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On the Sum of a Zonotope and an Ellipsoid

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1. Introduction

It is well known that, for $d \geqslant 3$, a d-dimensional convex body, all of whose $(d-1)$-dimensional sections are centrally symmetric, is an ellipsoid. P. W. Aitchison [1] has recently shown that a 3-dimensional strictly convex body, having all 2-dimensional sections "sufficiently close to the boundary" centrally symmetric, is an ellipsoid. However, this result does not extend to convex bodies. In this paper, we show that for $d \geqslant 3$, a d-dimensional convex body K has all its $(d-1)$-dimensional sections "sufficiently close to the boundary" centrally symmetric if and only if K is the sum of a zonotope and an ellipsoid.

2. Definitions and Statement of the Theorem

A *convex body* in E^d is a compact convex set having interior points. If $C \subset E^d$ is a closed convex set, the *support function* h_C of C is defined by

$$\mathrm{h}_C(\mathbf{u}) = \sup\{\mathbf{u} \cdot \mathbf{x} : \mathbf{x} \in C\}$$

for non-zero $\mathbf{u} \in E^d$. If $h_C(\mathbf{u}) < \infty$ and α is real, we define

$$H_C(\alpha, \mathbf{u}) = \{\mathbf{x} \in E^d : \mathbf{u} \cdot \mathbf{x} = h_C(\mathbf{u}) - \alpha\},$$

and then $H_C(0, \mathbf{u}\}$ is a support hyperplane of C; $H_C(0, \mathbf{u}) \cap C$ is called the *face of C in direction* $\mathbf{u}$, is denoted $f_C(\mathbf{u})$ and may be empty. These definitions extend in an obvious way to convex surfaces. We denote by S^{d-1} the unit sphere $\{\mathbf{x} \in E^d : \|\mathbf{x}\| = 1\}$, and we shall use it as an index set for directions.

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A convex body $K \subset E^d$ $(d \geq 3)$ is said to have *property* (A) if (A) *for each* $\mathbf{u} \in S^{d-1}$ *there exists* $\varepsilon(\mathbf{u}) > 0$ *such that* $H_K(\alpha, \mathbf{u}) \cap K$ *is centrally symmetric for* $0 < \alpha < \varepsilon(\mathbf{u})$.

A *zonotope* is a convex polytope of which all faces, of all dimensions, are centrally symmetric.

THEOREM. *Let* $K \subset E^d$ $(d \geq 3)$ *be a convex body. Then K has property* (A) *if and only if K is the sum of a (not necessarily d-dimensional) zonotope and a (d-dimensional) ellipsoid.*

COROLLARY. *Let* $K \subset E^d$ $(d \geq 3)$ *be a convex body. If K is strictly convex and K has property* (A), *then K is an ellipsoid.*

3. The 3-dimensional Case

Throughout this section, K will be a fixed 3-dimensional convex body which has property (A). Using methods which extend those of Aitchison, we will show that K is the sum of a polytope and an ellipsoid.

A *quadric surface* in E^3 is (the surface of) a paraboloid, an ellipsoid or one branch of a hyperboloid of two sheets. A *quadric curve* is a parabola, an ellipse or one branch of a hyperbola. A *cylindrical surface* in E^3 is the Minkowski sum of a quadric curve and a line which is not parallel to the affine hull of that curve. A *conical surface* in E^3 is the surface of an elliptical cone. A *quadric* (respectively *cylindrical, conical*) *piece* is a non-empty open connected subset of a quadric (respectively cylindrical, conical) surface which is contained in $bd\,K$ and which is maximal in the sense of set inclusion; additionally, we require that a conical piece should contain its apex. Whenever P is a non-empty open subset of a quadric, cylindrical or conical surface, we denote by $\hat{P}$ the unique quadric, cylindrical or conical surface which contains P.

A *disc* in S^2 with centre $\mathbf{a} \in S^2$ is the intersection of S^2 with an open ball with center $\mathbf{a}$ and radius less than 1. When A is a subset of a convex surface C, we shall denote by ∂A the boundary of A in the topology of C.

When C is a closed convex set and $\mathbf{u}$ is a unit vector with $h_C(\mathbf{u}) < \infty$, for positive real numbers α we write

$$N_C(\alpha, \mathbf{u}) = \{\mathbf{x} \in bdC : \mathbf{x} \cdot \mathbf{u} > h_C(\mathbf{u}) - \alpha\}$$

which is an open cap cut off from $bd\,C$ by $H_C(\alpha, \mathbf{u})$. This definition extends to convex surfaces.

Consider $\mathbf{u} \in S^2$, and write

$\varepsilon(\mathbf{u}) = \sup\{\beta > 0 : H_K(\alpha, \mathbf{u}) \cap K$ *is centrally symmetric and non-empty for* $0 < \alpha < \beta\}$

Observe that $f_K(\mathbf{u})$ is centrally symmetric, and that for $0 \leq \alpha < \varepsilon(\mathbf{u})$ the centre $\mathbf{c}(\alpha, \mathbf{u})$ of $H_K(\alpha, \mathbf{u}) \cap K$ is a continuous function of α. If $\mathbf{x} \in N_K(\varepsilon(\mathbf{u}), \mathbf{u})$ then $\mathbf{x} \in H_K(h_K(\mathbf{u}) - \mathbf{u} \cdot \mathbf{x}, \mathbf{u}) \cap K$, which has centre $\mathbf{c}(h_K(\mathbf{u}) - \mathbf{u} \cdot \mathbf{x}, \mathbf{u})$, and we write

$$\Omega_u(\mathbf{x}) = 2\mathbf{c}(h_K(\mathbf{u}) - \mathbf{u} \cdot \mathbf{x}, \mathbf{u}) - \mathbf{x},$$

which is the reflection of $\mathbf{x}$ in that centre. Then Ω_u is a homeomorphism, called the $\mathbf{u}$-*opposite map*, of $N_K(\varepsilon(\mathbf{u}), \mathbf{u})$ onto itself, and Ω_u^2 is the identity map.

We shall make use of a result of S. P. Olovjanischnikoff [**5**]:

LEMMA 1. *Let $C \subset E^3$ be a convex body, A an open connected subset of bd C and D a non-empty open set in S^2 such that $f_C(\mathbf{u}) \subset A$ for each $\mathbf{u}$ in D. Suppose that for each $\mathbf{u} \in D$, every plane H which is parallel to $H_C(0, \mathbf{u})$ and with $H \cap A \neq \emptyset$ has the property that $H \cap C$ is centrally symmetric. Then A is a subset either of a quadric surface or of a conical surface whose apex lies in A.*

LEMMA 2. *There are no conical pieces in bd K.*

Proof. Suppose false, and let P be a conical piece, whose apex we assume to be $\mathbf{o} \in P$. Consider an edge of $\hat{P}$, which must intersect P in a half-closed line segment $[\mathbf{o}, \mathbf{a})$. There is a unique outer unit normal $\mathbf{u}$ to K at $\frac{1}{2}\mathbf{a}$, and $f_K(\mathbf{u})$ is an edge. Given $0 < \delta < 1$, let H^+ and H^- be the half-spaces of points $\mathbf{x}$ for which $\mathbf{x} \cdot \mathbf{a} - (1 - \delta)\,\|\mathbf{a}\|^2$ is non-negative and non-positive respectively. For all sufficiently small positive α, $H^- \cap H_K(\alpha, \mathbf{u}) \cap bd\,K$ is contained in P and is therefore an arc of a parabola; since the outer unit normals of a parabola lie in a semi-circle of S^1, the centre $\mathbf{k}(\alpha)$ of $H_K(\alpha, \mathbf{u}) \cap K$ does not lie in $(\text{int}\,H^-) \cap H_K(\alpha, \mathbf{u}) \cap K$, so $\mathbf{k}(\alpha) \in H^+$. For all small $\alpha > 0$, $H_K(\alpha, \mathbf{u}) \cap K$ contains a point $\mathbf{x}(\alpha)$ with $\mathbf{x}(\alpha) \cdot \mathbf{a} < \delta\,\|\mathbf{a}\|^2$, and then $(2\mathbf{k}(\alpha) - \mathbf{x}(\alpha)) \cdot \mathbf{a} \geq (2 - 3\delta)\,\|\mathbf{a}\|^2$. By letting $\alpha \to 0$ we see that $(2 - 3\delta)\mathbf{a} \in bd\,K$, and hence $2\mathbf{a} \in \text{bd}\,K$. Thus $2P \subset bd\,K$, contradicting the maximality of P. This proves the Lemma.

After observing that the set of centrally symmetric compact convex sets is closed in the Hausdorff metric, the reader may prove:

LEMMA 3. *Let $\tau > 0$. Then $\{\mathbf{u} \in S^2 : \varepsilon(\mathbf{u}) \geq \tau\}$ is closed.*

LEMMA 4. *The set of* $\mathbf{u} \in S^2$ *such that* $f_K(\mathbf{u})$ *is not contained in any quadric piece is nowhere dense in* S^2.

Proof. Let D be a disc in S^2. Then for each natural number n, by Lemma 3, $F_n = \{\mathbf{u} \in D : \varepsilon(\mathbf{u}) \geq 2/n\}$ is closed in D, and $D = \bigcup_{n=1}^{\infty} F_n$. Since D is a locally compact metric space, by the Baire Category Theorem we can choose n so that F_n contains a disc $D_1 \subset D$. Choose an element $\mathbf{u}_1 \in D_1$, and let $A = N_K(1/n, \mathbf{u}_1)$ which is open and connected in $bd\,K$. Using the continuity of h_K we can find a disc $D_2 \subset D_1$ with centre $\mathbf{u}_1$ such that $A \subset N_K(2/n, \mathbf{u})$ for all $\mathbf{u} \in D_2$. Then we can choose a disc $D_3 \subset D_2$ with centre $\mathbf{u}_1$ such that $f_K(\mathbf{u}) \subset A$ for all $\mathbf{u} \in D_3$. It now follows from Lemmas 1 and 2 that A is a subset of a quadric surface, which establishes the Lemma.

LEMMA 5. *Let* $\mathbf{u} \in S^2$, $\varepsilon(\mathbf{u}) > \tau > 0$ *and suppose* F *is a compact subset of* $N_K(\tau, \mathbf{u}) \backslash f_K(\mathbf{u})$. *Then for each* $\delta > 0$ *there is a neighbourhood* D *of* $\mathbf{u}$ *in* S^2 *so that for all* $\mathbf{v} \in D$ *satisfying* $\varepsilon(\mathbf{v}) > \tau$ *we have* $F \subset N_K(\tau, \mathbf{v}) \backslash f_K(\mathbf{v})$ *and the Hausdorff distance of* $\Omega_v(F)$ *from* $\Omega_u(F)$ *is less than* δ.

