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Projective k -invariants

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1. Introduction

Let π be a group. A (π, m) -complex X is a finite connected m -dimensional CW complex having fundamental group π and trivial homotopy modules $\pi_i(X) = 0$ in dimensions $i = 2, \dots, m - 1$. A π -module π_m is said to be *topologically realizable* if $\pi_m \cong \pi_m(X)$ for some (π, m) -complex X . The classification problem for (π, m) -complexes is the problem of describing the set $HT(\pi, m)$ of homotopy types of (π, m) -complexes.

For π a finite group of order n , $H^{m+1}(\pi; \pi_m) \cong Z_n$ as a ring. An important aspect in this classification is the boundary operator $\partial: Z_n^* = \text{Units}(H^{m+1}(\pi; \pi_m)) \rightarrow \tilde{K}_0 Z\pi$, the (reduced) projective class group of the integral group ring $Z\pi$, associated with the Milnor Mayer-Vietoris sequence in algebraic K-theory [10].

This arises as follows. The cellular chain complex $C_*(\tilde{X})$ of the universal cover \tilde{X} is a truncated resolution of the trivial π -module Z :

$$0 \longrightarrow \pi_m \longrightarrow C_m(\tilde{X}) \xrightarrow{\partial_m} \cdots \xrightarrow{\partial_1} C_0(\tilde{X}) \xrightarrow{\epsilon} Z \longrightarrow 0.$$

The algebraic m -type $T(X)$ of X is the triple $(\pi, \pi_m(X), k(X))$ where $k(X) \in H^{m+1}(\pi, \pi_m)$ is the k -invariant which arises by comparing the truncated resolution above with a standard resolution (see section 6; also [5], [6]). One can show that $k(X) \in \text{Units}(H^{m+1}(\pi; \pi_m))$; furthermore any $k \in Z_n^*$ can be the k -invariant of a finitely generated truncated projective resolution

$$(*) \quad \mathcal{P}_k: 0 \rightarrow \pi_m \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow Z \rightarrow 0.$$

Also the assignment $(\pi, \pi_m, k) \rightarrow$ Euler characteristic $\chi(\mathcal{P}_k) = \sum_{i=0}^m (-1)^i [P_i]$ ($[P]$ is the class of the projective P in $\tilde{K}_0 Z\pi$) is the negative of the Milnor boundary ∂ . Then (π, π_m, k) ($k \in Z_n^*$, $m \geq 3$) is the m -type of a (π, m) -complex iff $k \in \ker \partial$ [4].

The purpose of this paper is to generalize the above to groups other than finite groups.

1.1. THEOREM. *Let π be a group and m be an integer $m \geq 0$ such that $H^{m+1}(\pi; Z\pi) = 0$. Let π_m be any finitely generated topologically realizable π -module. Then*

(a) $H^{m+1}(\pi; \pi_m)$ has the structure of a ring with identity such that the units $U(H^{m+1}(\pi, \pi_m))$ are the projective k -invariants, i.e., those k -invariants realizable by a resolution of the form (*).

(b) The function $\chi_m : U(H^{m+1}(\pi; \pi_m)) \rightarrow \tilde{K}_0 Z\pi$ which assigns to each $k \in U$ the Euler characteristic of a truncated resolution \mathcal{P}_k realizing the m -type (π, π_m, k) is a homomorphism.

We say that an m -type (π, π_m, k) comes from a (π, m) -complex if there exists a (π, m) -complex X such that $T(X) \cong (\pi, \pi_m, k)$ in the appropriate sense (see [4], [6] for a definition).

1.2. COROLLARY. *If $m \geq 3$ and $H^{m+1}(\pi; Z\pi) = 0$, then $\ker \chi_m$ is the set of k -invariants which come from (π, m) -complexes.*

The corollary follows from a theorem of J. Milnor [11, theorem 3.1] concerning the realizability of a resolution by a (π, m) -complex.

DEFINITION. The subgroup $\text{im } \chi_m \subset \tilde{K}_0 Z\pi$ is called the *Swan subgroup of $\tilde{K}_0 Z\pi$ in dimension m* .

If π is a finite group of order n , let $N = \sum_{x \in \pi} x \in Z\pi$ be the norm element. The left ideal (p, N) of $Z\pi$ is projective provided p is prime to n . For π finite, $\text{im } \chi_m = \text{im } \partial = \{[(p, N)] \in \tilde{K}_0 Z\pi \mid 1 \leq p < n, (p, n) = 1\}$. If π is a (Poincaré) duality group of cohomological dimension m , then $\text{im } \chi_{m-i} = 0$ ($2 \leq i \leq m$).

The Swan subgroup $\text{im } \chi_m$ is important because the *Wall obstruction* of any CW complex having fundamental group π and realizable π_m , which is dominated by a (π', m) -complex lies in $\text{im } \chi_m$ [12].

The organization of the paper is as follows. Let R be a ring. Section 2 gives certain constructions associated with the exact sequence of R -modules $0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$. We say that P is K -projective if $\partial : \text{End}(K) \rightarrow \text{Ext}(C, K)$ is *surjective*. Section 3 gives conditions under which $\text{Ext}(C, K)$ inherits a ring structure from $\text{End}(K)$, provided P is K -projective. Section 4 shows that elements in $\text{End}(K)$ which determine K -projective extensions are right units in $\text{Ext}(C, K)$. Section 5 studies conditions under which each K -projective element in $\text{End}(K)$ is a unit in $\text{Ext}(C, K)$. Theorem 1 is proved in section 6. In an appendix we study conditions under which $H^i(\pi; Z\pi) = 0$.

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2. Extensions as Pushouts and Pull-backs.

Let R be a ring. All modules are left R -modules. Let C be a given R -module and $\xi: 0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$ be an exact sequence of R -modules.

It is shown in [9, page 66] that given any module homomorphism $k: K \rightarrow K'$ there exists a module kP and a homomorphism $k\beta: P \rightarrow kP$ such that the following diagram commutes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{j} & C \longrightarrow 0 \\
 & & \downarrow k & & \downarrow k\beta & & \parallel \\
 0 & \longrightarrow & K' & \xrightarrow{i_k} & kP & \xrightarrow{j_k} & C \longrightarrow 0
 \end{array} \tag{2.1}$$

Here the bottom row is exact also. kP is defined as the pushout of i and k .

