}}$.*

Let μ, ν be two G-valued measures on $\mathfrak{R}$ and let $\mathfrak{A}$ be an atom of $\mu - \nu$. Since

$$\mathfrak{N}(\mu) \cap \mathfrak{N}(\nu) \subset \mathfrak{N}(\mu - \nu)$$

it follows that

$$\mathfrak{A} \cap \mathfrak{N}(\mu) = \varnothing \qquad \text{or} \qquad \mathfrak{A} \cap \mathfrak{N}(\nu) = \varnothing$$

By Corollary 1.4 we deduce that $\mathfrak{A}$ is an atom of either μ or ν. This shows that the set of strictly non-atomic G-valued measures is a subgroup of $G^{\mathfrak{R}}$.

Assume now that $\mathfrak{A}$ is a proper atom of $\mu - \nu$. By Proposition 1.3(d) $\mathfrak{A}$ is a maximal element of Λ. By Proposition 2.3, $\mathfrak{A}$ is a proper atom of either μ or ν. Hence the set of G-valued atomless measures on is a subgroup of G . ■

We say that a measure μ on $\mathfrak{R}$ satisfies the *atom condition* (abbreviated *ac*) if for any atom $\mathfrak{A}$ of μ, $\mu(\mathfrak{F}(\mathfrak{A}))$ is convergent; according to the general convention made above we denote by $\mu_{\mathfrak{A}}$ the limit of $\mu(\mathfrak{F}(\mathfrak{A}))$ which may be interpreted as the value of μ at $\mathfrak{A}$. By Proposition 1.1 any measure satisfying locally *ccc* satisfies *ac*. By Proposition 2.2 any measures with values in an ω-topological group and a fortiori in a complete topological group (Proposition 2.1) satisfies *ac*. The set of G-valued measures on $\mathfrak{R}$ satisfying *ac* is a subgroup of $G^{\mathfrak{R}}$ (Corollary 2.4).

A measure μ on $\mathfrak{R}$ satisfying *ac* is called *atomic* if for any $A \in \mathfrak{R}$, $\mu(A)$ is the sum of the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$, where Φ denotes the set of atoms $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$. By Propositions 1.3(d) and 2.3 we may replace Φ in the above definition by the set of maximal elements $\mathfrak{A}$ of Λ such that $A \in \mathfrak{A}$. From this remark it follows immediately that the set of G-valued atomic measures on $\mathfrak{R}$ is a subgroup of $G^{\mathfrak{R}}$. Any improper atom of an atomic measure is of the first kind. A measure which is at the same time atomic and atomless vanishes identically.

PROPOSITION 3.2. *Let μ, μ' be two atomless G-valued measures on $\mathfrak{R}$ and let ν, ν' be two atomic G-valued measures on $\mathfrak{R}$. If*

$$\mu + \nu = \mu' + \nu'$$

then $\mu = \mu'$ and $\nu = \nu'$.

By Proposition 3.1, $\mu - \mu'$ is an atomless measure on $\mathfrak{R}$. Since it is at the same time an atomic measure it vanishes identically. ■

We say that a measure μ on $\mathfrak{R}$ satisfying *ac* satisfies the *atomical summability condition* (abbreviated *asc*) if for any $A \in \mathfrak{R}$ the family $(\mu_{\mathfrak{A}})_{\mathfrak{A} \in \Phi}$ is summable, where Φ denotes the set of atome $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$. By Propositions 1.3(d) and 2.3 we may replace Φ in the above definition by the set of maximal elements $\mathfrak{A}$ for Λ for which $A \in \mathfrak{A}$. From this remark it follows immediately that the set of G-valued measures on $\mathfrak{R}$ satisfying *asc* is a subgroup of $G^{\mathfrak{R}}$. Any atomic measure satisfies *asc*.

PROPOSITION 3.3. *Let μ be a G-valued measure on $\mathfrak{R}$ satisfying ac and let Φ be a countable set of atoms of μ such that $\bigcap_{\mathfrak{A}\in\Phi}\mathfrak{A}\neq\emptyset$ and such that $\mu(\mathfrak{F}(\bigcap_{\mathfrak{A}\in\Phi}\mathfrak{A}))$ converges. Then the family $(\mu_{\mathfrak{A}})_{\mathfrak{A}\in\Phi}$ is summable and its sum is the limit of $\mu(\mathfrak{F}(\bigcap_{\mathfrak{A}\in\Phi}\mathfrak{A}))$.*