The proof, which is omitted, is a simple compactness and continuity argument, and uses the fact that the map $(\alpha, \mathbf{v}) \mapsto H_K(\alpha, \mathbf{v}) \cap K$ is continuous in the Hausdorff metric at $(\beta, \mathbf{w})$ if $H_K(\beta, \mathbf{w})$ intersects int K.

LEMMA 6. *Let* X_1 *and* X_2 *be quadric surfaces in* E^3 *and let* Λ *be a fixed plane such that for every plane* Λ' *parallel to* Λ, *the sections* $\Lambda' \cap X_1$ *and* $\Lambda' \cap X_2$ *are translates of the same ellipse, or are both empty, and for some planes* Π_1, Π_2 *not parallel to* Λ *or to one another, each of* Π_1, Π_2 *intersects* X_1 *and* X_2 *in non-empty sections which are translates of one another. Then* X_2 *is a translate of* X_1.

Proof. We may suppose that Λ is the x_2x_3 plane and that X_1 has equation

$$x_2^2 + x_3^2 = \varphi(x_1)$$

where φ is a quadratic form. Let l be the line of centres of the sections of X_2 which are parallel to Λ, and let m_1, m_2 be the lines through $\mathbf{o}$ parallel to $\Lambda \cap \Pi_1$, $\Lambda \cap \Pi_2$ respectively. For $i = 1, 2$ we may choose m_i to be the x_3 axis, so that Π_i has equation $x_2 = \xi_i x_1 + \eta_i$. For some constants w_2, w_3, a_2, a_3, the equation of X_2 is

$$(x_2 - w_2 x_1 + a_2)^2 + (x_3 - w_3 x_1 + a_3)^2 = \varphi(x_1)$$

and l has equations

$$x_2 = w_2 x_1 - a_2, \qquad x_3 = w_3 x_1 - a_3.$$

By comparing the equations of $\Pi_i \cap X_1$ and $\Pi_i \cap X_2$ we find $w_3 = 0$ and $w_2(w_2 - 2\xi_i) = 0$. If $m_1 = m_2$, then $w_2(w_2 - 2\xi_i) = 0$ for $i = 1, 2$ with $\xi_1 \neq \xi_2$, so $w_3 = w_2 = 0$. If $m_1 \neq m_2$, the condition $w_3 = 0$ holds in two different coordinate systems, showing that l is perpendicular to Λ. In each of these cases, we find that X_1 is a translate of X_2.

LEMMA 7. *Let X_1 be a paraboloid, X_2 a quadric surface in E^3, and let Λ_1, Λ_2, Λ_3 be pairwise non-parallel planes parallel to the axis of X_1, such that for $i = 1, 2, 3$ the section $\Lambda_i \cap X_2$ is a central reflection of $\Lambda_i \cap X_1$. Then X_2 is a central reflection of X_1.*

Proof. The sections of X_2 by Λ_1, Λ_2 and Λ_3 are parabolas, so X_2 is a paraboloid whose axis is parallel to that of X_1. We can assume that X_1 has equation $x_3 = x_2^2 + x_1^2$, that the sections of X_2 perpendicular to its axis have their principal axes parallel to the x_1 and x_2 axes and that the x_2x_3 plane is not parallel to any of Λ_1, Λ_2 and Λ_3 unless the x_1x_3 plane is also parallel to one of these planes. The remainder of the proof is left to the reader.

LEMMA 8. *Let X be a paraboloid and Y a cylindrical surface. Then there do not exist three pairwise non-parallel planes Π_1, Π_2, Π_3 all parallel to the axis of X, such that for $i = 1, 2, 3$, $\Pi_i \cap X$ is a central reflection of $\Pi_i \cap Y$.*

The proof is left to the reader.

LEMMA 9. *Let T be a set in S^2, $\mathbf{v}_1 \in T$ a limit point of T and let $\tau > 0$ be such that $\varepsilon(\mathbf{u}) > \tau$ for all $\mathbf{u} \in T$. Suppose that P is a quadric piece which intersects $N_K(\tau, \mathbf{v}_1)$, and let $Q = \Omega_{v_1}(N_K(\tau, \mathbf{v}_1) \cap P)$.*

If (i) *every member of T is an outer normal to $\hat{P}$ at some point, then Q is a subset of a quadric surface and $\hat{Q} = \mathbf{y} + \hat{P}$ for some $\mathbf{y} \in E^3$ having $\mathbf{y} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in T$ sufficiently close to $\mathbf{v}_1$.*

If (ii) *$\hat{P}$ is a paraboloid and every member $\mathbf{u}$ of T has* lin$\{\mathbf{u}\}$ *perpendicular to the axis of $\hat{P}$ then Q is a subset of a central reflection of $\hat{P}$.*

Proof. We remark that the significance of the positive lower bound for $\varepsilon(\mathbf{u})$ on T is that it ensures that every point of Q lies in infinitely many centrally symmetric sections of K. Let us suppose that either (i) or (ii) holds. We first consider a component Q^* of Q, so that $P^* = \Omega_{v_1}(Q^*)$ is a component of $N_K(\tau, \mathbf{v}_1) \cap P$. For any $\mathbf{u} \in T$, all sections of $\hat{P}$ perpendicular to lin$\{\mathbf{u}\}$ are directly homothetic quadric curves. We may suppose T to have sufficiently small diameter that there exists a positive β such that $H_K(\beta, \mathbf{u})$ intersects $\hat{P}$ in a proper section for each $\mathbf{u} \in T$. Then for each $\mathbf{u} \in T$ there is an open interval $\mathscr{C}_u$, which we take to be maximal, with

$$H_K(\alpha, \mathbf{u}) \cap \hat{P} = \psi(\alpha, \mathbf{u})(H_K(\beta, \mathbf{u}) \cap \hat{P}) + \mathbf{a}(\alpha, \mathbf{u})$$

for all $\alpha \in \mathscr{C}_u$, where $\mathbf{a}(\alpha, \mathbf{u})$ is a vector and $\psi(\alpha, \mathbf{u})$ is a positive scalar function having one of the forms $\lambda(\alpha+\kappa)^{1/2}$, $\lambda(\mu-(\alpha+\kappa)^2)^{1/2}$, $\lambda(\mu+(\alpha+\kappa)^2)^{1/2}$ in case (i), where λ, κ, μ depend on $\mathbf{u}$, or ψ is identically 1 in case (ii).

If $H_K(\alpha, \mathbf{u})$ intersects $P \cap (N_K(\tau, \mathbf{u}) \backslash f_K(\mathbf{u}))$, where $\mathbf{u} \in T$, then $\alpha \in \mathscr{C}_u$ and the $\mathbf{u}$-opposite set of $H_K(\alpha, \mathbf{u}) \cap P$ lies on a unique quadric curve $G(\alpha, \mathbf{u})$. Then $G(\alpha, \mathbf{u})$ is a central reflection of $H_K(\alpha, \mathbf{u}) \cap \hat{P}$ and satisfies

$$G(\alpha, \mathbf{u}) = -\psi(\alpha, \mathbf{u})(H_K(\beta, \mathbf{u}) \cap \hat{P}) + \mathbf{b}(\alpha, \mathbf{u})$$

for some vector $\mathbf{b}(\alpha, \mathbf{u})$.

We show that Q^* contains an exposed point of K. Suppose this is false. Since for $0 < \alpha < \tau$, every point of Q^* in $H_K(\alpha, \mathbf{v}_1)$ is an exposed point of $H_K(\alpha, \mathbf{v}_1) \cap K$, it follows that every point of Q^* belongs to an exposed edge of K which is not perpendicular to $\operatorname{lin}\{\mathbf{v}_1\}$. We can choose an open interval $\mathscr{L} \subset (0, \tau)$ and two distinct exposed edges I_1 and I_2 of K, such that for $\alpha \in \mathscr{L}$, $H_K(\alpha, \mathbf{v}_1) \cap Q^*$ intersects I_1 and I_2 at relatively interior points $\mathbf{d}_1(\alpha)$ and $\mathbf{d}_2(\alpha)$ respectively, and hence

$$\mathbf{d}_1(\alpha) - \mathbf{d}_2(\alpha) = \alpha \mathbf{g} + \mathbf{k} \tag{1}$$

where $\mathbf{g}$ and $\mathbf{k}$ are constant vectors. Let H_1, H_2 be support planes of K with $H_i \cap K = I_i$ for $i = 1, 2$. If $i = 1$ or 2, then the lines $H_K(\alpha, \mathbf{v}_1) \cap H_i$ are all parallel and support $G(\alpha, \mathbf{v}_1)$ in the same sense at the unique points $\mathbf{d}_i(\alpha)$, for $\alpha \in \mathscr{L}$. Then $\mathbf{d}_1(\alpha) - \mathbf{d}_2(\alpha) = \psi(\alpha, \mathbf{v}_1)\mathbf{g}'$ for all $\alpha \in \mathscr{L}$, where $\mathbf{g}'$ is a constant vector, not zero. This contradicts (1) in case (i). In case (ii), for all $\alpha \in \mathscr{L}$, $H_K(\alpha, \mathbf{v}_1) \cap Q^*$ lies in the cylindrical surface Y which has $G(\beta, \mathbf{v}_1)$ as a section and aff I_1 as a generator. By using Lemma 5, we may pick planes intersecting Y and $\hat{P}$ in a fashion which contradicts Lemma 8. Hence in both cases (i) and (ii) the assumption that Q^* contains no exposed points is false.