Furthermore, given any module homomorphism $s: C \rightarrow C$, there exists a module P_s and a homomorphism $\beta_s: P_s \rightarrow P$ such that the following diagram commutes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \xrightarrow{i^s} & P_s & \xrightarrow{j^s} & C \longrightarrow 0 \\
 & & \parallel & & \downarrow \beta_s & & \downarrow s \\
 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{j} & C \longrightarrow 0
 \end{array} \tag{2.2}$$

P_s is defined to be the pullback of j and s .

3. Ext_R(C, K) as a Ring.

Let R be a ring and

$$\xi: 0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$$

be an exact sequence of (left) R -modules.

DEFINITION We say that P is K -projective if

$$i^*: \text{Ext}_R^1(P, K) \rightarrow \text{Ext}_R^1(K, K)$$

is a monomorphism.

Of course, it follows from the long exact sequence for $\text{Ext}_R^i(-, K)$ [9, page 74] associated with ξ that P is K -projective iff the boundary operator $\partial: \text{End}_R(K) \rightarrow \text{Ext}_R^1(C, K)$ is surjective. Here $\partial(k)$ equals the equivalence class of the extension kP for any $k \in \text{End}(K)$. If $\text{Ext}_R(P, K) = 0$, then P is K -projective; in particular, any projective R -module is K -projective.

3.1. THEOREM. *If $0 \longrightarrow K \xrightarrow{i} P \xrightarrow{j} C \longrightarrow 0$ is an exact sequence of R -modules with P K -projective, then the boundary operator ∂ induces an isomorphism*

$$\bar{\partial}: \frac{\text{End}_R(K)}{i^*(\text{Hom}_R(P, K))} \rightarrow \text{Ext}_R^1(C, K).$$

For each $k \in \text{End}(K)$, let $\{k\}$ denote the element $\partial(k)$ in $\text{Ext}_R^1(C, K)$.

$\text{End}(K)$ has a ring structure under composition. The question is: when is $B = i^* \text{Hom}(P, K)$ a two-sided ideal? If we denote the composition $K \xrightarrow{\alpha} K \xrightarrow{\beta} K$ by $\beta\alpha$, then

$$B = \{\alpha: K \rightarrow K \mid \alpha \text{ extends to a map } \alpha': P \rightarrow K\}$$

is always a left ideal. For, if $\alpha \in B$, $\beta \in \text{End}(K)$ and $\alpha' \in \text{Hom}(P, K)$ extends α , then $\beta\alpha'$ extends $\beta\alpha$. Thus B is a right ideal and $B \neq \text{End}(K)$ implies that $\text{Ext}(C, K)$ is a ring with identity.

We will now delineate a sequence of sufficient conditions that imply that B is a right ideal.

3.2. (C). *The composition in $\text{End}(K)$ is commutative modulo B .*

3.3. (RE). Each homomorphism in $\text{End}(K)$ extends to a homomorphism in $\text{End}(P)$.

3.4. (E). Each homomorphism in $\text{Hom}(K, P)$ extends to a homomorphism in $\text{End}(P)$.

Note that the following implications hold:

$$(E) \Rightarrow (RE) \Rightarrow B \text{ is a right ideal} \Leftarrow (C).$$

3.5. If $\text{Ext}(C, P) = 0$, then (E) is true. This follows because $\text{Ext}(C, P) = 0$ implies $i^*: \text{End}(P) \rightarrow \text{Hom}(K, P)$ is surjective. If $\text{Ext}(P, P) = 0$, then (E) is equivalent to $\text{Ext}(C, P) = 0$. In particular, this is true if P is projective.

3.6. Also, one can easily see that (RE) iff the boundary homomorphism $\partial: \text{End}(C) \rightarrow \text{Ext}(C, K)$ is surjective iff $j_*: \text{Ext}(C, P) \rightarrow \text{Ext}(C, C)$ is a monomorphism.

Note that $\text{Ext}(C, K)$ is cyclic automatically implies (C).

We may call P C -injective if $j_*: \text{Ext}(C, P) \rightarrow \text{Ext}(C, C)$ is a monomorphism. Thus $\text{Ext}(C, K)$ has a ring structure as above if P is C -injective and K -projective.

More generally, we may proceed as follows: let P be K -projective.

DEFINITION. Let $\text{Ext}(C, K)_K$ denote the subset of $\text{Ext}(C, K)$ such that $\{k\} \in \text{Ext}(C, K)_K$ iff $Bk \subset B$.

It is clear that

- (a) $\text{Ext}(C, K)_K$ is a subgroup of $\text{Ext}(C, K)$.
- (b) $\text{Ext}(C, K)_K$ is a ring with identity under composition.
- (c) The image of the center of $\text{End}(K)$ is contained in $\text{Ext}(C, K)_K$.

$\text{Ext}(C, K)_K$ is called the *maximal K -ring of $\text{Ext}(C, K)$* .

Let $\partial_C: \text{End}(C) \rightarrow \text{Ext}(C, K)$ be the boundary operator in the exact sequence for $\text{Ext}^1(C, -)$ associated with the extension $\xi: 0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$. $\partial_C(r)$ is the equivalence class of the extension Pr (see 2.2).

3.7. PROPOSITION.

- (a) $\text{End}(C)$ always induces a ring structure on the subgroup $\text{im } \partial_C = {}_C\text{Ext}(C, K)$.
- (b) ${}_C\text{Ext}(C, K)$ is a subring of $\text{Ext}(C, K)_K$
- (c) If ∂_C is surjective, then ${}_C\text{Ext}(C, K) \cong \text{Ext}(C, K)_K$ as rings.

Proof.