Let M be a subset of $\mathbf{N}\setminus\{\emptyset\}$ and let $(\mathfrak{A}_n)_{n\in M}$ be a family of atoms of μ such $\mathfrak{A}_m\neq\mathfrak{A}_n$ for any different $m, n\in M$ and $\Phi=\{\mathfrak{A}_n \mid n\in M\}$. We set $\mathfrak{A}:=\bigcap_{n\in M}\mathfrak{A}_n$. Let V be an arbitrary 0-neighbourhood in G and let $(V_n)_{n\in\mathbf{N}}$ be a sequence of 0-neighbourhoods in G such that $V_0+V_0-V_0\subset V$ and such that $V_{n+1}+V_{n+1}\subset V_n$ for any $n\in\mathbf{N}$. For any $n\in M$ there exists $A_n\in\mathfrak{A}_n$ such that

$$\{\mu(B) \mid B\in\mathfrak{A}_n, B\subset A_n\}\subset\mu_{\mathfrak{A}_n}+V_n.$$

There exists $A\in\bigcap_{n\in M}\mathfrak{A}_n$ such that

$$\{\mu(B) \mid B\in\mathfrak{A}, B\subset A\}\subset\mu_{\mathfrak{A}}+V_0.$$

By Proposition 1.5 there exists a disjoint family $(B_n)_{n\in M}$ in $\mathfrak{R}$ such that $B_n\in\mathfrak{A}_n$ for any $n\in M$. Then $(A_n\cap B_n\cap A)_{n\in M}$ is a disjoint family in $\mathfrak{R}$ whose union belongs to $\mathfrak{R}$ and therefore

$$\mu\Big(\bigcup_{n\in M}(A_n\cap B_n\cap A)\Big)=\sum_{n\in M}\mu(A_n\cap B_n\cap A)$$

Since $\bigcup_{n\in M}(A_n\cap B_n\cap A)\in\mathfrak{A}$ (Proposition 1.3(a)) we have

$$\mu\Big(\bigcup_{n\in M}(A_n\cap B_n\cap A)\Big)\in\mu_{\mathfrak{A}}+V_0.$$

For any $n\in M$ we get

$$\mu(A_n\cap B_n\cap A)\in\mu_{\mathfrak{A}_n}+V_n.$$

Let M_0 be a finite subset of M such that

$$\sum_{n\in M'}\mu(A_n\cap B_n\cap A)-\sum_{n\in M}\mu(A_n\cap B_n\cap A)\in V_0$$

for any finite subset M' of M containing M_0. We deduce for any finite subset M' of M containing M_0.

$$\sum_{n\in M'} \mu_{\mathfrak{A}_n} - \mu_{\mathfrak{A}} \in V_0 + V_0 - V_0 \subset V.$$

Since V is arbitrary it follows that $(\mu_{\mathfrak{A}_n})_{n\in M}$ is summable and its sum is $\mu_{\mathfrak{A}_0}$. ■

PROPOSITION 3.4. *Any measure satisfying locally ccc satisfies asc.*

Let μ be a measure on $\mathfrak{R}$ satisfying locally *ccc*. By Proposition 1.1 $\mu(\mathfrak{F}(\mathfrak{A}))$ converges for any $\mathfrak{A} \in \Lambda$; in particular μ satisfies *ac*. Let $A \in \mathfrak{R}$ and let Φ be the set of atoms $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$. By Proposition 1.2(c) Φ is countable. By the preceding proposition $(\mu_{\mathfrak{A}})_{\mathfrak{A}\in\Phi}$ is summable. Hence μ satisfies *asc*. ■

PROPOSITION 3.5. *If G is an ω-space then any G-valued measure satisfies asc.*

Let μ be a G-valued measure on $\mathfrak{R}$. By Proposition 2.2 for any $\mathfrak{A} \in \Lambda$ the filter $\mu(\mathfrak{F}(\mathfrak{A}))$ converges; in particular μ satisfies *ac*. Let $A \in \mathfrak{R}$, let Φ be the set of atoms $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$, and let $\mathfrak{P}_c(\Phi)$ be the set of countable subsets of Φ ordered by the inclusion relation. By Proposition 3.3 for any $\Psi \in \mathfrak{P}_c(\Phi)$ the family $(\mu_{\mathfrak{A}})_{\mathfrak{A}\in\Psi}$ is summable; let us denote by f the map

$$\Psi \mapsto \sum_{\mathfrak{A}\in\Psi} \mu_{\mathfrak{A}} : \mathfrak{P}_c(\Phi) \to G.$$

Then $(\mathfrak{P}_c(\Phi), f)$ is an ω-net in G. Hence if $\mathfrak{F}$ denotes the section filter of $\mathfrak{P}_c(\Phi)$ then $f(\mathfrak{F})$ converges. We deduce that the family $(\mu_{\mathfrak{A}})_{\mathfrak{A}\in\Phi}$ is summable. Hence μ satisfies *asc*. ■