Let $\mathbf{w}'$ be an outer unit normal to K at an exposed point in Q^*. Then $f_K(\mathbf{w}) \subset Q^*$ for all $\mathbf{w} \in S^2$ sufficiently close to $\mathbf{w}'$. From Lemma 4 we deduce that Q^* intersects a quadric piece R say. We prove $Q^* \subset R$. Suppose this is false, and choose a boundary point $\mathbf{e}$ of $R \cap Q^*$ (in the topology of Q^*). We must have $\mathbf{e} \notin f_K(\mathbf{v}_1)$, for otherwise $\Omega_{v_1}(\mathbf{e}) \in f_K(\mathbf{v}_1) \cap P^*$ and so $f_K(\mathbf{v}_1)$ would be a point of P^*; this would imply $\mathbf{e} \in P^*$ and $R = P^*$ which is impossible as $\mathbf{e} \in \partial R$. Therefore we can choose an open connected set $B \subset bd\, K$ with $\mathbf{e} \in B$ and $\mathrm{cl}\, B \subset N_K(\tau, \mathbf{v}_1) \backslash f_K(\mathbf{v}_1)$. Using Lemma 5 we can choose $\mathbf{v}_2$ in $T \backslash \{\mathbf{v}_1, -\mathbf{v}_1\}$ such that $\Omega_{v_2}(B) \subset P^*$ and $B \subset N_K(\tau, \mathbf{v}_2) \backslash f_K(\mathbf{v}_2)$. Let R_1 be a component of $R \cap B$. For $i = 1$, 2, the set of α for which $H_K(\alpha, \mathbf{v}_i)$ intersects R_1 is an open interval $\mathcal{M}_i$, and for $\alpha \in \mathcal{M}_i$, $H_K(\alpha, \mathbf{v}_i) \cap Q \subset G(\alpha, \mathbf{v}_i) \subset \hat{R}$ by analytic continuation. Thus for $i = 1, 2$ the boundary Δ of R_1 in the topology of B lies in the union of 2 planes. Hence Δ lies in the union of 4 lines, and so by the strict convexity of $\hat{R}$, Δ is finite. If $R_1 \neq B$, we can choose $\mathbf{x} \in B \backslash \mathrm{cl}\, R_1$ (by the maximality of R), $\mathbf{z} \in R_1$ and infinitely many disjoint paths from $\mathbf{z}$ to $\mathbf{x}$ in B, so that Δ is infinite. We conclude that $R_1 = B$, but this contradicts the choice of $\mathbf{e} \in \partial R$, so $Q^* \subset R$.

Now suppose that Q is not connected, and let Q_1, Q_2 be two of its components. Then P contains a point outside $N_K(\tau, \mathbf{v}_1)$, and so for some positive $\sigma < \tau$, $H_K(\alpha, \mathbf{v}_1)$ intersects Q_1 and Q_2 for $\sigma < \alpha < \tau$. For $\sigma < \alpha < \tau$, $H_K(\alpha, \mathbf{v}_1) \cap Q \subset G(\alpha, \mathbf{v}_1)$ so $\hat{Q}_1$ and $\hat{Q}_2$ have an open subset in common and are therefore equal. Hence Q is contained in a quadric piece.

From the fact that $\varepsilon(\mathbf{u}) \geq \tau > 0$ for $\mathbf{u} \in T$ we deduce that for all $\mathbf{u} \in T$ close to $\mathbf{v}_1$, there is an interval $\mathcal{N}_u$ such that for $\alpha \in \mathcal{N}_u$, $H_K(\alpha, \mathbf{u})$ intersects $\hat{P}$ and $\hat{Q}$ in non-empty sections which are central reflections of one another. In case (ii), applying analytic continuation and Lemma 7, we conclude that $\hat{Q}$ is a central reflection of $\hat{P}$. Let us suppose that case (i) holds. Then for $\mathbf{u} \in T$ close to $\mathbf{v}_1$, for $\alpha \in \mathcal{N}_u$, $H_K(\alpha, \mathbf{u}) \cap \hat{P}$ and $H_K(\alpha, \mathbf{u}) \cap \hat{Q}$ are ellipses and are translates of one another. Applying analytic continuation and Lemma 6, we find $\hat{Q} = \mathbf{y} + \hat{P}$ for some vector $\mathbf{y}$. For $\mathbf{u} \in T$ close to $\mathbf{v}_1$, $H_K(0, \mathbf{u})$ supports $\hat{P}$ and therefore supports $\hat{Q}$ (by comparing sizes of sections), so $\mathbf{y} \cdot \mathbf{u} = 0$. This completes the proof of Lemma 9.

Now that we have given some preliminary results, we pause to summarize our methods. Lemma 10 will show that much of the boundary of a quadric piece adjoins cylindrical pieces. From Lemma 11 it follows that shadow boundaries of K which contain cylindrical pieces cannot cross quadric pieces; Lemma 15 shows that there are only finitely many such shadow boundaries, using Lemmas 12 and 13 (the necessity for Lemma 14 arises from the exceptional behaviour of parabolic cylinders in Lemma 13). It then follows that there are only finitely many quadric pieces, and these are all parts of translates of the same ellipsoid by Lemma 10. Lemma 16 then follows easily.

LEMMA 10. *Let P be a quadric piece in bd K, and let R be the set of unit outer normals to K at points of P. Suppose that an open set G intersects ∂R. Then there exists $\mathbf{v} \in G \cap \partial R$, $\tau > 0$, $\mathbf{y} \in E^3$, a cylindrical surface $\mathscr{C}$ and distinct closed half-spaces H^+, H^- bounded by a plane H, satisfying:*

(i) $\mathbf{y} \neq \mathbf{o}$ *and* $\mathbf{y} \cdot \mathbf{v} = 0$,
(ii) $H \cap \hat{P}$ *is the shadow boundary of* $\hat{P}$ *in direction* $\mathbf{y}$,
(iii) $\mathscr{C} = (H \cap \hat{P}) + \mathrm{lin}\,\{\mathbf{y}\}$,
(iv) $N_K(\tau, \mathbf{v}) \subset (H^- \cap \hat{P}) \cup (\mathbf{y} + (H^+ \cap \hat{P})) \cup (H^+ \cap (\mathbf{y} + H^-) \cap \mathscr{C})$.

Proof. The set R is open and connected in S^2, and for each $\mathbf{u}\in R$, $H_K(0,\mathbf{u})$ supports $\hat{P}$. Also R is maximal (under set inclusion) with these properties. This maximality ensures that R is not dense in any neighbourhood of a point of ∂R. Hence every point of ∂R is a limit point of ∂R.

Applying the Baire Category Theorem to $G\cap\partial R$, we can choose a disc $D_1\subset G$ with centre $\mathbf{v}\in\partial R$ and $\sigma>0$ such that $\varepsilon(\mathbf{u})>\sigma$ for all $\mathbf{u}\in D_1\cap\partial R$. For $\mathbf{u}\in D_1\cap\partial R$ let $Q_u=\Omega_u(N_K(\sigma,\mathbf{u})\cap P)$, and by Lemma 5 choose a disc $D_2\subset D_1$ with centre $\mathbf{v}$ such that $Q_u\cap Q_v\neq\emptyset$ for all $\mathbf{u}\in D_2\cap\partial R$. Then by Lemma 9 (i), for each $\mathbf{u}\in D_2\cap\partial R$, Q_u must be a subset of a quadric surface with $\hat{Q}_u=\hat{Q}_v=\mathbf{y}+\hat{P}$ for some $\mathbf{y}\in E^3$, and there is a disc $D_3\subset D_2$ with centre $\mathbf{v}$ such that $\mathbf{y}\cdot\mathbf{u}=0$ for all $\mathbf{u}\in D_3\cap\partial R$.

We prove $\mathbf{y}\neq\mathbf{o}$. Suppose this is false, so that $Q_u\subset\hat{P}$ for all $\mathbf{u}\in D_3\cap\partial R$. Let $0<\omega<\sigma$ and choose a disc $D_4\subset D_3$ with centre $\mathbf{v}$ such that $f_K(\mathbf{u})\subset N_K(\omega,\mathbf{v})\subset N_K(\sigma,\mathbf{u})$ for all $\mathbf{u}\in D_4$; then choose $\mathbf{x}^1\in D_4\cap R$. Let $\mathbf{x}^2$, $\mathbf{x}^3$ be unit vectors such that $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3$ are mutually orthogonal, and for $0\leq\theta\leq 2\pi$, $0\leq\phi\leq\pi$ let $\mathbf{z}(\theta,\phi)=\cos\phi\mathbf{x}^1+\sin\phi(\cos\theta\mathbf{x}^2+\sin\theta\mathbf{x}^3)\in S^2$, and $\mathbf{l}(\theta)=\sin\theta\mathbf{x}^2-\cos\theta\mathbf{x}^3\in S^2$. Then $f_K(\mathbf{z}(\theta,\phi))$ is contained in the shadow boundary $\mathcal{S}_\theta$ of K in direction $\mathbf{l}(\theta)$. Since $D_4\not\subset cl\,R$, there is a proper interval $[\delta,\eta]$ such that for each $\theta\in[\delta,\eta]$, there is a $\phi\in[0,\pi]$ with $\mathbf{z}(\theta,\phi)\in D_4\cap\partial R$; for $\theta\in[\delta,\eta]$ let $\nu(\theta)=\inf\{\phi\geq 0:\mathbf{z}(\theta,\phi)\in\partial R\}>0$. For each real α the set $\{\theta\in[\delta,\eta]:\nu(\theta)\leq\alpha\}$ is closed since ∂R is closed. If $0\leq\phi<\nu(\theta)$ then $f_K(\mathbf{z}(\theta,\phi))$ is an exposed point of K contained in $\mathcal{S}_\theta\cap P$; since we have $f_K(\mathbf{z}(\theta,\phi))\subset N_K(\sigma,\mathbf{z}(\theta,\nu(\theta)))$, the $\mathbf{z}(\theta,\nu(\theta))$-opposite point of $f_K(\mathbf{z}(\theta,\phi))$ lies in $\mathcal{S}_\theta\cap Q_{z(\theta,\nu(\theta))}\subset\mathcal{S}_\theta\cap\hat{P}$. Hence $H_K(0,\mathbf{z}(\theta,\phi))$ supports $\hat{P}$ for $0\leq\phi\leq g(\theta)$, where $g(\theta)$ is a number greater than $\nu(\theta)$. We suppose $g(\theta)$ is maximal with these properties; then, since $\mathbf{x}\notin\partial R$, there exists $\alpha>0$ such that $g(\theta)>\nu(\theta)+\alpha$ for all $\theta\in[\delta,\eta]$. Choose γ, δ', η' to satisfy $\delta<\delta'<\gamma<\eta'<\eta$ and $\nu(\theta)>\nu(\gamma)-\alpha/2$ for $\delta'<\theta<\eta'$. For $\delta'<\theta<\eta'$ and $|\phi-\nu(\gamma)|<\alpha/2$ we have $\phi\leq\alpha/2+\nu(\gamma)\leq\alpha+\nu(\theta)$, which shows that some neighbourhood of $\mathbf{z}(\gamma,\nu(\gamma))$ is contained in R, contradicting the definition of $\mathbf{z}(\gamma,\nu(\gamma))\in\partial R$. We conclude that $\mathbf{y}\neq\mathbf{o}$.