(a) P is K -projective implies that $\text{im} \{j_* : \text{Hom}(C, P) \rightarrow \text{End}(C)\}$ is a two-sided ideal. This follows because each homomorphism in $\text{End}(C)$ extends to a homomorphism in $\text{End}(P)$. Consider $l \in \text{End}(C)$ and the extension Pl . Then P is K -projective implies that there exists a $k \in \text{End}(K)$ such kP and Pl are equivalent extensions. Thus there is an isomorphism $\alpha : kP \rightarrow Pl$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
 0 & \rightarrow & K & \rightarrow & P & \rightarrow & C \rightarrow 0 \\
 & & \downarrow k & & \downarrow \beta & & \parallel \\
 0 & \rightarrow & K & \rightarrow & kP & \rightarrow & C \rightarrow 0 \\
 & & \parallel & & \downarrow \alpha & & \downarrow l \\
 & & & & Pl & & \\
 & & & & \downarrow \beta l & & \\
 0 & \rightarrow & K & \rightarrow & P & \rightarrow & C \rightarrow 0
 \end{array}$$

(b) Any $\{k\} \in \text{Ext}(C, K)$ ($k \in \text{End}(K)$) which is in the image of ∂_C clearly satisfies $Bk \subset B$. Let $\partial_C(l) = \{k\}$. Then we may choose an extension as in (a) so that the following commutes

$$\begin{array}{ccccccc}
 0 & \rightarrow & K & \rightarrow & P & \rightarrow & C \rightarrow 0 \\
 & & k \downarrow & & \downarrow \beta & & \downarrow l \\
 0 & \rightarrow & K & \rightarrow & P & \rightarrow & C \rightarrow 0
 \end{array}$$

Now $\alpha \in B$ iff α extends the zero map $0 : C \rightarrow C$, i.e., the following diagram commutes:

$$\begin{array}{ccccccc}
 0 & \rightarrow & K & \rightarrow & P & \rightarrow & C \rightarrow 0 \\
 & & \alpha \downarrow & & \downarrow \beta_\alpha & & \downarrow 0 \\
 0 & \rightarrow & K & \rightarrow & P & \rightarrow & C \rightarrow 0
 \end{array}$$

But $\alpha \in B$ and $\{k\} \in \text{im } \partial_C$ implies that $\alpha \circ k$ extends $0 \circ l = 0$. Thus (b) is proved.

(c) follows easily from (a) and (b). We only note that the ring isomorphism is given by the correspondence $\partial_C(l) \mapsto \{k\}$ where $k \in \text{End}(K)$ extends $l \in \text{End}(C)$. This completes 3.7.

Note that ∂_C is surjective iff condition (RE).

We now give a simple example to show that B is not always a right ideal. Let $R = Z$ and let the basic extension be given by

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Z \oplus Z & \xrightarrow{i} & Z \oplus Z & \xrightarrow{j} & Z_3 \oplus Z_2 \longrightarrow 0 \\
 & & \parallel & & \parallel & & \parallel \\
 & & K & & P & & C
 \end{array}$$

where i has matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ with respect to the natural bases. Then $B \subset \text{End}(Z \oplus Z)$ is the set of all 2×2 matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ over Z with the first column divisible by 3, the second by 2. $\text{Ext}(C, K) \cong Z_3^2 \oplus Z_2^2$. Representatives of the cosets modulo B are given by

$$\mathcal{R} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid \begin{array}{l} 0 \leq a_{i1} \leq 2 \\ 0 \leq a_{i2} \leq 1 \end{array}, \quad i = 1, 2 \right\}$$

It is easy to check that only the diagonal matrices in \mathcal{R} have the property that $B \circ k \subset B$. Hence $\text{Ext}(C, K)_K \cong Z_3 \oplus Z_2 \subset \text{Ext}(C, K)$ by embedding in the first and fourth coordinates.

4. K -Projective k -Invariants

Throughout this section we assume that $i^* : \text{End}(K) \rightarrow \text{Ext}(C, K)$ is surjective; i.e., that P is K -projective.

DEFINITION. The class $\{k\} \in \text{Ext}(C, K)$ determined by $k \in \text{End}(K)$ is called the k -invariant of the extension kP . A k -invariant $\{k\}$ is called K -projective if kP is a K -projective R -module. An element $k \in \text{End}(K)$ is also called K -projective if $\{k\}$ is K -projective. Let $\mathcal{P}_K(\text{Ext}(C, K))$ denote the set of K -projective k -invariants in $\text{Ext}(C, K)$, $\mathcal{P}_K(\text{End}(K))$ the set of K -projective elements $\text{End}(K)$.

DEFINITION. Let E be a ring with identity. An element $\alpha \in E$ is a *right unit* if there exists $\beta \in E$ such that $\beta\alpha = 1$. The set of (right) units of E is denoted by $(R)U(E)$.

For each $\alpha \in E$, let α^* denote the abelian group homomorphism $E \rightarrow E$ given by right multiplication by α . α is a right unit iff α^* is surjective.

4.1. THEOREM. Let $\text{Ext}(C, K)$ inherit a ring structure from $\text{End}(K)$. $\{k\}$ is a K -projective k -invariant iff $\{k\}$ is a right unit.

Proof. Suppose that k is K -projective. Then $\partial_k : \text{End}(K) \rightarrow \text{Ext}(C, K)$ ($\partial_k(\alpha) = (\alpha \circ k)P$, $\alpha \in \text{End}(K)$) is surjective. Thus there is a $k' \in \text{End}(K)$ such that $(k' \circ k)P$ is equivalent to P as extensions. Hence $k' \circ k - 1 \in B$, and k is a right unit.

If $k' \circ k - 1 \in B$, we will show that kP is K -projective. P and $(k' \circ k)P$ are

equivalent extensions, so there is a commutative diagram

$$\begin{array}{ccccccc}
 & & & P & & & \\
 & & i & \nearrow & j & & \\
 0 & \longrightarrow & K & & C & \longrightarrow & 0 \\
 & & \uparrow & \cong & \uparrow & & \\
 & & k' & & (k' \circ k)P & & \\
 & & & & \uparrow & & \\
 0 & \longrightarrow & K & \xrightarrow{i_k} & kP & \xrightarrow{j_k} & C \longrightarrow 0 \\
 & & & & & & \parallel \\
 & & & & & & \parallel \\
 & & & & & & \parallel
 \end{array}$$

Call the resulting map $\beta : kP \rightarrow P$. Apply $\text{Ext}(-, K)$ to this diagram to obtain the commutative diagram:

$$\begin{array}{ccccc}
 \text{Ext}(C, K) & \xrightarrow{j^*} & \text{Ext}(P, K) & \xrightarrow{i^*} & \text{Ext}(K, K) \\
 \parallel & & \downarrow \beta^* & & \downarrow k'^* \\
 \text{Ext}(C, K) & \xrightarrow{j_k^*} & \text{Ext}(kP, K) & \xrightarrow{i_k^*} & \text{Ext}(K, K)
 \end{array}$$

Thus $j_k^* = \beta^* j^* = 0$ because $j^* = 0$. Thus i_k^* is a monomorphism. This completes 4.1.