THEOREM 3.6. *Let μ be a $\mathfrak{K}$-regular G-valued measure on $\mathfrak{R}$ satisfying asc. We denote for any $A \in \mathfrak{R}$ by $\Phi(A)$ the set of atoms $\mathfrak{A}$ of μ such that $A \in \mathfrak{A}$ and by μ' the map*

$$A \mapsto \sum_{\mathfrak{A}\in\Phi(A)} \mu_{\mathfrak{A}} : \mathfrak{R} \to G.$$

Then μ' (resp. $\mu - \mu'$) is an atomic (resp. atomless) $\mathfrak{K}$-regular measure on $\mathfrak{R}$ absolutely continuous with respect to μ (i.e. $\mathfrak{N}(\mu) \subset \mathfrak{N}(\mu') \cap \mathfrak{N}(\mu - \mu')$). The proper atoms and the improper atoms of the first kind of μ and μ' coincide and we have

$\mu_{\mathfrak{A}}=\mu'_{\mathfrak{A}}$ *for any atom* $\mathfrak{A}$ *of* μ. *Any improper atom of* μ *of the second kind is an improper atom of* $\mu-\mu'$ *of the second kind.*

Let $(A_\iota)_{\iota\in I}$ be a countable disjoint family in $\mathfrak{R}$ whose union belongs to $\mathfrak{R}$. Then $(\Phi(A_\iota))_{\iota\in I}$ is a disjoint family and by Proposition 1.3(c) its union is $\Phi(\bigcup_{\iota\in I}A_\iota)$. We get

$$\mu'\Big(\bigcup_{\iota\in I}A_\iota\Big)=\sum_{\mathfrak{A}\in\Phi(\bigcup_{\iota\in I}A_\iota)}\mu_{\mathfrak{A}}=\sum_{\iota\in I}\sum_{\mathfrak{A}\in\Phi(A_\iota)}\mu_{\mathfrak{A}}=\sum_{\iota\in I}\mu'(A_\iota).$$

Hence μ' is a measure.

Let $A\in\mathfrak{R}$ and let U be a closed 0-neighbourhood in G. Then there exists a finite subset Ψ_0 of $\Phi(A)$ such that

$$\mu'(A)-\sum_{\mathfrak{A}\in\Psi}\mu_{\mathfrak{A}}\in U$$

for any finite subset Ψ of $\Phi(A)$ containing Ψ_0. By Proposition 2.5 there exists for any $\mathfrak{A}\in\Phi(A)$ a set $K_{\mathfrak{A}}\in\mathfrak{K}\cap\mathfrak{A}$ with $K_{\mathfrak{A}}\subset A$. We set

$$K:=\bigcup_{\mathfrak{A}\in\Psi_0}K_{\mathfrak{A}}\in\mathfrak{K}$$

Let B be a set of $\mathfrak{R}$ such that $K\subset B\subset A$. Then $\Phi(B)$ is a subset of $\Phi(A)$ containing Ψ_0 (Proposition 1.3(a)) and therefore

$$\mu'(A)-\mu'(B)=\mu'(A)-\sum_{\mathfrak{A}\in\Phi(B)}\mu_{\mathfrak{A}}\in U.$$

This shows that μ' is $\mathfrak{K}$-regular. We deduce that $\mu-\mu'$ is a $\mathfrak{K}$-regular measure on $\mathfrak{R}$. It is obvious that μ' and $\mu-\mu'$ are absolutely continuous with respect to μ.

Let $\mathfrak{A}$ be an atom of μ. Let U be a closed 0-neighbourhood in G and let V be a 0-neighbourhood in G such that $V-V-V\subset U$. There exists $A\in\mathfrak{A}$ such that

$$\{\mu(B)\mid B\in\mathfrak{A},B\subset A\}\subset\mu_{\mathfrak{A}}+V.$$

Let Ψ be a finite nonempty subset of $\Phi(A)\ \{\mathfrak{A}\}$. Then there exists

$$B\in\bigcap_{\mathfrak{A}'\in\Psi}\mathfrak{A}'\backslash\mathfrak{A}$$

such that $B \subset A$ and

$$\left\{\mu(C) \mid C \in \bigcap_{\mathfrak{A}' \in \Psi} \mathfrak{A}', C \subset B\right\} \subset \sum_{\mathfrak{A}' \in \Psi} \mu_{\mathfrak{A}'} + V.$$

Then $A \setminus B \in \mathfrak{A}$ (Proposition 1.3(c)) and therefore

$$\mu(A \setminus B) \in \mu_{\mathfrak{A}} + V,$$

$$\sum_{\mathfrak{A}' \in \Psi} \mu_{\mathfrak{A}'} \in \mu(B) - V = \mu(A) - \mu(A \setminus B) - V \subset V - V - V \subset U.$$