Since $\mathbf{y}\cdot\mathbf{u}=0$ for $\mathbf{u}\in D_3\cap\partial R$, it follows that $D_3\cap\partial R$ is contained in a great circle C of S^2 which divides D_3 into two regions D_3^+ and D_3^-. Suppose R intersects D_3^-. Then $D_3^-\subset R$ since $D_3\cap\partial R\subset C$, and so $D_3^+\cap\partial R=\emptyset$ as $D_3\not\subset R$. Thus $D_3\cap\partial R=D_3\cap C$. Let H be the plane such that $H\cap\hat{P}$ is the shadow boundary of $\hat{P}$ in direction $\mathbf{y}$, and let H^+, H^- be the closed half-spaces bounded by H. Then the outer unit normals to $\hat{P}$ at points of $H\cap\hat{P}$ form an arc (or possibly the whole) of C, and the points of $\hat{P}$ with outer unit normals in D_3^- lie in H^-. Since $\mathbf{v}\in D_3\cap C$ we may choose $\tau>0$ such that $\tau<\sigma$ and $H^-\cap N_K(\tau,\mathbf{v})=H^-\cap N_{\hat{P}}(\tau,\mathbf{v})$. Then

$$(\mathbf{y}+H^+)\cap N_K(\tau,\mathbf{v})=\Omega_v(H^-\cap N_K(\tau,\mathbf{v}))=\mathbf{y}+(H^+\cap N_{\hat{P}}(\tau,\mathbf{v})).$$

For any outer unit normal $\mathbf{u}$ to K at a point $\mathbf{x}$ of $(\partial P) \cap N_K(\tau, \mathbf{v})$ we have $\mathbf{u} \in C$ and $f_K(\mathbf{u})$ contains the line segment $[\mathbf{x}, \mathbf{x}+\mathbf{y}]$. Hence

$$H^+ \cap (\mathbf{y} + H^-) \cap N_K(\tau, \mathbf{v}) \subset \mathscr{C} = \operatorname{lin}\{\mathbf{y}\} + (H \cap \hat{P}).$$

This completes the proof of the Lemma.

For a unit vector $\mathbf{u}$ we write $\mathscr{S}_u$ for the shadow boundary of K in direction $\mathbf{u}$. A *band* is a shadow boundary which contains a cylindrical piece.

LEMMA 11. *Let $\mathscr{S}_u$ be a band. Then for some $\beta > 0$, for every point $\mathbf{x} \in \mathscr{S}_u$ the line $\mathbf{x} + \operatorname{lin}\{\mathbf{u}\}$ intersects K in a segment of length at least β.*

Proof. Choose orthogonal unit vectors $\mathbf{x}^1$, $\mathbf{x}^2$ which are orthogonal to $\mathbf{u}$, and write $\mathbf{x}(\theta) = \cos\theta \mathbf{x}^1 + \sin\theta \mathbf{x}^2$ for real θ. For $\beta > 0$, we say a convex set C is *β-wide* if for each $\mathbf{y} \in C$, $C \cap (\mathbf{y} + \operatorname{lin}\{\mathbf{u}\})$ has length at least β. By hypothesis there is a $\beta > 0$, a cylindrical piece $Q \subset \mathscr{S}_u$ and a closed interval, which we suppose to be $[0, \alpha]$, such that for $0 \leq \theta \leq \alpha$, $f_K(\mathbf{x}(\theta))$ is β-wide and is contained in $\hat{Q}$. Let

$$\alpha' = \sup\{\gamma \in (0, 2\pi) : f_K(\mathbf{x}(\theta)) \textit{ is } \beta\textit{-wide for } 0 \leq \theta \leq \gamma\} > 0, \quad \gamma = \alpha'.$$

Then $f_K(\mathbf{x}(\alpha'))$ is either a line segment of length at least β, or a centrally symmetric facet having an edge parallel to $\operatorname{lin}\{\mathbf{u}\}$ of length at least β; in either case, $f_K(\mathbf{x}(\alpha'))$ is β-wide. When $\theta > \gamma$ is sufficiently small, $f_K(\mathbf{x}(\theta))$ is contained in $N_K(\varepsilon(\mathbf{x}(\gamma)), \mathbf{x}(\gamma))$ and then $\Omega_{x(\gamma)} f_K(\mathbf{x}(\theta))$ is the union of line segments parallel to $\operatorname{lin}\{\mathbf{u}\}$ of length at least β, so $f_K(\mathbf{x}(\theta))$ is β-wide. We conclude that $\alpha' = 2\pi$ and the Lemma follows.

LEMMA 12. *Let $T \subset S^2$ be a set with a limit point $\mathbf{v} \in T$, and suppose $\tau > 0$ is such that $\varepsilon(\mathbf{u}) > \tau$ for all $\mathbf{u} \in T$. If C is a cylindrical piece whose generators are not parallel to $H_K(0, \mathbf{v})$, then $Q = \Omega_v(C \cap N_K(\tau, \mathbf{v}))$ is a subset of a cylindrical surface.*

Proof. We first show that Q contains no extreme points of K. Suppose this is false. Then Q must contain an exposed point of K, so $f_K(\mathbf{u}) \subset Q$ for all $\mathbf{u}$ in some disc on S^2, and then by Lemma 4 Q intersects a quadric piece P. By using Lemma 5 and analytic continuation, there exists a disc D in S^2 with centre $\mathbf{v}$ such that for each $\mathbf{u} \in D \cap T$, there is an interval $\mathscr{N}_u$ such that $H_K(\alpha, \mathbf{u}) \cap \hat{P}$ is non-empty and is a central reflection of $H_K(\alpha, \mathbf{u}) \cap \hat{C}$ for all $\alpha \in \mathscr{N}_u$. Since parallel sections of a cylindrical surface are translates of one another, we conclude that $\hat{P}$ is a paraboloid whose axis is parallel to $H_K(0, \mathbf{u})$ for all $\mathbf{u} \in D \cap T$. Applying analytic continuation again, we have a contradiction to Lemma 8. Hence Q contains no extreme points.

Consider any point $\mathbf{x} \in Q$. Then $\mathbf{x}$ is relatively interior to a line segment $I \subset bd\, K$, and $\mathbf{x} \in H_K(h_K(\mathbf{v}) - \mathbf{v} \cdot \mathbf{x}, \mathbf{v}) \cap Q \subset \mathscr{C}$, where $\mathscr{C}$ is some quadric curve. Then a neighbourhood of $\mathbf{x}$ in Q lies in the cylindrical surface with generators parallel to I which has $\mathscr{C}$ as a section. We conclude that Q is contained in a cylindrical surface if Q is connected; when Q is not connected we compare components as in Lemma 9.

LEMMA 13. *Let Y_1, Y_2 be two cylindrical surfaces in E^3 and let Π_1, Π_2, Π_3 be three planes parallel to a line l, such that no two of Π_1, Π_2, Π_3 are parallel, and l is not parallel to the generators of Y_1. Suppose that for $i = 1, 2, 3$, Π_i is not a tangent to Y_1, and $\Pi_i \cap Y_1$ is a non-empty central reflection of $\Pi_i \cap Y_2$. Then the generators of Y_1 and Y_2 are parallel, except possibly when Y_1 is parabolic and l is parallel to its plane of axes, in which case l and the generators of Y_1 and Y_2 are all parallel to a single plane.*

Proof. We may suppose that $\Pi_1 \cap \Pi_2 \cap \Pi_3 = l$, and we may assume l to pass through an arbitrary point. Thus we take l to be the x_3 axis and Π_1 to be the x_2x_3 plane. We suppose Π_2, Π_3 to meet Y_1 in proper quadric curves and to have equations $x_2 = \xi_2 x_1$, $x_2 = \xi_3 x_1$ respectively with $\xi_2 \neq \xi_3$. For $i = 2, 3$ the orthogonal projections on the x_1x_3 plane of $\Pi_i \cap Y_1$ and $\Pi_i \cap y_2$ are reflections of one another in a point $(b_1^i, 0, b_3^i)$.

First suppose l is neither parallel to an asymptote plane if Y_1 is hyperbolic nor parallel to the plane of axes if Y_1 is parabolic. We take Y_1 to have equation

$$x_3^2 = \alpha + \beta x_2 + \gamma x_2^2$$

where either $\alpha = \gamma = 0$ or $\alpha\gamma \neq 0$ and $\beta = 0$. Then Y_2 has equation

$$(c_3 + w_3 x_1 - x_3)^2 = \alpha + \beta(c_2 + w_2 x_1 - x_2) + \gamma(c_2 + w_2 x_1 - x_2)^2$$

say. Hence $w_3 = 0$. If $\gamma \neq 0$ then $\gamma(w_2 - \xi_i)^2 = \gamma \xi_i^2$ for $i = 2, 3$, while if $\gamma = 0$ and $\beta \neq 0$ then $-\beta\xi_i = \beta(w_2 - \xi_i)$ for $i = 2, 3$; in either case $w_2 = w_3 = 0$.

Next suppose Y_1 is hyperbolic and l is parallel to an asymptote plane. Then Y_1 has equation $x_2 x_3 = 1$ say, and Y_2 has equation

$$(c_2 + w_2 x_1 - x_2)(c_3 + w_3 x_1 - x_3) = 1$$

say, and for some $\lambda \neq 0$ we find

$$w_3(w_2 - 1) = 0, \qquad (w_3 - 1) = -\lambda, \qquad c_2 = 2\lambda b_1^2,$$
$$c_3(w_2 - 1) = -2\lambda b_3^2, \qquad c_3 c_2 - 1 = \lambda(4 b_1^2 b_3^2 - 1)$$

which shows that $w_2 = w_3 = 0$.

Lastly we take Y_1 to be parabolic cylinder $x_3 = x_2^2$, so Y_2 has equation

$$c_3 + w_3x_1 - x_3 = (c_2 + w_2x_1 - x_2)^2.$$

We find $(w_2 - \xi_i)^2 = \xi_i^2$ for $i = 2, 3$, so $w_3 = 0$. This completes the proof.

LEMMA 14. *No quadric piece in bd K is a subset of a paraboloid.*

Proof. Suppose P is a quadric piece such that $\hat{P}$ is a paraboloid, and let $\mathcal{F}$ be the set of quadric pieces Q such that $\hat{Q}$ is a translate of $\hat{P}$. Write $T = \operatorname{int}\operatorname{cl}\{\mathbf{u} \in S^2 : f_K(\mathbf{u}) \subset Q \textit{ for some } Q \in \mathcal{F}\}$. Then T is contained in a hemisphere of S^2, so $\partial T \neq \varnothing$; it easily follows that every point of ∂T is a limit point of ∂T. Applying the Baire Category Theorem to ∂T, we can find a disc $D \subset S^2$ whose centre $\mathbf{v}$ lies in ∂T, and $\tau > 0$, such that $\varepsilon(\mathbf{u}) > \tau$ for all $\mathbf{u} \in D \cap \partial T$. Let $\mathbf{a}$ be a unit vector parallel to the axis of $\hat{P}$.

