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# On Meromorphic Solutions of First-Order Differential Equations

STEVEN B. BANK AND ROBERT P. KAUFMAN

## 1. Introduction

In the first part of this paper, we consider first-order differential equations of the form,

$$\sum_{n=0}^N R_n(z, w)(w')^n = 0, \quad (1)$$

where for each  $n$ ,  $R_n(z, w) = \sum_{k=0}^{B(n)} a_{kn}(z)P_{kn}(w)$ , the  $a_{kn}(z)$  are analytic functions in a neighborhood of  $\infty$ , having no essential singularity at  $\infty$ , and the functions  $P_{kn}(w)$  are all defined and analytic on some common open set in the plane. Such equations were treated by A. A. Gol'dberg in [1], and he showed [1; Th 4] that any solution, meromorphic in a neighborhood of  $\infty$ , is of finite order of growth, and he obtained estimates (which depend on the equation) for the growth. The main technique used by Gol'dberg in proving this result is 0. Frostman's generalization of the Ahlfors-Shimizu formula (see [3; p. 180] or [10; p. 42]). In the first part of our paper (§§ 3, 4 below), we present an alternate proof of Gol'dberg's result which seems to be more elementary and more transparent than Gol'dberg's proof. In our proof, we determine disks around the sufficiently large  $a$ -points of the solution (for most values of  $a$ ) on which the solution is univalent. From this we obtain estimates for the growth of the counting functions for the  $a$ -points, and hence an estimate on the growth of the solution by Nevanlinna's Second Fundamental Theorem.

The second part of the paper deals with algebraic differential equations (i.e. equations of the form  $F(z, y, y', \dots, y^{(n)}) = 0$ , where  $F$  is a polynomial in all its variables.) In [5], Pólya proved that an entire transcendental function of order zero cannot be a solution of a first-order algebraic differential equation. This result was generalized by Valiron [8], who showed that in the first-order case, the order of an entire transcendental solution must be a positive rational number (and it is now known (see Strelitz [7; p. 70]) that the order must be at least  $\frac{1}{2}$ ). In addition, Valiron (see [9; pp. 223–225]) found a third-order algebraic differential

equation with a transcendental solution of order zero. The remaining case of second-order algebraic differential equations was settled by Zimogljad [11], who showed that these equations cannot possess transcendental entire solutions of order zero. At the present time, the situation concerning transcendental meromorphic solutions of order zero is less clear. In the third-order equation,  $F(z, y, y', y'', y''') = 0$ , constructed by Valiron in [9], it is easily seen that  $F$  is homogeneous as a polynomial in  $y, y', y''$  and  $y'''$ , and hence the logarithmic derivative of Valiron's solution does provide an example of a transcendental meromorphic solution of order zero of a second-order algebraic differential equation. However, to the authors' knowledge, it was not known whether transcendental meromorphic functions of order zero can satisfy first-order algebraic differential equations. (The result of Gol'dberg [1, Th. 4] hints at the possibility that such solutions may exist, but none had been constructed.) In the second part of our paper (see § 5 below), we construct an example of a transcendental meromorphic function  $f(z)$  of order zero which satisfies a first-order algebraic differential equation. The characteristic of our function  $f(z)$  satisfies  $T(r, f) = O((\log r)^2)$  as  $r \rightarrow +\infty$  (which is the estimate suggested by Gol'dberg's result), and from the construction of  $f(z)$ , it is easy to see that  $T(r, f) \neq o((\log r)^2)$  as  $r \rightarrow +\infty$ . We conjecture that first-order algebraic differential equations cannot possess transcendental meromorphic solutions whose characteristic is  $o((\log r)^2)$  as  $r \rightarrow +\infty$ .