4.2. THEOREM. *If $\{k \circ k'\} = \{k' \circ k\} = \{1\}$ in $\text{Ext}(C, K)$, then $\text{Ext}(kP, M) = 0$ iff $\text{Ext}(P, M) = 0$, where M is an R -module.*

If we were to define the “degree of projectivity” of k by the class of modules \mathcal{M}_k such that $M \in \mathcal{M}_k$ iff $\text{Ext}(kP, M) = 0$, then the above says that $\{k\}$ is a unit implies that $\mathcal{M}_k = \mathcal{M}_1$; i.e., kP is “just as projective” as P is.

Proof. Because $k' \circ k - 1 \in B$, the argument of (4.1) implies the existence of the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \xrightarrow{i_k} & kP & \xrightarrow{j_k} & C \longrightarrow 0 \\
 & & \uparrow k & & \uparrow \beta & & \parallel \\
 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{j} & C \longrightarrow 0 \\
 & & \uparrow k' & & \uparrow \beta' & & \parallel \\
 0 & \longrightarrow & K & \xrightarrow{i_k} & kP & \xrightarrow{j_k} & C \longrightarrow 0
 \end{array}$$

Now $k \circ k' = 1 + \alpha' \circ i$, where $\alpha' \in \text{Hom}(P, K)$. Let M be any R -module such that $\text{Ext}(P, M) = 0$. Apply the functor $\text{Ext}(-, M)$ to the above diagram.

$$\begin{array}{ccccc}
 \text{Ext}(C, M) & \xrightarrow{i_k^*} & \text{Ext}(kP, M) & \xrightarrow{i_k^*} & \text{Ext}(K, M) \\
 \parallel & & \downarrow \beta^* & & \downarrow (k \circ k')^* \\
 & & \text{Ext}(P, M) & & \\
 & & \downarrow \beta'^* & & \\
 \text{Ext}(C, M) & \xrightarrow{j_k^*} & \text{Ext}(kP, M) & \xrightarrow{i_k^*} & \text{Ext}(K, M)
 \end{array}$$

The rows are exact at $\text{Ext}(kP, M)$. $(k \circ k')^* = (1 + \alpha' \circ i)^* = 1 + (\alpha' \circ i)^* = 1$, since $(\alpha' \circ i)^* = 0$. Thus $\beta'^* \circ \beta^*$ is an isomorphism. Then $\text{Ext}(P, M) = 0$ implies $\text{Ext}(kP, M) = 0$. A similar argument shows the converse. This completes (4.2).

Since the set of right units is a semigroup under composition, the following is clear.

4.3. COROLLARY. *Let $\text{Ext}(C, K)$ have a ring structure as above. Then the set $\mathcal{P}_k(\text{Ext}(C, K))$ of K -projective k -invariants is a semigroup with identity under composition. \mathcal{P}_k is a group iff each K -projective k -invariant is a unit.*

5. k -Invariants as Units.

In this section we will study conditions under which right units are units in the ring $\text{Ext}(C, K)$. We continue our assumption that P is K -projective. We also assume in this section that B is a right ideal.

DEFINITION. For each $k \in \text{End}(K)$, let $B_k = \text{im}\{\text{Hom}(kP, K) \rightarrow \text{End}(K)\} = \ker\{\partial_k : \text{End}(K) \rightarrow \text{Ext}(C, K)\}$, where $\partial_k(\alpha) = (\alpha \circ k)P$ ($\alpha \in \text{End}(K)$).

5.1. LEMMA. $B = \text{im}\{\text{Hom}(P, K) \rightarrow \text{End}(K)\}$ is a right ideal iff $B \subset B_k$ for all $k \in \text{End}(K)$.

Proof. Let $\alpha \in B$. For any $k \in \text{End}(K)$, $\alpha \circ k \in B$ since B is a right ideal. Thus $(\alpha \circ k)P \cong \alpha(kP)$ is trivial implies that $\alpha \in B_k$. Conversely, if $B \subset B_k$ for all $k \in \text{End}(K)$, then let $\alpha \in B$, and consider $\alpha \circ k$ ($k \in \text{End}(K)$). $\alpha \in B_k$ implies $\alpha(kP) \cong (\alpha \circ k)P \cong K \times C$ which in turn implies that $\alpha \circ k \in B$.

We say that $\{k\} \in \text{Ext}(C, K)$ is a *right zero divisor* if there exists a $\{k'\} \neq 0$ such that $\{k' \circ k\} = 0$.

5.2. PROPOSITION. $\{k\} \in \text{Ext}(C, K)$ is not a right zero divisor iff $B = B_k$. If k is K -projective, then $\{k\}$ is a unit iff $B = B_k$.

Proof. For each $k \in \text{End}(K)$, let $k^* : \text{Ext}(C, K) \rightarrow \text{Ext}(C, K)$ be the function defined by right multiplication by $\{k\}$. It is a homomorphism of the underlying abelian group structure. Thus $\{k\}$ is not a right zero divisor iff k^* is a monomorphism. But k^* is a monomorphism iff $B = B_k$ follows from the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & B & \longrightarrow & \text{End}(K) & \xrightarrow{\partial} & \text{Ext}(C, K) \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow k^* \\
 0 & \longrightarrow & B_k & \longrightarrow & \text{End}(K) & \xrightarrow{\partial_k} & \text{Ext}(C, K) \longrightarrow \text{Ext}(kP, K) \longrightarrow \dots
 \end{array}$$

Here $\partial(\alpha) = \alpha P$, $\partial_k(\alpha) = \alpha(kP) = (\alpha \circ k)P$ and the horizontal sequences are exact. Furthermore, k^* is an isomorphism implies that ∂_k is surjective and hence $B = B_k$. $B = B_k$ together with ∂_k surjective implies k^* is an isomorphism.

5.3. LEMMA. Let $k \in \text{End}(K)$ and suppose there exists $k' \in \text{End}(K)$ such that $k' \circ k - 1 \in B$. Then $B = B_{k'}$.

Proof. Consider the homomorphisms k^*, k'^* as in the proof of (5.2). The composite $k^* \circ k'^* = (k' \circ k)^* = 1$. Thus k'^* is a monomorphism and, by (5.2), $B = B_{k'}$.

We will now give several conditions under which K -projective k -invariants are units. Clearly, if $\text{Ext}(C, K)$ is commutative or has no zero divisors, then every right unit is a unit. Furthermore a theorem of N. Jacobson [7] shows that any ring having right units which are not units must be very large. The following is just a restatement of theorem 1 of [7].