We first consider the case when $\mathbf{u} \cdot \mathbf{a} = 0$ for all $\mathbf{u} \in D \cap \partial T$. Then $D \cap \partial T = D \cap C$ where C is some great circle of S^2, and $D \cap T = D^+$, where D^+ and D^- are the components of $D \backslash C$. Choose σ with $0 < \sigma < \tau$ and a disc $D_1 \subset D$ with centre $\mathbf{v}$ such that $f_K(\mathbf{u}) \subset N_K(\sigma, \mathbf{v}) \subset N_K(\tau, \mathbf{u})$ for all $\mathbf{u} \in D_1 \cap C$. Since D_1 intersects D^+ we may choose $Q \in \mathcal{F}$ which intersects $N_K(\sigma, \mathbf{u})$. We have $\partial Q \neq \varnothing$.

We construct a closed convex cylinder $\mathscr{C}$ with generators parallel to $\operatorname{lin}\{\mathbf{a}\}$ such that $Q = \hat{Q} \cap \operatorname{int} \mathscr{C}$ and $\partial Q = Q \cap bd\, \mathscr{C}$. Consider a plane Λ such that $\Lambda \cap \hat{Q}$ is the shadow boundary of $\hat{Q}$ in the direction of a unit vector $\mathbf{w}$, and suppose $\Lambda \cap \hat{Q}$ contains an arc Γ which is common to ∂Q and to the boundary of some cylindrical piece whose generators are parallel to $\operatorname{lin}\{\mathbf{w}\}$. (Here we envisage the situation which arises in Lemma 10). Then $\mathscr{S}_w$ is a band, and so from Lemma 11 $\mathscr{S}_w$ does not intersect Q. Consequently $\Lambda \cap Q = \varnothing$, and Q lies in a closed half-space Λ^+ bounded by Λ. By Lemma 10, such arcs Γ (for various Λ) are dense in ∂Q. We take $\mathscr{C}$ to be the intersection of all such half-spaces Λ^+, and $\mathscr{C}$ has the required properties. Notice that $\mathscr{C}$ has a bounded cross-section, and its facets are dense in $bd\, \mathscr{C}$.

We may choose a disc $D_2 \subset D_1$ with centre $\mathbf{v}$ so that $H_{\mathscr{C}}(0, \mathbf{u}) \cap \partial Q \subset N_K(\sigma, \mathbf{v})$ for all $\mathbf{u} \in D_2 \cap C$. We can then choose $\mathbf{u} \in D_2 \cap C$, non-parallel planes Λ_1 and Λ_2 which intersect $\mathscr{C}$ in facets, and numbers β, γ with $0 < \beta < \gamma < \tau$ such that $H_K(\alpha, \mathbf{u})$ intersects $\Lambda_1 \cap \mathscr{C}$ and $\Lambda_2 \cap \mathscr{C}$ for $\beta < \alpha < \gamma$. For $i = 1, 2$ there is a unit vector $\mathbf{w}_i$ such that $\Lambda_i \cap \hat{Q}$ is the shadow boundary of $\hat{Q}$ in direction $\mathbf{w}_i$. By Lemma 10 we can find numbers β_1, γ_1 with $\beta < \beta_1 < \gamma_1 < \gamma$ and cylindrical pieces Z_1, Z_2 having generators parallel to $\operatorname{lin}\{\mathbf{w}_1\}$ and $\operatorname{lin}\{\mathbf{w}_2\}$ respectively, such that $H_K(\alpha, \mathbf{u})$ intersects $\Gamma_i = (\partial Z_i) \cap \Lambda_i \cap \partial Q$ for $i = 1, 2$ when $\beta_1 < \alpha < \gamma$. By slight alterations of $\mathbf{u}$, β_1, γ_1 we may suppose that $\mathbf{u} \cdot \mathbf{w}_1$ and $\mathbf{u} \cdot \mathbf{w}_2$ are both non-zero.

For sets $A \subset bd\, K$ write $A' = \Omega_u(A \cap N_K(\tau, \mathbf{u}))$. By Lemma 9, Q' is a subset of a central reflection of $\hat{Q}$. By Lemmas 12 and 13, for $i = 1, 2$ Z'_i is a subset of a cylindrical surface whose generators are parallel to a unit vector $\mathbf{g}_i \in \operatorname{lin}\{\mathbf{a}, \mathbf{w}_i\}$. Thus the shadow boundary of $\hat{Q}'$ in direction $\mathbf{g}_i$ lies in a plane Λ'_i parallel to Λ_i. Applying Lemma 10, we find that $\Gamma'_i \subset \Lambda'_i$. For $\beta_1 < \alpha < \gamma_1$, $i = 1, 2$ let $\mathbf{s}_i(\alpha) = \Gamma_i \cap H_K(\alpha, \mathbf{u})$, $\mathbf{s}'_i(\alpha) = \Gamma'_i \cap H_K(\alpha, \mathbf{u})$ so that $\mathbf{s}_i(\alpha) + \mathbf{s}'_i(\alpha) = 2\mathbf{c}(\alpha)$, where $\mathbf{c}(\alpha)$ is the centre of $H_K(\alpha, \mathbf{u}) \cap K$. Let π be the orthogonal projection on $(\operatorname{lin}\{\mathbf{a}\})^{\perp}$. Since $\mathbf{s}_i(\alpha)$, $\mathbf{s}'_i(\alpha)$ move in planes parallel to $\operatorname{lin}\{\mathbf{a}\}$, there are constant vectors $\mathbf{k}_i$, $\mathbf{t}_i$, $\mathbf{t}'_i$ with $\pi\mathbf{s}_i(\alpha) = \alpha\mathbf{k}_i + \mathbf{t}_i$, $\pi\mathbf{s}'_i(\alpha) = \alpha\mathbf{k}_i + \mathbf{t}'_i$. Thus $2\pi\mathbf{c}(\alpha) = 2\alpha\mathbf{k}_i + \mathbf{t}_i + \mathbf{t}'_i$ for $i = 1, 2$, $\beta_1 < \alpha < \gamma_1$. Hence $\mathbf{k}_1 = \mathbf{k}_2$, which implies that Λ_1 is parallel to Λ_2, contrary to the way they were chosen.

We may therefore suppose that $\hat{P}$ has an outer normal in some direction $\mathbf{v} \in D \cap \partial T$. Then there is a disc $D_3 \subset D$ with centre $\mathbf{v}$ such that every point of D_3 is an outer normal to $\hat{P}$. Let $0 < \sigma < \tau$ and choose a disc $D_4 \subset D_3$ with centre $\mathbf{v}$ such that $f_K(\mathbf{u}) \subset N_K(\sigma, \mathbf{v}) \subset N_K(\tau, \mathbf{u})$ for all $\mathbf{u} \in D_4$. Then we can choose $Q_1 \in \mathcal{F}$ such that the open connected set U_1 of outer unit normals to K at points of Q_1 satisfies $U_1 \cap D_4 \cap T \neq \emptyset$. By Lemma 10 we may choose $\mathbf{w} \in D_4 \cap \partial U_1$, $\nu > 0$ and a member $Q_2 \in \mathcal{F}$ with $\hat{Q}_2 = \mathbf{y} + \hat{Q}_1$ $(\mathbf{y} \neq \mathbf{o})$ such that $N_K(\nu, \mathbf{w})$ is contained in the union of Q_1, Q_2 a cylindrical piece Z with generators parallel to $\operatorname{lin}\{\mathbf{y}\}$, and two arcs Γ_1, Γ_2 of the shadow boundaries of $\hat{Q}_1$, $\hat{Q}_2$ respectively in direction $\mathbf{y}$. Notice that $U_2 \cap D_4 \cap T \neq \emptyset$ where U_2 is the set of outer unit normals to K at points of Q_2. Since $f_K(w) \subset N_K(\sigma, \mathbf{v})$ we may also suppose that $N_K(\nu, \mathbf{w}) \subset N_K(\sigma, \mathbf{v})$. For sets $A \subset bd\, K$, write $A' = \Omega_v(N_K(\nu, \mathbf{w}) \cap A)$. From Lemma 9, Q'_1 and Q'_2 are contained in translates of $\hat{Q}_1$ and $\hat{Q}_2$ respectively, so $\hat{Q}'_2 = \mathbf{y}' + \hat{Q}'_1$ for some vector $\mathbf{y}'$.

We wish to choose $i \in \{1, 2\}$ such that $\hat{Q}'_i \neq \hat{Q}_i$. Suppose this is impossible, so $\mathbf{y} = \mathbf{y}'$. First consider the possibility that $\mathbf{v} \cdot \mathbf{y} = 0$. Then we may choose $\alpha \in (0, \tau)$ such that $H = H_K(\alpha, \mathbf{v})$ intersects Q'_1 and Q'_2. Let $\mathbf{c}$ be the centre of $H \cap K$. Then $H \cap \hat{Q}'_i = 2\mathbf{c} - (H \cap \hat{Q}_i)$ for $i = 1, 2$, $H \cap \hat{Q}_2 = \mathbf{y} + (H \cap \hat{Q}_1)$ and $H \cap \hat{Q}'_2 = \mathbf{y} + (H \cap \hat{Q}'_1)$, which is impossible as $\mathbf{y} \neq \mathbf{o}$.