## 2. Notation

If  $f$  is a meromorphic function and  $\lambda$  is a complex number or  $\infty$ , we will use the standard notation for the Nevanlinna functions,  $T(r, f)$ ,  $N(r, \lambda, f)$  and  $m(r, \lambda, f)$ , (see [2; p. 6] or [4; pp. 6, 12]). We will also use the notation  $n(r, \lambda, f)$  to denote the number of roots of  $f(z) = \lambda$  (counting multiplicity) in  $|z| \leq r$ .

**3. THÉOREM.** *Given the equation,*

$$\sum_{n=0}^N R_n(z, w)(w')^n = 0, \quad (2)$$

where each  $R_n(z, w)$  is a polynomial in  $z$  whose coefficients are meromorphic functions of  $w$ , say

$$R_n(z, w) = \sum_{j=0}^{A(n)} z^j P_{jn}(w), \quad (3)$$

where  $P_{jn}(w)$  is a meromorphic function of  $w$ . Let  $J$  be the set of integers  $n$ ,  $0 \leq n \leq N$ , for which  $R_n(z, w)$  is not identically zero, and we may assume that  $0$  and  $N$  belong to  $J$ , and if  $n$  belongs to  $J$ , then  $P_{A(n),n}(w)$  is not identically zero. Assume also that  $N \geq 1$  (see § 4(b)), and set

$$L = \max \{((A(j) - A(N))/(N - j)) : j \in J - \{N\}\}. \tag{4}$$

Let  $w = f(z)$  be a meromorphic function on the plane which satisfies equation (2) at every point of analyticity. Then, as  $r \rightarrow +\infty$ ,

(a)  $T(r, f) = O(\log r)$  if  $L < -1$ ,

(b)  $T(r, f) = O((\log r)^2)$  if  $L = -1$ ,

(c)  $T(r, f) = O(r^{2L+2})$  if  $L > -1$ .

*Proof.* Let  $E$  be the set of all complex numbers  $w_0$  with the property that if  $P_{jn}(w)$  is not identically zero, then  $P_{jn}(w)$  is analytic and nonzero at  $w_0$ . We now prove a sequence of four lemmas from which the theorem will immediately follow.

LEMMA A. Let  $w_0$  belong to  $E$ . Then there exist real numbers,  $b > 0$ ,  $r_1 > 1$ ,  $K_2 > K_1 > 0$  and  $a_1 < a_2 < \dots < a_q \leq L$ , such that if  $1 \leq j < s \leq q$ , then

$$2K_2 |z|^{a_j} < K_1 |z|^{a_s} \quad \text{if} \quad |z| > r_1, \tag{5}$$

and in addition, if  $z$  is a complex number satisfying  $|z| > r_1$  and  $|f(z) - w_0| \leq b$ , then there is a unique element  $j$  in the set  $\{1, 2, \dots, q\}$  such that,

$$K_1 |z|^{a_j} \leq |f'(z)| \leq K_2 |z|^{a_j}. \tag{6}$$

*Proof.* If  $w_0$  belongs to  $E$ , then there exist real numbers  $b > 0$  and  $d_2 > d_1 > 0$  such that on  $|w - w_0| \leq b$ , we have,

$$d_1 \leq |P_{jn}(w)| \leq d_2, \tag{7}$$

for all  $P_{jn}$  which are not identically zero. It now easily follows from (3) that there exists  $r_0 > 1$ , such that if  $n$  belongs to  $J$ , then for any  $z$  satisfying  $|z| > r_0$  and  $|f(z) - w_0| \leq b$ , we have,

$$c_1 |z|^{A(n)} \leq |R_n(z, f(z))| \leq c_2 |z|^{A(n)}, \tag{8}$$



where  $c_1 = d_1/2$  and  $c_2 = 2d_2$ . For convenience, denote  $R_n(z, f(z))$  by  $B_n(z)$ . Now if  $z$  satisfies  $|z| > r_0$  and  $|f(z) - w_0| \leq b$ , let  $k$  be the largest element of  $J$  for which,

