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Ordinal Bounds for Well Ordered Extensions of the Coordinatewise Partial Ordering

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1. Introduction

The *coordinatewise* partial ordering of any subset of N^n is defined by $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$ if and only if $x_i \leq y_i$ for each $1 \leq i \leq n$. We shall show that for any such subset, all extensions of this partial ordering to a well ordering have order types which do not exceed ω^n, the order type of the lexicographical ordering of N^n.

Quite often the calculation of order types proceeds from an effective rule for comparing terms. Our result might be useful when no such effective rule is at hand. We show an application of this kind at the end.

The following open problems are suggested by our result. Is ω^ω a bound if we work with the set of *finite*-tuples instead of n-tuples. Indeed it is not clear to the writer that there is a countable bound at all. Also, there is the question of whether α^n is a bound if we work with the set of n-tuples of elements from an arbitrary well ordered set of order type α.

At this point a word of caution is in order. Although the embedding $(x_1, x_2, \ldots, x_n) \to (x_1, x_2, \ldots, x_n, k)$ from N^n into N^{n+1} is order preserving with respect to the coordinatewise and lexicographical orderings, it need not be order preserving with respect to arbitrary well ordered extensions of the coordinatewise ordering. A simple counter example is the embedding from N^2 into N^3 where N^3 has the lexicographical ordering from the left, and N^2 has an ordering obtained from the lexicographical ordering from the left by inserting the ordered pair $(1, 0)$ between $(0, 0)$ and $(0, 1)$.

2. Conventions

N will denote the set of *non-negative* integers. k and m range over N. n ranges over $N-\{0\}$. i ranges over $\{1, 2, \ldots, n\}$. N^n denotes the set of n-tuples of elements from N. If a relation $\leq$ on Y induces a well ordering on $X \subseteq Y$, then $|X{:}\leq|$ denotes the order type of X under this induced ordering. When the ordering is understood from the context, we simply write $|X|$. If $s \in X$, we will denote the initial segment $\{x \in X \mid x < s\}$ by X_s. If $A \subseteq N^n$ let $A_i^k = \{(x_1, x_2, \ldots, x_n) \in A \mid x_i = k\}$ and $\tilde{A}_i^m = \bigcup_{k=0}^m A_i^k$. $\leq_A$ will denote the coordinatewise ordering on A, and $\leq'_A$ will be used in denoting extensions of $\leq_A$ to well orderings of A. For $n \geq 2$, $(x_1, \ldots, \hat{x}_i, \ldots, x_n)$ is the $n-1$ tuple obtained from $(x_1, x_2, \ldots, x_n)$ by deleting the ith coordinate. For example $(2, 4, \hat{6}, 3) = (2, 4, 3)$. $\alpha \# \beta$ denotes the Hessenberg "natural sum" of the ordinals α and β.

3. Main Results

LEMMA. *Suppose that $A \cup B$ is a well ordered set and that under the induced ordering $|A| = \alpha$ and $|B| = \beta$, then $|A \cup B| \leq \alpha \# \beta$.*

Proof. By transfinite induction on $\alpha \# \beta$ over the usual ordering of the ordinals. Let A, B, α, β, be given as stated above. Our induction hypothesis is that for any well ordered set $C \cup D$ where $|C| = \gamma$, $|D| = \delta$ and $\gamma \# \delta < \alpha \# \beta$, it is the case that $|C \cup D| \leq \gamma \# \delta$. If $A \cup B = \phi$ the desired result is trivial. Assume $A \cup B \neq \phi$. We shall show that $|A \cup B| \leq \alpha \# \beta$ by showing that $|(A \cup B)_s| < \alpha \# \beta$ for each initial segment $(A \cup B)_s$. Let $s \in (A \cup B)$ be given.

Case 1. $s \in A$; then $(A \cup B)_s = A_s \cup D$ where $D \subseteq B$. Thus

$$|(A \cup B)_s| = |A_s \cup D|. \tag{1}$$

Now $|A_s| < \alpha$ and $|D| \leq \beta$, so $|A_s| \# |D| < \alpha \# \beta$. This gives us the right to invoke the induction hypothesis to get

$$|A_s \cup D| \leq |A_s| \# |D| < \alpha \# \beta.$$

This with (1) gives $|(A \cup B)_s| < \alpha \# \beta$ as desired.

Case 2. $s \in B$; then owing to the fact that set union and natural sum of ordinals are commutative, we can argue the same as in case 1. but with the roles of A and B interchanged.

The author has learned from the referee that this lemma had been proved earlier in a different way. See Carruth [2]. For results related to this lemma see Neumer [3].

THEOREM. $|A : \leq'_A| \leq \omega^n$ *for all $A \subseteq N^n$ and all orderings $\leq'_A$ which are extensions of the coordinatewise ordering $\leq_A$ to a well ordering of A.*

Proof. By induction on n. When $n=1$ the theorem is trivial because all the orderings in question are induced by the usual ordering of N. Let $n>1$, $A\subseteq N^n$ and $\leq'_A$ as stated in the theorem be given. We first show

$$|A_i^k:\leq'_A|\leq\omega^{n-1}\quad\text{all}\quad i,k. \tag{2}$$

Let i, k be given. Let ϕ be the mapping from N^n into N^{n-1} defined by

$$\phi(x_1,x_2,\ldots,x_n)=(x_1,\ldots,\hat{x}_i,\ldots,x_n).$$

Now ϕ induces a one to one mapping ϕ^* from A_i^k into N^{n-1}. Letting B denote the range of ϕ^* define an ordering $\leq'_B$ on B by the condition that ϕ^* be order preserving. Clearly $\leq'_B$ is a well ordering of B and it is easy to see that it is an extension of the coordinatewise ordering $\leq_B$. Now from the very definition of B and $\leq'_B$

$$|A_i^k:\leq'_A|=|B:\leq'_B|. \tag{3}$$

On the other hand since $B\subseteq N^{n-1}$ we can invoke the induction hypothesis to get

$$|B:\leq'_B|\leq\omega^{n-1}. \tag{4}$$

From (3) and (4) we get (2) as desired.