5.4. THEOREM. If $E = \text{Ext}(C, K)$ has either the ascending or descending chain condition for principal right ideals generated by idempotent elements, then right units are units.

Thus it follows that if E is finitely generated as a left (or right) E module, then right units are units in E . For example, if R is commutative and K is a finitely generated R -module, then $\text{Ext}(C, K)$ is a finitely generated R -module and hence, by (5.4), right units are units.

Now let P be a *projective* R -module and consider any exact sequence

$$0 \longrightarrow K_1 \xrightarrow{i_1} P_1 \xrightarrow{j_1} K \longrightarrow 0$$

of R -modules where P_1 is projective. The boundary operator

$$\partial: \text{Ext}^1(C, K) \rightarrow \text{Ext}^2(C, K_1) = \text{Ext}^1(K, K_1)$$

is given by $\partial(\{k\}) = \text{class of the extension } P_1 k$ (see 2.2).

5.5. THEOREM. *If $\partial: \text{Ext}^1(C, K) \rightarrow \text{Ext}^2(C, K_1)$ is a monomorphism, then projective k -invariants are units in $\text{Ext}(C, K)$.*

5.6. COROLLARY. *If $\text{Ext}(C, R) = 0$ and K is finitely generated as an R -module, then projective k -invariants are units in $\text{Ext}(C, K)$.*

The proof of (5.5) is postponed to (6.13). The corollary follows because K is finitely generated implies P_1 may be chosen to be finitely generated. $\text{Ext}(C, R) = 0$ then yields $\text{Ext}(C, P_1) = 0$ and this implies that ∂ is a monomorphism.

6. The k -Invariant of a Truncated Resolution.

Let M be an R -module. Choose a *projective* resolution

$$\mathcal{F}(M): \cdots \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \longrightarrow \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

of M , where each C_i is projective R -module. $\mathcal{F}(M)$ is called the *base resolution*; each $\pi_m = \ker \partial_m$ ($m \geq 0$) is called an *M -realizable R -module*. If $M = Z$, the trivial R -module, then π_m is *realizable* means it is *Z -realizable*. We say that a resolution \mathcal{F} is of *finite type* if each C_i is a finitely generated R -module.

Let

$$\mathcal{G}(M): \cdots \longrightarrow G_m \xrightarrow{g_m} G_{m-1} \xrightarrow{g_{m-1}} \cdots \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \longrightarrow 0$$

be a (not necessarily projective) resolution of M . Let π'_m denote $\ker g_m$. The *k -invariant of \mathcal{G} in dimension m relative to \mathcal{F}* is the element $\{k\} \in \text{Ext}_R^{m+1}(M, \pi'_m)$ determined by a chain map $f: \mathcal{F}(M) \rightarrow \mathcal{G}(M)$ covering the identity on M . Thus f is a sequence of maps making the following diagram commute:

$$\begin{array}{ccccccccccc} C_{m+1} & \xrightarrow{\partial_{m+1}} & C_m & \xrightarrow{\partial_m} & C_{m-1} & \longrightarrow & \cdots & \longrightarrow & C_0 & \longrightarrow & M & \longrightarrow & 0 \\ \vdots & & \downarrow f_m & & \downarrow f_{m-1} & & & & \downarrow f_0 & & \parallel & & \\ 0 & \longrightarrow & \pi'_m & \longrightarrow & G_m & \xrightarrow{g_m} & G_{m-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

The map $k = f_m \circ \partial_{m+1} : C_{m+1} \rightarrow \pi'_m$ determines an element $\{k\} \in \text{Ext}_R^{m+1}(M, \pi'_m)$. This is well-defined by a standard argument [5].

6.1. LEMMA. For each $m \geq 0$ and each element $\bar{k} \in \text{Ext}_R^{m+1}(M, \pi'_m) \exists$ a resolution $\mathcal{G}_{\bar{k}}$ of M realizing \bar{k} . If C_i ($i = 0, 1, \dots, m$) and π'_m are finitely generated, then $\mathcal{G}_{\bar{k}}^{(m)}$ may be chosen to be of finite type.

Proof. Consider $k : C_{m+1} \rightarrow \pi'_m$ realizing \bar{k} ; $k \cdot \partial_{m+2} = 0$ implies that k defines a map $k' : \pi_m \rightarrow \pi'_m$. Use the construction of section 2 to build

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_m & \longrightarrow & C_m & \longrightarrow & \pi_{m-1} \longrightarrow 0 \\ & & \downarrow k' & & \downarrow & & \parallel \\ & & \pi'_m & \xrightarrow{i'} & k' C_m & \xrightarrow{j'} & \pi_{m-1} \longrightarrow 0 \end{array}$$

Then the m -skeleton $\mathcal{G}_{\bar{k}}^{(m)}$ is given by

$$0 \longrightarrow \pi'_m \xrightarrow{i'} k' C_m \xrightarrow{\partial'_m} C_{m-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

where ∂'_m is the composite $k' C_m \xrightarrow{j'} \pi_{m-1} \hookrightarrow C_{m-1}$. This completes 6.1.

DEFINITION. An element $k \in \text{Ext}^{m+1}(M, \pi'_m)$ is called *projective* if k can be realized as the k -invariant of a truncated projective resolution:

$$\mathcal{P}_k^{(m)} : 0 \rightarrow \pi'_m \rightarrow P_m \rightarrow P_{m-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$$

when compared with the base resolution $\mathcal{F}(M)$. The set of projective k -invariants of $\text{Ext}^{m+1}(M, \pi'_m)$ is denoted by $\mathcal{P}(\text{Ext}^{m+1}(M, \pi'_m))$.

6.2. THEOREM. Let M be any R -module and π_m be M -realizable for $m \geq 0$. Then

(a) $\text{Ext}_R^{m+1}(M, \pi_m) \cong \frac{\text{End}(\pi_m)}{\text{im Hom}(C_m, \pi_m)}$.