$$|B_k(z)(f'(z))^k| = \max \{|B_n(z)(f'(z))^n| : n \in J\}. \tag{9}$$

Then there must exist an element  $m$  in  $J - \{k\}$  such that,

$$|B_m(z)(f'(z))^m| \geq N^{-1} |B_k(z)(f'(z))^k|, \tag{10}$$

or else equation (2) would clearly be violated at  $z$ . If  $m$  is the smallest element of  $J - \{k\}$  with property (10), we will say that the pair  $(k, m)$  is the *index* for  $z$ . Denote by  $\alpha_{km}$  the number  $(A(k) - A(m))/(m - k)$ . In addition, let  $K_2$  denote the maximum of all numbers  $(Nc_2/c_1)^{1/(j-n)}$ , and  $K_1$  the minimum of all numbers  $(c_1/Nc_2)^{1/(j-n)}$ , where  $j$  and  $n$  belong to  $J$  and  $j > n$ . (Clearly  $K_1$  and  $K_2$  are independent of  $z$ .) From (8), (9) and (10), it easily follows that if  $z$  satisfies  $|z| > r_0$  and  $|f(z) - w_0| \leq b$ , and if  $(k, m)$  is the index for  $z$ , then

$$K_1 |z|^{\alpha_{km}} \leq |f'(z)| \leq K_2 |z|^{\alpha_{km}}. \tag{11}$$

Let  $F$  be the set of distinct numbers of the form  $\alpha_{km}$  for which there exists a complex number  $z$  satisfying  $|z| > r_0$  and  $|f(z) - w_0| \leq b$  having  $(k, m)$  for its index. Let  $F_1$  be the subset of  $F$  consisting of those elements of  $F$  which are larger than  $L$ . Let  $r_2$  be so large that  $r_2 > r_0$  and

$$r_2^{\alpha - L} > K_2/K_1 \quad \text{for all } \alpha \text{ in } F_1. \tag{12}$$

We now claim that if  $z$  satisfies  $|z| > r_2$  and  $|f(z) - w_0| \leq b$ , and if  $(k, m)$  is the index for  $z$ , then

$$\alpha_{km} \leq L. \tag{13}$$

If  $k = N$ , then (13) is clear, so we may assume that  $k < N$ . If  $(k, m)$  is the index for  $z$ , then by (9), we have

$$|B_k(z)(f'(z))^k| \geq |B_N(z)(f'(z))^N|. \tag{14}$$

In view of (8) and (4), it follows that  $|f'(z)| \leq K_2 |z|^L$ . But since  $(k, m)$  is the index for  $z$ , (11) holds, so that  $|z|^{\alpha_{km} - L} \leq K_2/K_1$ . Hence by (12), we see that  $\alpha_{km}$  cannot belong to  $F_1$  which proves (13). Now if  $a_1 < a_2 < \dots < a_q$  are the elements of  $F - F_1$ , and if  $r_1$  is chosen so large that  $r_1 > r_2$  and (5) holds for  $|z| > r_1$ , then the proof of the lemma is complete.

DEFINITION. Let  $w_0$  belong to  $E$ , and let  $b, r_1, K_1, K_2$  and  $a_1, \dots, a_q$  be as in Lemma A. If  $z$  is a complex number satisfying  $|z| > r_1$  and  $|f(z) - w_0| \leq b$ , then the unique  $a_j$  for which (6) holds will be said to be *associated with  $z$* .

LEMMA B. Let  $w_0$  belong to  $E$  with  $b, r_1, K_1, K_2$  and  $a_1, \dots, a_q$  as in Lemma A. Assume that some  $a_k$  is less than  $-1$ . Then there exists  $R_0 > r_1$  such that if there is at least one point  $z_0$  satisfying  $|z_0| > R_0$  and  $f(z_0) = w_0$ , whose associated  $a_j$  is less than  $-1$ , then  $f(z)$  is a rational function.