It remains to consider the possibility that $\mathbf{v} \cdot \mathbf{y} \neq 0$, in which case $\mathbf{u} \cdot \mathbf{y} \neq 0$ for all $\mathbf{u} \in S^2$ close to $\mathbf{v}$. Lemma 12 then shows that Z' is contained in a cylindrical surface, and by Lemma 10 the generators of $\hat{Z}'$ are parallel to $\operatorname{lin}\{\mathbf{y}'\} = \operatorname{lin}\{\mathbf{y}\}$. Since $H_K(0, \mathbf{v})$ is not parallel to $\operatorname{aff} \Gamma_1$, there is an interval $\mathcal{L} \subset (0, \tau)$ such that $H_K(\alpha, \mathbf{v})$ intersects Q'_1 and Z' for all $\alpha \in \mathcal{L}$. Let $\mathbf{c}(\alpha)$ be the centre of $H_K(\alpha, \mathbf{v}) \cap K$; then since $H_K(\alpha, \mathbf{v})$ intersects Q_1 and Q'_1 in $\mathbf{v}$-opposite arcs of $H_K(\alpha, \mathbf{v}) \cap \hat{Q}_1$, $\mathbf{c}(\alpha)$ is the centre of $H_K(\alpha, \mathbf{v}) \cap \hat{Q}_1$ for $\alpha \in \mathcal{L}$. Hence $\{\mathbf{c}(\alpha) : \alpha \in \mathcal{L}\}$ lies in a line parallel to $\operatorname{lin}\{\mathbf{a}\}$. Choose a generator l of $\hat{Z}$. Then $(l \cap Z)'$ is contained in a generator l' of $\hat{Z}'$, so $\{\mathbf{c}(\alpha) : \alpha \in \mathcal{L}\}$ lies on a line parallel to l. This shows that $\mathbf{y}$ is a scalar multiple of $\mathbf{a}$, which is impossible since a paraboloid has no support lines parallel to its axis. Hence we may make the requisite choice of i.

We therefore suppose that $\hat{Q}_1' = \mathbf{z} + \hat{Q}_1$ with $\mathbf{z} \neq \mathbf{o}$. Then by Lemma 9 there is a disc $D_5 \subset D_4$ with centre $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{z} = 0$ for all $\mathbf{u} \in D_5 \cap \partial T$. Then there is a great circle $C \subset S^2$ which divides D_5 into two open components D_5^+, D_5^- with $D_5 \cap \partial T = D_5 \cap C$ and $D_5 \cap T = D_5^+$.

We may choose $\omega \in (0, \tau)$ such that the set V of outer unit normals to $\hat{P}$ at points of $N_{\hat{P}}(\omega, \mathbf{v})$ is contained in D_5. Then V is open and contains $\mathbf{v}$, so we may choose a non-empty open connected subset $R \subset bd\, K$ of a translate of $\hat{P}$, such that the outer unit normals to K at points of R lie in $D_5^+ \cap V$. It follows that $R \subset N_{\hat{R}}(\omega, \mathbf{v})$ and that $\Omega_v(R)$ is a translate of a subset of $\hat{R}$ by a vector orthogonal to $\mathbf{v}$ (from Lemma 9), so $\Omega_v(R)$ is a translate of a subset of $N_{\hat{R}}(\omega, \mathbf{v})$. Therefore the outer unit normals to K at the points of $\Omega_v(R)$ are contained in $D_5^- \cap V$ which is impossible as $D_5 \cap T = D_5^-$. This contradiction proves the Lemma.

LEMMA 15. *There are only finitely many $\mathbf{y} \in S^2$ for which $\mathcal{S}_y$ is a band.*

Proof. Let $\mathbf{a} \in S^2$ and let C be the great circle $\{\mathbf{u} \in S^2 : \mathbf{u} \cdot \mathbf{a} = 0\}$. Then $\mathcal{S}_a = \bigcup f_K(C)$. By applying the Baire Category Theorem to C, there is a closed arc $C_1 \subset C$ and $\tau > 0$ such that $\varepsilon(\mathbf{u}) > \tau$ for all $\mathbf{u} \in C_1$. We claim there is a $\delta > 0$ and $\mathbf{v} \in C_1$ such that no two points of $N_K(\delta, \mathbf{v})$ belong to distinct parallel support planes of K.

Suppose this is false, so that, since K has a positive minimum width w, for each $\mathbf{v} \in C_1$, K has distinct parallel support planes H_v^1, H_v^2 which intersect $f_K(\mathbf{v})$. The orthogonal projection of K on lin C has at most countably many edges. Thus there is a countable set $F \subset C_1$ such that $f_K(\mathbf{u})$ is contained in a line parallel to lin $\{\mathbf{a}\}$, for each $\mathbf{u} \in C_1 \backslash F$. By rechoosing C_1 to be a small closed neighbourhood of an interior point of C_1 not in F, we can ensure that for no $\mathbf{v} \in C_1$ are the outer unit normals to H_v^1 or H_v^2 members of C. Then for each $\mathbf{v} \in C_1$, all lines parallel to lin $\{\mathbf{a}\}$ intersect the region bounded by H_v^1 and H_v^2 in segments of equal finite length $\beta(\mathbf{v}) \geq w$; when for some $\mathbf{v}$ there is more than one possible choice for H_v^1 and H_v^2, we choose these planes to minimize $\beta(\mathbf{v})$. Define $\beta = \inf\{\beta(\mathbf{v}) : \mathbf{v} \in C_1\}$, so that $\beta \geq w$ and there exists $\mathbf{v}^* \in C_1$ with $\beta(\mathbf{v}^*) = \beta$. For $\mathbf{v} \in C_1 \backslash F$, $f_K(\mathbf{v})$ is a line segment with end points in H_v^1 and H_v^2, so $f_K(\mathbf{v})$ has length $\beta(\mathbf{v})$. But $f_K(\mathbf{v})$ lies between $H_{v^*}^1$ and $H_{v^*}^2$, so $\beta(\mathbf{v}) = \beta$ and $f_K(\mathbf{v})$ has end points in $H_{v^*}^1$ and $H_{v^*}^2$. It then follows that $f_K(\mathbf{v})$ intersects $H_{v^*}^1$ and $H_{v^*}^2$ for all $\mathbf{v} \in C_1$. Let $\mathbf{u}^*$ be the unit outer normal to $H_{v^*}^1$. Then $\mathbf{u}^* \notin C_1$, and for $\mathbf{v} \in C_1$, $0 < \lambda < 1$ we have

$$f_K(\lambda \mathbf{u}^* + (1-\lambda)\mathbf{v}) = f_K(\mathbf{u}^*) \cap f_K(\mathbf{v})$$

so that the set of outer normals to K at points of $f_K(\mathbf{u}^*)$ contains a disc on S^2. This contradicts Lemma 4.

Hence there exist δ and $\mathbf{v}$ as claimed, and we suppose $\delta<\tau$. Let $\Sigma=N_K(\delta,\mathbf{v})\cap\mathcal{S}_a$. Then $N_K(\delta,\mathbf{v})\backslash\Sigma$ is the union of two dijoint open connected sets A_1, A_2 in $bd\,K$ with $\Omega_v(A_1)=A_2$. We have $f_K(\mathbf{u})\subset N_K(\delta,\mathbf{v})$ for all $\mathbf{u}\in S^2$ close to $\mathbf{v}$, so by Lemma 4 $N_K(\delta,\mathbf{v})$ intersects a quadric piece P. We may assume $P\cap A_1\neq\varnothing$. We claim that $(\partial P)\cap N_K(\delta,\mathbf{v})\subset\Sigma$.

Suppose this is false, so that by Lemma 10 $N_K(\delta,\mathbf{v})$ intersects a cylindrical piece Z whose generators are not parallel to $\operatorname{lin}\{\mathbf{a}\}$, and such that some section of $\hat{Z}$ is a shadow boundary of $\hat{P}$. By Lemma 14 $\hat{P}$ is not a paraboloid, so $\hat{Z}$ is not parabolic. The generators of $\hat{Z}$ are orthogonal to at most two elements of C_1, so by Lemma 5 we may choose an arc $C_2\subset C_1$ and a non-empty open set $Z_1\subset Z\cap N_K(\delta,v)$ such that $\Omega_u(Z_1)\subset N_K(\delta,\mathbf{v})$ and $H_K(0,\mathbf{v})$ is not parallel to the generators of $\hat{Z}$, when $\mathbf{u}\in C_2$. Fix $\mathbf{u}'\in C_2$. Then by Lemmas 12 and 13, $Z'=\Omega_{u'}(Z_1)$ is a subset of a cylindrical surface with generators parallel to those of $\hat{Z}$. Choose α so that $H_K(\alpha,u')\cap Z_1$ contains a point $\mathbf{x}$, and let Λ be the support plane of $\hat{Z}$ at $\mathbf{x}$. Then Λ contains a generator l of $\hat{Z}$ and a support line m of $H_K(\alpha,\mathbf{u}')\cap\hat{Z}$; then m is not parallel to l. The support plane Λ' to K at $\Omega_{u'}(\mathbf{x})$ contains lines parallel to l and m, so Λ' is parallel to Λ, contrary to our choice of δ and $\mathbf{v}$. We conclude that $(\partial Z)\cap N_K(\delta,\mathbf{v})\subset\Sigma$.

Hence $A_1\subset P$. If $(\partial P)\cap\Sigma=\varnothing$ then A_2 intersects P, and by the same argument as above $A_2\subset P$, so that $N_K(\delta,\mathbf{v})\subset P$. If $(\partial P)\cap\Sigma\neq\varnothing$ then there exist $\mathbf{x}\in(\partial P)\cap\Sigma$, an outer unit normal $\mathbf{u}$ to $\hat{P}$ at $\mathbf{x}$ and $\nu>0$, such that $N_K(\nu,\mathbf{u})$ has the form described in Lemma 10; in this case $\mathbf{x}$ is a smooth point of K, so $\mathbf{u}\in C$ and the cylindrical piece which intersects $N_K(\nu,\mathbf{u})$ has generators parallel to $\operatorname{lin}\{\mathbf{a}\}$. In either of these cases, there exist $\delta'>0$ and $\mathbf{v}'\in C$ such that $N_K(\delta',\mathbf{v}')$ contains no line segments which are not parallel to $\operatorname{lin}\{\mathbf{a}\}$.

But if $\mathbf{y}\in S^2$ is close to $\mathbf{a}$, then $\mathcal{S}_y$ intersects $N_K(\delta',\mathbf{v}')$ so $\mathcal{S}_y$ cannot be a band unless $\mathbf{y}=\pm\mathbf{a}$, by Lemma 11. This shows that $\{\mathbf{y}\in S^2:\mathcal{S}_y \textit{ is a band}\}$ has no limit points, and is therefore finite.