(b) If $B^m = \text{im}\{\text{Hom}(C_m, \pi_m) \rightarrow \text{End}(\pi_m)\}$ is a right ideal, then $\text{Ext}^{m+1}(M, \pi_m)$ has a ring structure induced from that of $\text{End}(\pi_m)$ such that the projective k -invariants lie between the units and right units of $\text{Ext}^{m+1}(M, \pi_m)$:

$$U(\text{Ext}^{m+1}(M, \pi_m)) \subset \mathcal{P}(\text{Ext}^{m+1}(M, \pi_m)) \subset RU(\text{Ext}^{m+1}(M, \pi_m)).$$

(c) If B^m is a right ideal, $\mathcal{P}(\text{Ext}^{m+1}(M, \pi_m)) = U(\text{Ext}^{m+1}(M, \pi_m))$, and each C_i

($i = 0, 1, \dots, m+1$) a finitely generated free module, then the function

$$\chi_m : \mathcal{P}(\text{Ext}^{m+1}(M; \pi_m)) \rightarrow \tilde{K}_0 R$$

which assigns to each $k \in \mathcal{P}$ the Euler characteristic $\chi_m(\mathcal{P}_k^{(m)}) = \sum_{i=0}^m (-1)^i [P_i] \in \tilde{K}_0 R$ of $\mathcal{P}_k^{(m)}$ is a homomorphism.

Note. (1) Theorem 6.2 is theorem 1.1 in the case $R = Z\pi$ and $M = Z$. This follows because $H^{m+1}(\pi; Z\pi) = 0$ and C_m finitely generated implies that $H^{m+1}(\pi; C_m) = 0$. Thus $H^{m+1}(\pi; \pi_m)$ is a ring (3.5) and by (5.6) right units are units because π_m is finitely generated.

(2) It follows from [11, theorem 3.1] that if $m \geq 3$, any π -module π_m realizable by a truncated free resolution over Z is topologically realizable as well.

(3) It follows from (4.1) that the set \mathcal{P}_{π_m} of π_m -projective k -invariants is equal to the set of right units of $\text{Ext}^{m+1}(M; \pi_m)$. Furthermore, (4.2) implies that any unit in $\text{Ext}^{m+1}(M, \pi_m)$ must be projective. We do not know whether in general \mathcal{P} is distinct from U or RU (see 5.4).

The following lemma is useful in the subsequent work:

6.3. LEMMA OF COCKCROFT-SWAN [3, Appendix]. Let $\xi_i^{(m)} : 0 \rightarrow \pi_m \rightarrow E_m^i \rightarrow P_{m-1}^i \rightarrow \dots \rightarrow P_0^i \rightarrow M \rightarrow 0$ ($i = 1, 2$) be resolutions of M with each P_j^i ($j = 0, 1, \dots, m-1$) projective. Let $f : \xi_1^{(m)} \rightarrow \xi_2^{(m)}$ be a chain map covering the identity on M and inducing an isomorphism on π_m . Then

$$E_m^1 \oplus P_{m-1}^2 \oplus P_{m-2}^1 \oplus \dots \cong E_m^2 \oplus P_{m-1}^1 \oplus P_{m-2}^2 \oplus \dots$$

Note the similarity between this and Schanuel's lemma [11].

6.4. COROLLARY. Let $\xi_1^{(m)}$ be projective (i.e., E_m^1 is projective) and suppose $k(\xi_1^{(m)}) = k(\xi_2^{(m)})$ when compared to \mathcal{F} . Then

$$E_m^1 \oplus P_{m-1}^2 \oplus P_{m-2}^1 \oplus \dots \cong E_m^2 \oplus P_{m-1}^1 \oplus P_{m-2}^2 \oplus \dots$$

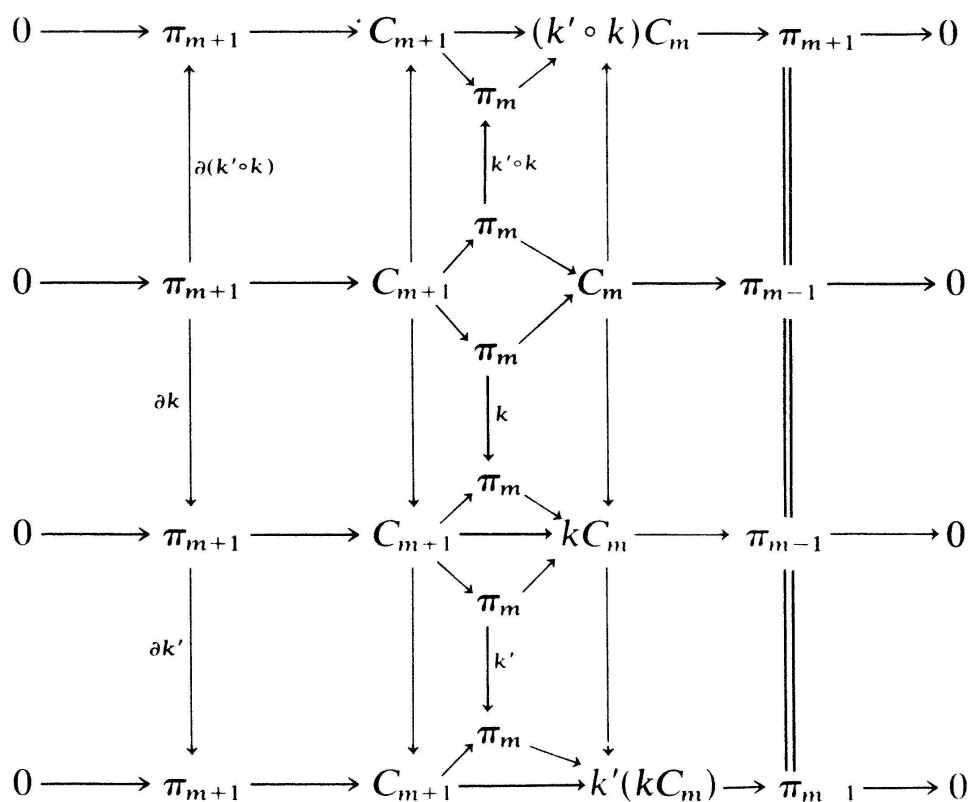
and hence $\xi_2^{(m)}$ is projective also.

Proof. By standard obstruction arguments, there exists a chain map $\xi_1^{(m)} \rightarrow \xi_2^{(m)}$ inducing the identity on M and π_m . Then apply (6.3).