*Proof.* Let  $a_m$  be the largest  $a_k$  less than  $-1$  say  $a_m = -1 - \eta$ , where  $\eta > 0$ . Choose  $R_0$  so large that  $R_0 > r_1$  and,

$$R_0^{-\eta} \leq \min \{b\eta/4K_2, b/8\pi K_2\}. \tag{15}$$

Now let  $z_0$  satisfy  $|z_0| > R_0$ ,  $f(z_0) = w_0$ , and have associated  $a_j < -1$ . Then  $a_j = -1 - \sigma$ , where

$$\eta \leq \sigma. \tag{16}$$

If  $z_0 = |z_0| e^{i\varphi}$ , we now assert that,

$$|f'(re^{i\varphi})| \leq K_2 r^{a_j} \quad \text{for all } r \geq |z_0|. \tag{17}$$

If (17) fails to hold, then clearly we can find  $\epsilon$  satisfying  $0 < \epsilon < K_2$ , and a point  $z_2 = |z_2| e^{i\varphi}$ , with  $|z_2| > |z_0|$ , such that,

$$|f'(z_2)| = (K_2 + \epsilon) |z_2|^{a_j}, \tag{18}$$

while,

$$|f'(re^{i\varphi})| \leq (K_2 + \epsilon) r^{a_j} \quad \text{for } |z_0| \leq r \leq |z_2|. \tag{19}$$

Hence from (19),

$$|f(z_2) - f(z_0)| \leq (K_2 + \epsilon) |z_0|^{-\sigma} / \sigma, \tag{20}$$

so in view of (15) and (16),

$$|f(z_2) - w_0| \leq b/2. \tag{21}$$

By Lemma A, let  $a_s$  be associated with  $z_2$ . Then by (6) and (18), we have,

$$K_1 |z_2|^{a_s} \leq (K_2 + \epsilon) |z_2|^{a_j} \leq K_2 |z_2|^{a_s}. \quad (22)$$

From the second inequality, it follows easily that  $a_s \leq a_j$  is impossible, so  $a_j < a_s$ . But then the first inequality contradicts (5), thus proving (17). From (17) (together with (15) and (16)), it follows that

$$|f(re^{i\varphi}) - w_0| \leq b/2 \quad \text{for all } r \geq |z_0|. \quad (23)$$

We now assert that if  $r > R_0$ , then

$$|f'(z)| \leq K_2 |z|^{a_j} \quad \text{on } |z| = r. \quad (24)$$

By (17), we know that (24) holds at  $z_1 = re^{i\varphi}$ . Hence if (24) failed to hold at some point on  $|z| = r$ , then we can find  $\epsilon$ , with  $0 < \epsilon < K_2$ , and a point  $z_2 = re^{i\psi}$ , with  $\varphi < \psi < \varphi + 2\pi$ , such that,

$$|f'(z_2)| = (K_2 + \epsilon) |z_2|^{a_j}, \quad (25)$$

while for  $\varphi \leq \theta \leq \psi$ ,

$$|f'(re^{i\theta})| \leq (K_2 + \epsilon) r^{a_j}. \quad (26)$$

Hence,  $|f(z_2) - f(z_1)| \leq 4\pi K_2 r^{a_j+1}$ , which with (15), (16) and (23), yields  $|f(z_2) - w_0| \leq b$ . Thus if  $a_s$  is associated with  $z_2$  (by Lemma A), then using (25), we again obtain (22), which as before is impossible. This proves (24), and it easily follows that  $f$  is rational.

**LEMMA C.** *Let  $w_0$  belong to  $E$ , and let  $b, r_1, K_1, K_2$  and  $a_1, \dots, a_q$  be as in Lemma A. Assume  $f$  is transcendental. Then there is a constant  $\delta_1$ , with  $0 < \delta_1 < \frac{1}{2}$ , with the property that if  $z_0$  satisfies  $|z_0| > 2r_1$  and  $f(z_0) = w_0$ , and if the  $a_j$  associated with  $z_0$  is at least  $-1$ , then  $f$  is univalent on the disk  $|z - z_0| \leq \delta_1 |z_0|^{-L}$ .*