Since Ψ is arbitrary we get

$$\sum_{\mathfrak{A}' \in \Phi(A) \setminus \{\mathfrak{A}\}} \mu_{\mathfrak{A}'} \in U, \quad \mu'(A) \in \mu_{\mathfrak{A}} + U.$$

Since U is arbitrary we deduce $\mu'_{\mathfrak{A}} = \mu_{\mathfrak{A}}$. Hence the proper atoms of μ and μ' coincide (Corollary 2.4). We deduce further that the improper atoms of μ and μ' of the first kind coincide (Corollary 1.4). Moreover for any $A \in \mathfrak{R}$ we get

$$\mu'(A) = \sum_{\mathfrak{A} \in \Phi} \mu_{\mathfrak{A}} = \sum_{\mathfrak{A} \in \Phi} \mu'_{\mathfrak{A}},$$

where Φ denotes the set of maximal elements $\mathfrak{A}$ of Λ such that $A \in \mathfrak{A}$ (Propositions 1.3(d) and 2.3). Hence μ' is an atomic measure.

Let $\mathfrak{A}$ be an atom of $\mu - \mu'$. Then $\mathfrak{A}$ is an atom of μ (Corollary 1.4) and by the above considerations it follows that $\mathfrak{A}$ is an improper atom of $\mu - \mu'$. Hence $\mu - \mu'$ is atomless.

Let $\mathfrak{A}$ be an improper atom of μ of the second kind. Then there exists $A \in \mathfrak{A}$ such that $A \in \mathfrak{N}(\mu')$ and therefore $\mathfrak{A}$ is an atom of $\mu - \mu'$ (Corollary 1.7)). Since $\mu - \mu'$ is atomless it is an improper atom of $\mu - \mu'$ of the second kind. ■

Example. We want to give an example of a locally convex space E and of an E-valued measure on a σ-algebra of sets, possessing an improper atom of the second kind. Let X be a set. For any $A \subset X \times [0, 1]$ and for any $x \in X$ we set

$$A(x) := \{y \in [0, 1] \mid (x, y) \in A\}.$$

We denote by $\mathfrak{R}$ the set of $A \subset X \times [0, 1]$ such that: (a) $A(x)$ is a Borel set for any $x \in X$; (b) the set $\{x \in X \mid A(x) \neq \emptyset$ and $A(x) \neq [0, 1]\}$ is countable. It is obvious that $\mathfrak{R}$ is a σ-algebra of subsets of $X\times[0, 1]$. For any $A\in\mathfrak{R}$ we denote by $\mu(A)$ the map

$$x\mapsto\lambda(A(x)):X\to\mathbf{R},$$

where λ denotes the Lebesque measure on $[0, 1]$. It is easy to see that μ is an atomless $\mathbf{R}^X$-valued measure on $\mathfrak{R}$. Let $\mathfrak{F}$ be a non-trivial ultrafilter on X such that the intersection of any countable family in $\mathfrak{F}$ belongs to $\mathfrak{F}$ (we assume that such an ultrafilter exists). We set

$$\mathfrak{A}:=\{A\in\mathfrak{R}\mid\{x\in X\mid A(x)=[0, 1]\}\in\mathfrak{F}\}.$$

Then $\mathfrak{A}\in\Lambda(\mu)$. Let $\mathfrak{A}'\in\Lambda(\mu)$ with $\mathfrak{A}\subset\mathfrak{A}'$, let $A\in\mathfrak{A}'$, and let

$$X_0:=\{x\in X\mid A(x)\neq\emptyset\}$$

If $X_0\notin\mathfrak{F}$ then $X\backslash X_0\in\mathfrak{F}$ and therefore $(X\backslash X_0)\times[0, 1]\in\mathfrak{A}$ and this leads to the contradictory relation

$$\emptyset=A\cap((X\backslash X_0)\times[0, 1])\in\mathfrak{A}'.$$

Hence $X_0\in\mathfrak{F}$. Since $A\in\mathfrak{R}$ the set

$$\{x\in X\mid A(x)\neq\emptyset \quad\text{and}\quad A(x)\neq[0, 1]\}$$

is countable and therefore it does not belong to $\mathfrak{F}$. Hence

$$\{x\in X\mid A(x)=[0, 1]\}\in\mathfrak{F}$$

and we deduce successively $A\in\mathfrak{A}$, $\mathfrak{A}=\mathfrak{A}'$ and $\mathfrak{A}$ is an atom of μ. The measure μ being atomless $\mathfrak{A}$ is an improper atom of the second kind.

Example. Let X be an uncountable set, let $\mathfrak{P}(X)$ be the set of subsets of X, and for any $A\in\mathfrak{P}(X)$ let l_A be the characteristic function of A. Then

$$A\mapsto l_A:\mathfrak{P}(X)\to R^X$$

is an example of a measure satisfying *asc* and not satisfying *ccc*.

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