Proof of 6.2. We will only show that if $\mathcal{P} = U$, then $\chi : \mathcal{P} \rightarrow \tilde{K}_0 R$ is a homomorphism. Let $k, k' \in \text{End}(\pi_m)$ represent projective k -invariants in $\text{Ext}^{m+1}(M; \pi_m)$. We will show that

$$(k' \circ k)C_m \oplus C_m \oplus C_{m+1} \cong kC_m \oplus k'C_m \oplus C_{m+1}.$$

three ways:

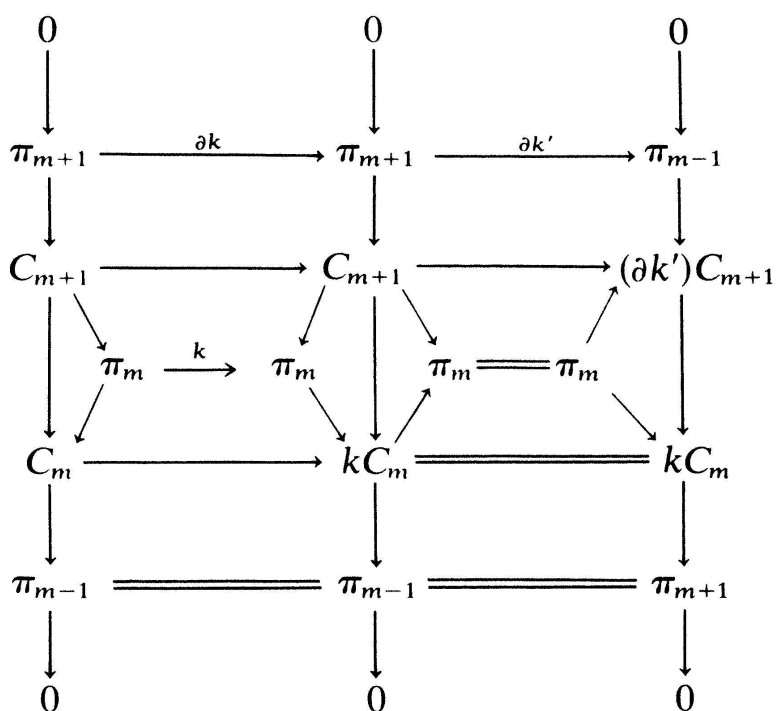


It follows that

$$(k' \circ k)C_m \cong k'(kC_m)$$

via a map inducing identity on π_{m-1} and π_m because the k -invariants are the same. Thus $\{\partial(k' \circ k)\} = \{\partial k' \circ \partial k\}$. Note that $k' \circ k$ is projective because it is a unit.

Furthermore, the following also has k -invariant $\partial k' \circ \partial k$:



Thus, by another application of lemma 6.4, we have $C_{m+1} \oplus kC_m \cong (k' \circ k)C_m \oplus (\partial k')C_{m+1}$. (6.5) and (6.6) taken together prove (c).

CONJECTURE (see [11, lemma 6.1 (c)]).

$$(k' \circ k)C_m \oplus C_m \cong kC_m \oplus k'C_m.$$

Let $\partial: \text{Ext}^{m+1}(M, \pi_m) \rightarrow \text{Ext}^{m+2}(M, \pi_{m+1})$ be the boundary operator in the coefficient exact sequence associated with the functor $\text{Ext}^i(M, -)$ and the exact sequence

$$0 \rightarrow \pi_{m+1} \rightarrow C_{m+1} \rightarrow \pi_m \rightarrow 0.$$

The previous proof shows that ∂ is a ring homomorphism, provided the domain and range are rings.

Furthermore, we see that because C_i is finitely generated and free for $i = 0, \dots, m + 1$, then $\text{im } \chi_m \subset \text{im } \chi_{m+1}$. This follows from the commutative diagram:

$$\begin{array}{ccc} \mathcal{P}(\text{Ext}^{m+2}(M, \pi_{m+1})) & & \\ \uparrow \partial & \searrow \chi_{m+1} & \\ \mathcal{P}(\text{Ext}^{m+1}(M, \pi_m)) & \xrightarrow{\chi_m} & \tilde{K}_0 R \end{array}$$

The conditions of section 3 have obvious analogs in this setting:

6.7. $(C(m))$. The composition in $\text{End}(\pi_m)$ is commutative modulo B^m .

6.8. (RE_m) . Each map $k \in \text{End}(\pi_m)$ extends to a map in

$$\begin{aligned} \text{End}(C_m) \Leftrightarrow \partial: \text{Ext}^m(M, \pi_{m-1}) \rightarrow \text{Ext}^{m+1}(M, \pi_m) \text{ is surjective} \\ \Leftrightarrow \text{Ext}^{m+1}(M; C_m) \rightarrow \text{Ext}^{m+1}(M; \pi_{m-1}) \text{ is monic.} \end{aligned}$$

6.9. (E_m) . Each map $f \in \text{Hom}(\pi_m, C_m)$ extends to a map in

$$\text{End}(C_m) \Leftrightarrow \text{Ext}^1(\pi_{m-1}, C_m) = \text{Ext}^{m+1}(M; C_m) = 0$$

Again: $(E_m) \Rightarrow (RE_m) \Rightarrow B^m$ is a right ideal $\Leftarrow (C(m))$

At the present writing, I know of no examples where $C(m)$ is not satisfied.

We can “dualize” RE_m as follows:

6.10. (RE^m) . Any map $k \in \text{End}(\pi_m)$ which coextends to C_{m+1} extends to C_m .

Thus, in the following diagram,

$$\begin{array}{ccc} & \pi_m & \xrightarrow{i} C_m \\ \exists \alpha \swarrow & \downarrow k & \searrow \exists \beta \\ C_{m+1} & \xrightarrow{j} \pi_m & \end{array}$$

the existence of α such that $j \circ \alpha = k$ implies the existence of a β such that $\beta \circ i = k$. The converse is always true because C_m is projective.

6.11. PROPOSITION. *Any map $k \in \text{End}(\pi_m)$ which coextends to C_{m+1} extends to C_m iff $\partial: \text{Ext}^{m+1}(M, \pi_m) \rightarrow \text{Ext}^{m+2}(M, \pi_{m+1})$ is a monomorphism iff $i_*: \text{Ext}^{m+1}(M, \pi_{m+1}) \rightarrow \text{Ext}^{m+1}(M, C_{m+1})$ is surjective.*

6.12. PROPOSITION. *If each $k \in \text{End}(\pi_m)$ which coextends to C_{m+1} also extends to C_m , then $\text{Ext}^{m+1}(M; \pi_m)$ is a ring.*

Proof. Let $k, \bar{k} \in \text{End}(\pi_m)$, let k extend to C_m . We must show that $k \circ \bar{k}$ extends to C_m . But k extends to C_m implies that k coextends to C_{m+1} by (6.10). Thus $k \circ k'$ coextends to C_{m+1} . But condition RE^m implies that $k \circ k'$ extends to C_m . This proves (6.12).