*Proof.* Let  $\lambda = \max \{|a_k| : k = 1, \dots, q\}$ . Let  $\delta$  be a positive real number such that,

$$\delta < \min \left\{ \frac{1}{4}, (b/K_2)2^{-\lambda} \right\}, \quad (27)$$

and set,

$$\delta_1 = \delta (K_1/K_2) 2^{-(2\lambda+2)}. \quad (28)$$

Let  $z_0$  satisfy  $|z_0| > 2r_1$ ,  $f(z_0) = w_0$ , and let  $a_j$  be associated with  $z_0$  and satisfy  $a_j \geq -1$ . Since  $f$  is transcendental, the set

$$H = \{z : |z| \geq 3r_1/2, \quad |f(z) - w_0| = b\}, \tag{29}$$

is not empty. Let  $z_1$  be a point in  $H$  such that  $|z_0 - z_1| = \min \{|z_0 - z| : z \in H\}$ .

We now assert that,

$$|z_1 - z_0| \geq \delta |z_0|^{-a_j}. \tag{30}$$

To prove (30), we assume the contrary, so that

$$|z_1 - z_0| \leq \delta |z_0|^{-a_j} \leq \delta |z_0|. \tag{31}$$

Let  $D$  be the disk  $|z - z_0| \leq |z_1 - z_0|$ . In view of our assumption (31), it follows that for all  $z$  in  $D$ , we have  $|z| \geq (\frac{3}{2})r_1$  (by (27)) and  $|f(z) - w_0| \leq b$  (or else the definition of  $z_1$  would be violated). Hence by Lemma A, each  $z$  in  $D$  has some  $a_k$  associated with it. Using (5) and (6), it follows by an argument very similar to that used to prove (17), that for every  $z$  in  $D$ , the  $a_k$  associated with  $z$  is the original  $a_j$  associated with  $z_0$ , so that (6) holds on  $D$ . Now from (27) and (31), we have,  $|z_0|/2 \leq |z| \leq 2|z_0|$  on  $D$ , so it follows from (6) that  $|f'(z)| \leq K_2 2^{|a_j|} |z_0|^{a_j}$ . Since  $z_1$  belongs to  $H$ , we thus have  $b \leq K_2 2^{|a_j|} |z_0|^{a_j} |z_1 - z_0|$ , which in view of (31) and (27) is impossible. This proves (30).

Let  $D_1$  be the disk  $|z - z_0| \leq \delta |z_0|^{-a_j}$ . Then using (30) and (27), it follows that for all  $z$  in  $D_1$ , we have  $|z| \geq 3r_1/2$  and  $|f(z) - w_0| \leq b$  (or else the definition of  $z_1$  would be violated). As before, the  $a_k$  associated with each  $z$  in  $D_1$  must be the original  $a_j$  associated with  $z_0$ , so that,

$$K_1 |z|^{a_j} \leq |f'(z)| \leq K_2 |z|^{a_j} \quad \text{on } D_1. \tag{32}$$

Since  $|z_0|/2 \leq |z| \leq 2|z_0|$  on  $D_1$ , we thus have,

$$K_1 2^{-|a_j|} |z_0|^{a_j} \leq |f'(z)| \leq K_2 2^{|a_j|} |z_0|^{a_j}, \quad \text{on } D_1. \tag{33}$$

Now let  $D_2$  be the disk,  $|z - z_0| \leq (\delta/2) |z_0|^{-a_j}$ . If  $z$  belongs to  $D_2$ , then by Cauchy's formula for derivatives (using (33) and the circle of radius  $(\delta/2) |z_0|^{-a_j}$  around  $z$ ), we obtain,

$$|f''(z)| \leq K_2 2^{|a_j|+1} |z_0|^{2a_j} / \delta \quad \text{on } D_2. \tag{34}$$