In the remainder of the argument the only ordering at issue will be $\leq'_A$, so we shall suppress the relation symbol $\leq'_A$.

We now show by a subsidiary induction on m that

$$|\tilde{A}_i^m|\leq\omega^{n-1}(m+1)\quad\text{all}\quad i,m. \tag{5}$$

Let i be given, when $m=0$, (5), just reads $|\tilde{A}_i^0|\leq\omega^{n-1}$; but after recognizing that $\tilde{A}_i^0=A_i^0$ we see that this is a special case of (2). For $m>0$, we have $\tilde{A}_i^m=\tilde{A}_i^{m-1}\cup A_i^m$. Using this, the lemma tells us that

$$|\tilde{A}_i^m|\leq|\tilde{A}_i^{m-1}|\#|A_i^m|. \tag{6}$$

By the subsidiary induction hypothesis

$$|\tilde{A}_i^{m-1}|\leq\omega^{n-1}m \tag{7}$$

and we have already seen in (2) that

$$|A_i^m| \leq \omega^{n-1}. \tag{8}$$

From (6), (7), and (8) follows

$$|\tilde{A}_i^m| \leq \omega^{n-1} m \# \omega^{n-1} = \omega^{n-1}(m+1)$$

as desired.

We now proceed to show $|A| \leq \omega^n$ as required in the statement of the theorem. We do this by showing that for each $\vec{a} = (a_1, a_2, \ldots, a_n) \in A$, the initial segment $A_{\vec{a}}$ has order type less than ω^n. Let $\vec{a} = (a_1, a_2, \ldots, a_n) \in A$ be given. If $(b_1, b_2, \ldots, b_n) \in A$ and $(b_1, b_2, \ldots, b_n) <'_A (a_1, a_2, \ldots, a_n)$, then $b_i < a_i$ for at least one i, and for such an i, $(b_1, b_2, \ldots, b_n) \in \tilde{A}_i^{a_i}$. Thus $A_{\vec{a}} \subseteq \bigcup_{i=1}^n \tilde{A}_i^{a_i}$ from which it follows that

$$|A_{\vec{a}}| \leq \left| \bigcup_{i=1}^{n} \tilde{A}_i^{a_i} \right|. \tag{9}$$

Using the lemma we get

$$\left| \bigcup_{i=1}^{n} \tilde{A}_i^{a_i} \right| \leq |\tilde{A}_1^{a_1}| \# \cdots \# |\tilde{A}_n^{a_n}|. \tag{10}$$

But we have already seen in (5) that $|\tilde{A}_i^{a_i}| \leq \omega^{n-1}(a_i + 1)$ for each i; and this with (9) and (10) gives

$$|A_{\vec{a}}| \leq \omega^{n-1}(a_1+1) \# \cdots \# \omega^{n-1}(a_n+1) < \omega^n.$$

4. Application

Consider the family $\mathscr{F}$ of functions from N into N of the form

$$(a_1x^2 + a_2x + a_3)^x + (a_4x^2 + a_5x + a_6)^x \tag{11}$$

where $a_j \in N$ for each j. It is known [1] that this family is well ordered by the *majorization* relation $\preceq$ defined by $f \preceq g$ if and only if there exists $n_0 \in N$ such that $f(x) \leq g(x)$ for all $x \geq n_0$. We shall show that ω^6 is a bound for this ordering. Consider the functional from N^6 into $\mathscr{F}$ which sends each 6-tuple

$(a_1, a_2, a_3, a_4, a_5, a_6)$ to the function (11). This functional, of course, is many to one. We can get a one to one "inverse" functional ψ from $\mathcal{F}$ back into N^6 by choosing for each $f \in \mathcal{F}$ just one of its inverse images. Let A denote the range of ψ. Define a well ordering $\leq'_A$ of A by the condition that ψ be order preserving. It is easy to verify that $\leq'_A$ is an extension of the coordinatewise ordering $\leq_A$. Since $A \subseteq N^6$ we have by the main theorem that ω^6 is a bound for $|A:\leq'_A|$ and, consequently, for $|\mathcal{F}:\leq|$.

Similar results can be obtained for functions involving higher degree polynomials and more terms.

REFERENCES

[1] A. Ehrenfeucht, *Polynomial Functions with Exponentiation are Well Ordered*, Algebra Universalis, vol. 3, (1973), pp. 261–262.
[2] P. W. Carruth, *Arithmetic of ordinals with application to the theory of ordered abelian groups*, Bull. Amer. Math. Soc., *48* (1942), 262–271.
[3] W. Neumer, *Ueber Mischsummen von Ordnungszahlen*, Arch. Math., *5* (1954), 244–248.

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