Note that $(RE_m) \leftarrow (E_m) \rightarrow (RE^m)$.

Notice that it follows from (6.6) that if $\{k\} \in \text{Ext}^m(M, \pi_{m-1})$ is projective and $\{k' \circ k\} = 1$, then $\{\partial k'\} \in \text{Ext}^{m+1}(M; \pi_m)$ is projective. Also, (6.5) implies that $\partial\{k\}$ is projective if $\{k\}$ is.

6.13. COROLLARY. *If $\partial: \text{Ext}^{m+1}(M; \pi_m) \rightarrow \text{Ext}^{m+2}(M; \pi_{m+1})$ is a monomorphism (condition RE^m), then each projective k -invariant is a unit.*

Proof. Let $\{k\} \in \text{Ext}^{m+1}(M; \pi_m)$ be projective. By (5.3), there is a $k' \in \text{End}(\pi_m)$ such that $k' \circ k - 1 \in B^m$. Thus $\partial k' \circ \partial k - 1 \in B^{m+1}$. By (6.6), $\{\partial k'\}$ is projective. By (5.3) again, $\{\partial k \circ \partial k'\} = 1 = \{\partial k' \circ \partial k\}$. Since ∂ is a monomorphism, $\text{im } \partial$ a ring, and $\partial\{k \circ k'\} = \{\partial k \circ \partial k'\}$, then $\{k \circ k'\} = 1 = \{k' \circ k\}$. This completes (6.13).

The proof of the following corollary is similar to 6.13.

6.14. COROLLARY. *If $\partial|_{\mathcal{P}}: \mathcal{P}(\text{Ext}^m(M, \pi_{m-1})) \rightarrow \mathcal{P}(\text{Ext}^{m+1}(M, \pi_m))$ is surjective, then each projective k -invariant in $\text{Ext}^{m+1}(M, \pi_m)$ is a unit.*

Questions. (a) If $M = Z$, is B^m always a right ideal? For example, if $A(\pi)$ is the augmentation ideal in $Z\pi$, is $H^1(\pi; A(\pi))$ a ring?

(b) If B^m is a right ideal, is $\mathcal{P}(\text{Ext}^{m+1}(M; \pi_m))$ a semigroup under composition?

Appendix: Groups Having $H^i(\pi; Z\pi) = 0$

We will give some results that show that the hypothesis of theorem 1.1 is often satisfied.

(a) If π is a finite group, then $H^i(\pi; Z\pi) = 0$ ($i > 0$). This follows because any projective π -module is weakly injective.

(b) If π is a (Poincare) duality group with cohomological dimension m , then $H^i(\pi; Z\pi) = 0$ ($i \neq m$) [1].

(c) If F is a free abelian group of countable rank, then $H^i(F; ZF) = 0$ for all $i \geq 0$.

(d) [1, Proposition 3.1] *If S is a subgroup of G with finite index (not necessarily normal), then $H^i(S; ZS) \cong H^i(G; ZG)$ as right S -modules. Thus if $S < G$ such that $[G : S] < \infty$, then $H^k(S; ZS) = 0 \Leftrightarrow H^k(G; ZG) = 0$.*

For example, if $0 \rightarrow C \rightarrow G \rightarrow T \rightarrow 0$ is an exact sequence of groups where C is a group of cohomological dimension n and T is finite, then $H^i(G; ZG) = 0$ for $i > n$. Thus, any finitely generated abelian group A of rank n has $H^i(A; ZA) = 0$ for $i \neq n$.

(e) The following theorem is an easy consequence of the spectral sequence of a group extension: *Let $1 \rightarrow N \rightarrow \pi \rightarrow G \rightarrow 1$ be an exact sequence of groups. Let N be finite. Then $H^i(\pi; Z\pi) \cong H^i(G; ZG)$ for all $i > 0$.*

For example, if π is an extension of a finite group by a duality group of cohomological dimension n , then $H^i(\pi; Z\pi) = 0$ for $i \neq n$. Also any one relator group G [8] is such that $H^i(G; ZG) = 0$ for $i \geq 3$.

(f) We say that a group π has *property \mathcal{P}^n* if $H^i(\pi; Z\pi) = 0$, $0 < i < n$. The functor $H^*(\pi, -)$ is *strongly additive* if it commutes with arbitrary direct sums. For example, if π admits a projective resolution of finite type

$$\cdots \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow Z \rightarrow 0$$

of the trivial π -module Z (i.e., each P_i is a finitely generated projective π -module), then $H^*(\pi; -)$ is strongly additive. The following is then true: *Let $1 \rightarrow A \rightarrow \pi \rightarrow B \rightarrow 1$ be an exact sequence of groups such that $H^*(A; -)$ is strongly additive. Then A has \mathcal{P}^i and B has \mathcal{P}^j implies that π has \mathcal{P}^k , where $k = \min(i, j)$.*

(g) Let $n(G)$ denote the smallest integer $\leq \infty$ such that $H^i(G; ZG) = 0$ for all $i > n(G)$. Let \mathcal{L} be the class of all groups G such that $n(G)$ is finite. It follows easily from (d) and (e) that \mathcal{L} contains all polycyclic (=soluble with maximum condition on subgroups) groups. More generally, if \mathcal{A} is a class of groups, we say that a group G is *poly(\mathcal{A})* if there exists a *finite* sequence of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = 1$$

such that $G_{i+1} \triangleleft G_i$ and G_i/G_{i+1} is a member of \mathcal{A} . Let *fcd* denote the class of

groups of finite cohomological dimension. By the use of (d) and (e) one may show the following:

THEOREM. *If G is poly (finitely generated abelian) or poly (finite or fcd) then G is a member of \mathcal{L} .*

Furthermore, it follows from [13, page 138] that \mathcal{L} is closed under finite sums. It is closed under infinite sums provided that each of the summands G_i has $n(G_i) < k$, k being independent of i . \mathcal{L} is closed under amalgamated sums by [2]. If $G = \bigcup_{i \in \mathbb{Z}} G_i$ is a countable union of subgroups G_i such that $n(G_i) \leq M < \infty$ for all $i \in \infty$, then $n(G) \leq M + 1$ (R. Bieri). Thus any countable torsion group G has $n(G) \leq 1$, because G is the countable union of finite subgroups. There are simple examples to show that \mathcal{L} is not closed under arbitrary direct limits.

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