Now let  $D_3$  be the disk  $|z - z_0| \leq \delta_1 |z_0|^{-a}$ , where  $\delta_1$  is as in (28). Then  $D_3$  is contained in  $D_2$ , and by (34) and (33), it easily follows that on  $D_3$ ,

$$|f'(z) - f'(z_0)| \leq |f'(z_0)|/2. \tag{35}$$

Writing  $f'(z) = f'(z_0) + (f'(z) - f'(z_0))$ , it now easily follows from (35) (and (33)) that if  $\zeta$  and  $\sigma$  are distinct points in  $D_3$ , then  $\int_{\sigma}^{\zeta} f'(z) dz$  (where the contour is the line segment joining  $\sigma$  to  $\zeta$ ) cannot be zero, and hence  $f$  is univalent on  $D_3$ . Since  $a_j \leq L$ ,  $D_3$  contains the disk  $|z - z_0| \leq \delta_1 |z_0|^{-L}$  and hence the result is proved.

**LEMMA D.** *Let  $L$  and  $\delta_1$  be real numbers with  $L \geq -1$  and  $0 < \delta_1 < 1$ . Let  $\{z_k\}$  be a sequence of complex numbers such that each disk  $|z - z_k| \leq \delta_1 |z_k|^{-L}$  contains no other  $z_m$ . Then as  $R \rightarrow +\infty$ , the number of points  $z_k$  in the annulus  $1 \leq |z| \leq R$  is  $O(R^{2L+2})$  if  $L > -1$ , and is  $O(\log R)$  if  $L = -1$ .*

*Proof.* Let  $n$  be a nonnegative integer and let  $A_n = \{k : 2^n \leq |z_k| \leq 2^{n+1}\}$ . Let  $\delta_2 = 2^{-|L|}\delta_1$ . Then it is easy to see that,

$$\delta_1 |z_k|^{-L} \geq \delta_2 2^{-nL} \quad \text{for } k \text{ in } A_n. \tag{36}$$

For  $k$  in  $A_n$ , let  $w_k = z_k 2^{-n}$ , so  $1 \leq |w_k| \leq 2$ . In view of the hypothesis and (36), it follows that for  $k$  in  $A_n$ , the disk  $|w - w_k| \leq r_n$ , where  $r_n = \delta_2 2^{-nL} 2^n$ , contains no other  $w_m$  for  $m$  in  $A_n$ . Thus the disks  $|w - w_k| \leq r_n/3$ , for  $k$  in  $A_n$ , are all disjoint and all lie in  $|w| \leq 3$ . By an area argument, it follows that if  $\sigma(n)$  is the cardinal number of  $A_n$ , then,

$$\sigma(n) \leq (81/\delta_2^2) 4^{n(L+1)}. \tag{37}$$

Now if  $R > 2$  is given, let  $m$  be such that  $2^m \leq R \leq 2^{m+1}$ . Then if  $\nu(R)$  is the number of  $z_k$  in  $1 \leq |z| \leq R$ , clearly,

$$\nu(R) \leq \sum_{n=0}^m \sigma(n). \tag{38}$$

From (37) and (38), Lemma D immediately follows.

By Nevanlinna's Second Fundamental Theorem [4; p. 69], it follows that if  $w_1, w_2$  and  $w_3$  are distinct complex numbers, then as  $r \rightarrow +\infty$ ,

$$T(r, f) = O\left(\sum_{j=1}^3 N(2r, w_j, f) + \log r\right), \tag{39}$$

and hence it is now clear that the theorem of § 3 follows immediately from Lemmas A–D.