LEMMA 16. *K is the sum of a polytope and an ellipsoid.*

Proof. Let $\mathbf{a}(1),\ldots,\mathbf{a}(n)$ be distinct points of S^2, no pair being antipodal, such that $\mathcal{S}_{a(1)},\ldots,\mathcal{S}_{a(n)}$ are all the bands in $bd\,K$. Write C_i for the great circle $\{\mathbf{u}\in S^2:\mathbf{u}\cdot\mathbf{a}(i)=0\}$ so that $\mathcal{S}_{a(i)}=\bigcup f_K(C_i)$ $(i=1,\ldots,n)$. Let $R_1,\ldots,R_k$ be the components of $S^2\backslash\bigcup_{i=1}^n C_i$, and let $P_i=\bigcup f_K(R_i)$ $(i=1,\ldots,k)$. Consider $j\in\{1,\ldots,k\}$. From Lemma 4, P_j intersects a quadric piece P_j^*; let R_j^* be the set of outer unit normals to K at points of P_j^*. Lemma 10 shows that $\partial R_j^*\subset\bigcup_{i=1}^n C_i$, so $R_j\subset R_j^*$, while Lemma 11 shows that P_j^* intersects no band, which ensures that $R_j^*\cap\bigcup_{i=1}^n C_i=\varnothing$. Hence $R_j=R_j^*$ and $P_j=P_j^*$. This shows that $P_1,\ldots,P_k$ are the quadric pieces in $bd\,K$.

Let C be a great circle on S^2 which is not one of $C_1, \ldots, C_n$, and which contains no points which belong to more than one of $C_1, \ldots, C_n$. After relabelling we may suppose that C is contained in the union of $R_1, \ldots, R_s$ with a finite set, and that for $i=1, \ldots, s-1$, $R_i \cap C$ and $R_{i+1} \cap C$ are arcs with an end in common, so that $(\partial R_i) \cap (\partial R_{i+1})$ contains an arc of C_j for some j. Then by Lemma 10 $\hat{P}_i$ is a translate of $\hat{P}_{i+1}$ for $i=1, \ldots, s-1$. Since such a great circle C may be chosen to intersect any given pair of $R_1, \ldots, R_k$, all quadric pieces must be translates of subsets of the same quadric surface, which must be an ellipsoid, since this is the only quadric surface with outer normals in all directions. Hence there exist a solid ellipsoid E with centre $\mathbf{o}$ and points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ with $\hat{P}_i = \mathbf{x}_i + bd\, E$ for $i=1, \ldots, k$.

Write $X = \operatorname{conv}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$. We shall prove $K = X+E$. If $U \subset bd\, K$ is a neighbourhood of an exposed point of K, the set of outer unit normals to K at points of U contains a disc in S^2, so U intersects a quadric piece. Thus (see for example [4])

$$K = \operatorname{conv} \operatorname{cl} \exp(K) = \operatorname{conv} \operatorname{cl} \bigcup_{i=1}^{k} P_i \subset X+E.$$

To prove $K = X+E$ it will be sufficient to prove $\mathbf{x}_i + E \subset K$ for $i=1, \ldots, k$. For $\mathbf{u} \in S^2$ we write $H^+(\mathbf{u})$ for the closed half-space bounded by $H_K(0, \mathbf{u})$ which contains K. Let R, R' be two of $R_1, \ldots, R_k$, and let $\mathbf{u} \in R$. Choose a great circle $D \subset S^2$ with $\mathbf{u} \in D$ and $D \cap R' \neq \emptyset$, such that D contains no point belonging to more than one of $C_1, \ldots, C_n$. Let $\mathbf{w} \in R' \cap D$ and choose $\mathbf{v} \in D$ such that $\mathbf{v} \cdot \mathbf{u} = 0$ and $\mathbf{v} \cdot \mathbf{w} \geq 0$. Write $\mathbf{z}(\theta) = \cos\theta\mathbf{u} + \sin\theta\mathbf{v} \in D$ for $0 \leq \theta \leq \pi$, and let $R_1, \ldots, R_t$ be those members of $R_1, \ldots, R_k$ (relabelled if necessary) which intersect $\mathbf{z}[0, \pi]$, ordered so that $\mathbf{z}(\theta) \in R_i$ for $\theta_{i-1} < \theta < \theta_i$ $(i=1, \ldots, t)$ where $0 = \theta_0 < \theta_1 < \cdots < \theta_t = \pi$. We fix $i \in \{2, \ldots, t\}$ and prove $\mathbf{x}_i \cdot \mathbf{u} \leq \mathbf{x}_{i-1} \cdot \mathbf{u}$. Writing $\mathbf{c} = f_E(\mathbf{z}(\theta_{i-1}))$ we find that $H_K(0, \mathbf{z}(\theta_{i-1}))$ intersects $\operatorname{cl} P_{i-1}$ and $\operatorname{cl} P_i$ in the points $\mathbf{x}_{i-1} + \mathbf{c}$ and $\mathbf{x}_i + \mathbf{c}$ respectively. For $\theta_{i-2} < \alpha < \theta_{i-1} < \beta < \theta_i$ let

$$\mathbf{p}(\alpha) = f_K(\mathbf{z}(\alpha)) \in P_{i-1}, \qquad \mathbf{q}(\beta) = f_K(\mathbf{z}(\beta)) \in P_i, \text{ so that}$$

$$\lim_{\alpha \to \theta_{i-1}^-} \mathbf{p}(\alpha) = \mathbf{x}_{i-1} + \mathbf{c}, \qquad \lim_{\beta \to \theta_{i-1}^+} \mathbf{q}(\beta) = \mathbf{x}_i + \mathbf{c}.$$

We have $\mathbf{q}(\beta) \cdot \mathbf{z}(\alpha) \leq \mathbf{p}(\alpha) \cdot \mathbf{z}(\alpha)$ and $\mathbf{p}(\alpha) \cdot \mathbf{z}(\beta) \leq \mathbf{q}(\beta) \cdot \mathbf{z}(\beta)$; writing $\mathbf{z}(\alpha)$ and $\mathbf{z}(\beta)$ out in full and combining the inequalities we find $(\mathbf{q}(\beta) - \mathbf{p}(\alpha)) \cdot \mathbf{u} \sin(\beta - \alpha) \leq 0$. Hence $(\mathbf{x}_i - \mathbf{x}_{i-1}) \cdot \mathbf{u} \leq 0$ as required. This shows that $\mathbf{x}_i \cdot \mathbf{u} \leq \mathbf{x}_1 \cdot \mathbf{u}$ for $i=2, \ldots, t$. Hence for $i=1, \ldots, k$ we have $\mathbf{x}_i + E \subset H^+(\mathbf{u})$ for a set of $\mathbf{u}$ dense in S^2, so $\mathbf{x}_i + E \subset K$. This completes the Lemma, since $K = X+E$ where X is a polytope and E is an ellipsoid.

4. Preliminaries for the Higher Dimensional Cases

LEMMA 17. *Let $C \subset E^d$ $(d \geq 4)$ be a convex body with $\mathbf{o} \in \operatorname{int} C$, and let A be a non-empty open connected set in bd C. If for every 3-dimensional orthogonal projection π, $(\pi A) \cap (\operatorname{relbd} \pi C)$ is contained in the surface of a 3-dimensional ellipsoid having $\mathbf{o}$ in its relative interior, then A is contained in the boundary of an ellipsoid.*

Proof. Writing C^* for the polar dual of C and letting

$$A^* = \{\mathbf{y} \in bd\, C^* : \mathbf{x} \cdot \mathbf{y} = 1 \text{ for some } \mathbf{x} \in A\},$$

we see that A^* is open and connected in $bd\, C^*$ (since the points of A are smooth and exposed on C), and for every 3-flat Λ containing $\mathbf{o}$, $\Lambda \cap A^*$ is contained in the relative boundary of a 3-dimensional ellipsoid whose relative interior contains $\mathbf{o}$. It will be sufficient to show that A^* is contained in the boundary of an ellipsoid whose interior contains $\mathbf{o}$.

Every point of A^* is a smooth exposed point of C^*. Let H_1 be a support hyperplane of C^* at a point $\mathbf{p} \in A^*$, and let H_2 be a translate of H_1 with $H_2 \cap \operatorname{int} C^* \neq \emptyset$ and so that, writing $B = (bd\, C^*) \cap \operatorname{conv}(H_1 \cup H_2)$ we have $B \subset A^*$. Let H_3 and H_4 be distinct translates of H_1 which lie strictly between H_1 and H_2. Then all 2-dimensional sections of $H_3 \cap C^*$ and $H_4 \cap C^*$ are ellipses, being sections of 3-dimensional ellipsoids containing $\mathbf{o}$. Hence $H_3 \cap C^*$ and $H_4 \cap C^*$ are ellipsoids (see for example [2]). Further, by choosing parallel 2-dimensional sections of $H_3 \cap C^*$ and $H_4 \cap C^*$ which lie in the same 3-flat through $\mathbf{o}$, and using the fact that parallel sections of an ellipsoid are homothetic, we can show that parallel central 2-dimensional sections of $H_3 \cap C^*$ and $H_4 \cap C^*$ are directly homothetic. Hence $H_3 \cap C^*$ is directly homothetic to $H_4 \cap C^*$ by a result of Rogers [6].

Let l be the line through $\mathbf{p}$ and the centre of $H_3 \cap C^*$. When Θ is a 2-flat containing l, $\Theta \cap A^*$ lies on an ellipse $E(\Theta)$, and the chords of $E(\Theta)$ parallel to H_1 are bisected by l. Hence the centre of $E(\Theta)$ lies on l. It now follows that the centre of $H_4 \cap C^*$ lies on l. Consequently all the ellipses $E(\Theta)$, for 2-flats Θ containing l, have the same centre $\mathbf{b} \in l$. It is now clear that $B \subset bd\, E$, where E is the unique ellipsoid with centre $\mathbf{b}$ supported by H_1 at $\mathbf{p}$ and having $H_3 \cap C^*$ as a section. By analytic continuation, $A^* \subset bd\, E$, which completes the proof.

LEMMA 18. *Let $C \subset E^d$ $(d \geq 4)$ be a convex body and suppose that every 3-dimensional orthogonal projection of C is the sum of a polytope and a 3-dimensional ellipsoid. Then C is the sum of a polytope and a d-dimensional ellipsoid.*

Proof. Clearly C is smooth. Consider a unit vector $\mathbf{u}$ and let $F = f_C(\mathbf{u})$, $H = H_C(0, \mathbf{u})$. Let S be the intersection of S^{d-1} with the hyperplane through $\mathbf{o}$ parallel to H, and for $\mathbf{v} \in S$ let $\pi(\mathbf{v}) = \text{lin}\{\mathbf{u}, \mathbf{v}\}$, Φ_v be the orthogonal projection on $\pi(\mathbf{v})$. For $\mathbf{v} \in S$, $\Phi_v(C)$ has an expression as $E(\mathbf{v}) + P(\mathbf{v})$ where $E(\mathbf{v})$ is an ellipse in $\pi(\mathbf{v})$ with centre $\mathbf{o}$ and $P(\mathbf{v})$ is a polytope in $\pi(\mathbf{v})$. Further, the expression of $\Phi_v(C)$ as the sum of an ellipse with centre $\mathbf{o}$ and a non-smooth compact convex set is unique. For $\mathbf{v} \in S$, let $t(\mathbf{v}) = \sup\{\theta \in [0, \pi]: f_{P(v)}(\mathbf{u}) \cap f_{P(v)}(\cos\theta\mathbf{u} + \sin\theta\mathbf{v}) \neq \emptyset\}$. By the non-smoothness of polytopes, $t(\mathbf{v}) > 0$ for each $\mathbf{v}$. We show that for $\tau > 0$, the set $G = \{\mathbf{v} \in S: t(v) \geq \tau\}$ is closed. Let $\{\mathbf{v}(i)\}_{i=1}^{\infty}$ be a sequence in G with limit $\mathbf{v}$. The bodies $E(\mathbf{v}(i))$, $P(\mathbf{v}(i))$, $i = 1, 2, \ldots$ are contained in a bounded region, and so there is an infinite set N of natural numbers so that $E(\mathbf{v}(i))$ and $P(\mathbf{v}(i))$ converge to limits E and P respectively as $i \to \infty$ through N, from Blaschke's Selection Theorem. Then E is an ellipsoid with centre $\mathbf{o}$ having dimension at most 2, and $f_P(\mathbf{u}) \cap f_P(\cos\theta\mathbf{u} + \sin\theta\mathbf{v}) \neq \emptyset$ for $0 \leq \theta \leq \tau$, so that P is non-smooth. Also $E + P = \Phi_v(C)$ which is smooth, so E must have dimension 2. Then by the uniqueness of expression, $E = E(\mathbf{v})$ and $P = P(\mathbf{v})$, so $\mathbf{v} \in G$ as required.

By taking $\tau = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and applying the Baire Category Theorem to S, we can choose $\tau > 0$ and a non-empty open cap $D \subset S$ such that $t(\mathbf{v}) > \tau$ for all $\mathbf{v} \in D$. We may assume that D lies in a hemisphere of S. Write $U = \{\cos\theta\mathbf{u} + \sin\theta\mathbf{v} : \mathbf{v} \in D, 0 < \theta < \tau\}$, which is an open connected set in S^{d-1}, and has the property that $\pi(\mathbf{v}) \cap U$ is an arc with $\mathbf{u}$ as an end point for all $\mathbf{v} \in S$. Let $V = \bigcup f_C(U)$, which by smoothness is an open connected set in $bd\, C$, and which has a limit point in F. If I is a line segment contained in V then $\Phi_v(I)$ is an exposed point of $\Phi_v(C)$ for some $\mathbf{v} \in D$, so that I is perpendicular to $\pi(\mathbf{v})$. Hence I is parallel to H. Notice that we can rechoose V to lie within an arbitrarily small distance of F by intersecting U with a sufficiently small ball with centre $\mathbf{u}$.

By successive application of constructions similar to the one given above, we can choose support hyperplanes $H_1, \ldots, H_d$ with $H = H_1$, having outer unit normals $\mathbf{u} = \mathbf{u}_1, \ldots, \mathbf{u}_d$ and open sets $U = U_1, \ldots, U_d$ in S^{d-1}, open sets $V = V_1, \ldots, V_d$ in $bd\, C$ satisfying $U_i \subset U_{i-1}$, $\mathbf{u}_i \in U_{i-1} \setminus \text{lin}\{\mathbf{u}_1, \ldots, \mathbf{u}_{i-1}\}$ for $i = 2, \ldots, d$, $V_j = \bigcup f_C(U_j)$ and every line segment in V_j is parallel to H_j for $j = 1, \ldots, d$. Then $V_d \subset V$ and every point of V_d is an exposed point of C. Let $U' \subset U_d$ be a cap on S^{d-1}, and let $V' = \bigcup f_C(U') \subset V_d$. Then for every 3-dimensional orthogonal projection Ψ, $(\Psi V') \cap \text{relbd}\, \Psi C$ is a connected subset of $\exp \Psi C$, and is therefore contained in the relative boundary of a 3-dimensional ellipsoid; by a change of origin and by further reducing V' we may suppose that $\mathbf{o}$ lies in the relative interiors of all ellipsoids arising in this manner. It now follows from Lemma 17 that V' lies in the boundary of a d-dimensional ellipsoid $\mathscr{E}$ say. Let $U'' = U \cap \bigcup \{\text{lin}\{\mathbf{u}, \mathbf{v}\} : \mathbf{v} \in U'\}$ which is an open connected set in S^{d-1}, and let $V'' = \bigcup f_C(U'')$. For each $\mathbf{w} \in U''$ the support plane of C with outer normal $\mathbf{w}$ also supports $\mathscr{E}$. Since U'' is open this ensures that $V'' \subset bd\,\mathscr{E}$. Notice that F contains a limit point of V''. We have now shown that every face of C intersects the closure of an open subset of $bd\,C$ which is contained in the surface of an ellipsoid.

Let $\mathscr{H}$ be the family of maximal open connected non-empty subsets of $bd\,C$ which are contained in the surfaces of ellipsoids; for $P \in \mathscr{H}$ let $\hat{P}$ be the ellipsoid surface which contains P. If $P, Q \in \mathscr{H}$ then there is an open non-empty set $\mathscr{L}$ of 3-dimensional orthogonal projections such that relbd $\varphi\hat{P}$ and relbd $\varphi\hat{Q}$ contain open non-empty subsets of relbd φC for all $\varphi \in \mathscr{L}$. Hence $\varphi\hat{P}$ is a translate of $\varphi\hat{Q}$ for all $\varphi \in \mathscr{L}$, so $\hat{P}$ is a translate of $\hat{Q}$. Let X be the set of centres of ellipsoids $\hat{P}$ with $P \in \mathscr{H}$ and let $\mathscr{E}$ be the solid ellipsoid with centre $\mathbf{o}$ such that $bd\,\mathscr{E}$ is a translate of $\hat{P}$ for each $P \in \mathscr{H}$. Write $K = \operatorname{conv} \operatorname{cl} X$.

Every exposed point of C is a face of C, and so must be contained in the closure of a member of $\mathscr{H}$. Thus $C = \operatorname{conv} \operatorname{cl} \exp C \subset K + \mathscr{E}$. Consider any $\mathbf{q} \in X$, let $Q \in \mathscr{H}$ be a subset of $\mathbf{q} + bd\,\mathscr{E}$, and let H be a hyperplane which supports C at a point of Q. Let H' be any other support hyperplane of C and let H'_+ be the closed half-space bounded by H' which contains C. Consider a 3-dimensional orthogonal projection Ψ in a $(d-3)$-dimensional direction parallel to $H \cap H'$, and let $\mathscr{C}$ be the cylinder $\Psi^{-1}\Psi C$ which is supported by H and H'. Then relbd ΨC contains a non-empty open subset of relbd $\Psi\hat{Q}$, so $\Psi C = Y + \Psi\mathscr{E}$ where Y is some polytope with $\Psi\mathbf{q}$ as a vertex. Hence $\Psi(\mathbf{q} + \mathscr{E}) \subset \Psi C$, so $\mathbf{q} + \mathscr{E} \subset \mathscr{C} \subset H'_+$. This holds for all such H', so $\mathbf{q} + \mathscr{E} \subset C$. Hence $K + \mathscr{E} = C$.

It remains to show that K is a polytope. Let φ be any 3-dimensional orthogonal projection. Choose a non-empty open set $M \subset \text{relbd}\,\varphi C$ which is contained in the surface of a 3-dimensional ellipsoid W. For each $\mathbf{x} \in M$, $C \cap \varphi^{-1}(\mathbf{x})$ is a face of C which meets the closure of some $P(\mathbf{x}) \in \mathscr{H}$. Since $\mathscr{H}$ is countable, we can apply the Baire Category Theorem to choose a non-empty open subset $M' \subset M$ and $P \in \mathscr{H}$ with $\varphi^{-1}(\mathbf{x}) \cap \operatorname{cl} P \neq \varnothing$ for all $\mathbf{x} \in M'$. Thus $\varphi\hat{P} = W$. But $\hat{P} = bd(\mathbf{y} + \mathscr{E})$ for some $\mathbf{y} \in X$ and φC is expressible as $\varphi C = Z + W$ for some polytope Z, so $\rho\mathbf{y} + \varphi\mathscr{E} + Z = \varphi C$. Now $\varphi C = \varphi K + \varphi\mathscr{E}$, so by comparing the support functions, we see that φK is a polytope. Hence every 3-dimensional orthogonal projection of K is a polytope. By a result of Klee [3], K is a polytope as required.

5. Proof of the Theorem

Let $K \subset E^d$ $(d \geqslant 3)$ have property (A). We first show that K is the sum of a polytope and an ellipsoid. This was established for $d = 3$ by Lemma 16. If $d \geqslant 4$, then every 3-dimensional orthogonal projection of K has property (A), and is therefore the sum of a polytope and an ellipsoid; hence by Lemma 18 K is the sum of a polytope and an ellipsoid. Thus we can write $K = X + E$ where X is a polytope and E is an ellipsoid. Every face of X (including X itself if $\dim X < d$) is a translate of a face of K, and is therefore centrally symmetric, since every face of K is a limit of centrally symmetric sections of K. Hence X is a zonotope.

Suppose $K \subset E^d$ $(d \geq 3)$ is a convex body with $K = X + E$, where X is a zonotope and E is an ellipsoid. Then $f_X(\mathbf{u})$ is centrally symmetric for each $\mathbf{u} \in S^{d-1}$, even when $f_X(\mathbf{u}) = X$, for every zonotope is centrally symhetric (see Shephard [7]). By an elementary calculus argument we can show that for each $\mathbf{u} \in S^{d-1}$ there exists $\varepsilon(\mathbf{u}) > 0$ such that

$$H_K(\alpha, \mathbf{u}) \cap K = f_X(\mathbf{u}) + (H_E(\alpha, \mathbf{u}) \cap E)$$

for $0 < \alpha < \varepsilon(\mathbf{u})$, from which it follows that K has property (A).

6. Proof of the Corollary

If $K \subset E^d$ $(d \geq 3)$ is a strictly convex body which has property (A), then from the Theorem $K = X + E$ where X is some zonotope and E is some ellipsoid. Every face of X is a translate of a face of K and so is a single point by strict convexity. Hence X is reduced to a point and K is an ellipsoid.

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