#### 4. Remarks

(a) It is easy to see that the proof of the theorem of § 3 is valid when the hypotheses are relaxed as follows. In equation (2), for each  $n$ ,

$$R_n(z, w) = \sum_{k=0}^{B(n)} a_{kn}(z)P_{kn}(w), \tag{40}$$

where the  $a_{kn}(z)$  are analytic functions in a neighborhood of  $\infty$ , having no essential singularity at  $\infty$ , the functions  $P_{kn}(w)$  are all defined and analytic on some common open set in the plane, and  $f(z)$  is a solution meromorphic in a neighborhood of  $\infty$ , say  $|z| \geq R_0$ . (In the definition of the characteristic  $T(r, f)$  of such a function (see [10; p. 49]), only the  $a$ -points lying in  $R_0 \leq |z| \leq r$  are considered in defining  $N(r, a, f)$ , and the Second Fundamental Theorem still holds for such functions (see [10; p. 50]).) Let  $J$  be as in the statement of the theorem, and for  $n$  in  $J$ , rearrange terms in  $R_n(z, w)$  so that  $R_n(z, w)$  has the form,

$$R_n(z, w) = z^{A(n)}g_n(w) + \sum_{k=0}^{B(n)} b_{kn}(z)P_{kn}(w), \tag{41}$$

where  $g_n(w)$  is not identically zero, and the highest power of  $z$  in the Laurent expansion for each  $b_{kn}(z)$  at  $\infty$  is less than  $A(n)$ . Then with  $L$  as defined in (4), the conclusions (a), (b), (c) of the theorem hold. (In this formulation, the theorem is now fully equivalent to Gol'dberg's result [1; Theorem 4].) The proof in this formulation is easily seen to be identical to the proof we gave in § 3, with three minor changes. First, the set  $E$  would consist of all complex numbers  $w_0$  with the property that each  $g_n(w)$  (for  $n$  in  $J$ ) is analytic and nonzero at  $w_0$ , and if  $P_{jn}(w)$  is not identically zero, then  $P_{jn}(w)$  is analytic and nonzero at  $w_0$ . Secondly, the conclusion of Lemma B would be  $T(r, f) = O(\log r)$  as  $r \rightarrow +\infty$ , which follows easily from (24) since then  $f(z)$  has a finite limit at  $\infty$ . Finally, in the hypothesis of Lemma C, we would assume that  $T(r, f) \neq O(\log r)$  as  $r \rightarrow +\infty$ , instead of assuming  $f$  is transcendental.

(b) If  $N = 0$  in the theorem of § 3, then  $f$  must be a rational function. This is easily seen as follows. If  $A(0) = 0$ , clearly  $f$  must be a constant, so we may assume  $A(0) > 0$ . In this case, letting  $E$  be as in the proof, we see that if  $w_0$  belongs to  $E$ , then there are positive constants  $b, d_1$  and  $d_2$  such that on  $|w - w_0| \leq b$ , we have

(7). It easily follows that if  $|z|$  is sufficiently large, then  $|f(z) - w_0| > b$  and hence  $f$  must be rational.

**5. EXAMPLE.** In this section, we construct an example of a transcendental meromorphic function on the plane of order zero, which satisfies a first-order algebraic differential equation.

Let  $P(z)$  denote the Weierstrass  $Pe$ -function having primitive periods 1 and  $2\pi i$  (see [6; p. 368]). If  $z$  is a nonzero complex number, and  $\alpha_1$  and  $\alpha_2$  are two values of  $\log z$ , then clearly  $P(\alpha_1) = P(\alpha_2)$ . Thus,  $w(z) = P(\log z)$  is single-valued on the punctured plane. Now if  $\zeta$  is a complex number, then clearly there is at least one complex number  $z$  such that  $z + z^{-1} = \zeta$ , and if  $z_j + z_j^{-1} = \zeta$  for  $j = 1, 2$ , then either  $z_1 = z_2$  or  $z_1 = z_2^{-1}$ . In either case,  $w(z_1) = w(z_2)$  since  $P(z)$  is an even function. Thus, the function  $u(\zeta)$ , defined by,

$$u(\zeta) = w(z) \quad \text{where} \quad z + z^{-1} = \zeta, \tag{42}$$

is single-valued on the plane.

We assert that  $u(\zeta)$  is meromorphic on the plane. First, if  $\zeta_0 \neq \pm 2$ , then there is an analytic function  $h(\zeta)$  around  $\zeta_0$  such that  $h(\zeta) + (h(\zeta))^{-1} \equiv \zeta$ . Since  $h(\zeta_0) \neq 0$ , there is an analytic branch  $L(z)$  of  $\log z$  on a neighborhood of  $h(\zeta_0)$ , so that  $L(h(\zeta))$  is analytic on a neighborhood of  $\zeta_0$ . Thus  $u(\zeta) = P(L(h(\zeta)))$  is meromorphic on a neighborhood of  $\zeta_0$ . Now suppose  $\zeta_0 = 2$ . By the above argument,  $u(\zeta)$  is meromorphic on  $0 < |\zeta - \zeta_0| < 4$ . Let  $\{\zeta_n\}$  be a sequence converging to  $\zeta_0$  such that  $\zeta_n \neq \zeta_0$  for  $n = 1, 2, \dots$ , and let  $z_n$  be such that  $z_n + z_n^{-1} = \zeta_n$ . Then clearly  $\{z_n\} \rightarrow 1$  and  $z_n \neq 1$  for each  $n$ . Let  $L(z)$  be an analytic branch of  $\log z$  on  $|z - 1| < 1$  such that  $L(1) = 0$ . Then for all sufficiently large  $n$ ,  $0 < |L(z_n)| < 1$ , and hence  $u(\zeta_n)$  is finite. This shows that  $\zeta_0 = 2$  is an isolated singularity of  $u(\zeta)$ , and since  $P(0) = \infty$ , we see that  $\zeta_0 = 2$  is a pole of  $u(\zeta)$ . A similar argument shows that  $\zeta_0 = -2$  is a removable singularity of  $u(\zeta)$ , and hence  $u(\zeta)$  is meromorphic on the plane.

From the differential equation for  $P(z)$  (see [6; 372]), it easily follows that  $u(\zeta)$  satisfies the first-order algebraic differential equation,

$$(\zeta^2 - 4)(u')^2 = 4u^3 - g_2u - g_3, \tag{43}$$

where  $g_2$  and  $g_3$  are certain constants. Let  $z_1 = \frac{1}{2}$ ,  $z_2 = \pi i$ ,  $z_3 = (1 + 2\pi i)/2$ , and  $e_j = P(z_j)$  for  $j = 1, 2, 3$ . It follows (see [6; pp. 366, 371]) that  $e_1, e_2$  and  $e_3$  are distinct and that for each  $j$ , the set of points where  $P(z) = e_j$  is

$$\{z_j + m + 2\pi in : m, n = 0, \pm 1, \pm 2, \dots\}. \tag{44}$$

From this it easily follows that the set of points where  $u(\zeta) = e_j$  is

$$\{\exp(z_j + m) + \exp(-(z_j + m)) : m = 0, \pm 1, \dots\}. \quad (45)$$

If  $j = 1, 2, 3$ , then under the change of variable  $v = (u - e_j)^{-1}$ , it follows (using [6; p. 373. equation (5.9)]) that equation (43) is transformed into an equation of the form,

$$(\zeta^2 - 4)(v')^2 = av^3 + bv^2 + cv + d, \quad (46)$$

where  $a, b, c, d$  are constants and  $a \neq 0$ . It easily follows that no pole of  $v$  can have multiplicity more than 2. Thus the multiplicity of each root of  $u(\zeta) = e_j$  is at most 2. It then follows from (45) that for  $j = 1, 2, 3$ , we have  $n(r, e_j, u) = O(\log r)$  as  $r \rightarrow +\infty$ , and hence by Nevanlinna's Second Fundamental Theorem, we have  $T(r, u) = O((\log r)^2)$  as  $r \rightarrow +\infty$ . Thus  $u(\zeta)$  is a transcendental meromorphic solution of equation (43), whose order of growth is zero. (From (45), it easily follows that  $T(r, u) \neq o((\log r)^2)$  as  $r \rightarrow +\infty$ .)

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