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Fundamental Groups, Γ-groups, and Codimension two Submanifolds

SYLVAIN E. CAPPELL AND JULIUS L. SHANESON

Introduction and Statement of Results

Let $\phi: M^n \to W^{n+2}$ be a smooth or piecewise linear (P.L.) embedding (possibly not locally flat) of the connected, compact, oriented smooth or P.L. manifold M in the compact connected smooth or P.L. manifold W. Assume that a regular neighborhood of $\phi(M)$ meets the boundary in a regular neighborhood of $\phi(\partial M) = \partial W \cap \phi(M)$; possibly $\partial M = \emptyset$ or $\partial W = \emptyset$. If $\phi(M)$ is smooth or P.L. locally flat, the usual geometric procedure, using transversality and counting intersection points together with orientations and associated elements of $\pi_1 W$ (see, e.g. [W, §5]), gives an intersection number $x \cdot \phi_*[M]$, $x \in \pi_2 W$, with values in $Z \otimes_{Z\pi_1 M} Z\pi_1 W$. This tensor product is defined using the left $Z\pi_1 M$-module structure on the integral group ring $Z\pi_1 W$ given by[1] $\alpha\beta = \overline{\beta\phi_*(\alpha)}$, $\alpha \in Z\pi_1 M$, $\beta \in Z\pi_1 W$. For the general ϕ, $x \cdot \phi_*[M]$ is still defined homologically as in §1 or, alternatively, using geometric intersection numbers defined on the chain level [ST].

THEOREM 1. *Let K be the kernel of the map $\pi_1(W - \phi M) \to \pi_1 W$ induced by inclusion. Then $K_{ab} = K/[K, K]$, $[K, K]$ the commutator subgroup, is isomorphic as a $Z\pi_1 W$ module ($\pi_1 W$ acts by conjugation) to*

$$Z \otimes_{Z\pi_1 M} Z\pi_1 W / \{x \cdot \phi_*[M] / x \in \pi_2 W\}.$$

In theorem 1, Z has the trivial $Z\pi_1 M$-module structure. In the non-orientable case the result is still valid, but the $Z\pi_1 M$-module structure on Z is given by the difference of the orientation characters of M and W. Theorem 1 also applies to Poincaré embeddings of Poincaré complexes, and hence to locally flat topological

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[1] In $Z\pi_1 W$, $(\sum \alpha_g g)^- = \sum \alpha_g g^{-1}$. Thus $Z \otimes_{Z\pi_1 M} Z\pi_1 W$ is the free abelian group on the left cosets of $\phi_*(\pi_1 M)$ in $\pi_1 W$. If we used the module structure $\alpha\beta = \phi_*(\alpha)\beta$, we would get right cosets, corresponding to writing covering translations as right rather than left operators.

embeddings. In [CS2, §5], it is shown how a general P.L. embedding, locally flat or not, induces an underlying Poincaré embedding.

We apply theorem 1 to the codimension two splitting problem. Suppose $\phi(M)=P$ is a smooth, P.L. locally flat or topologically locally flat submanifold of W. (Again, it suffices to consider Poincaré embeddings of Poincaré complexes.) Let

$$f:(Q^{n+2}, \partial Q)\to(W, \partial W)$$

be a degree 1 map that induces a (simple) homotopy equivalence[(2)] of Q with W. Assume that $f\,|\,\partial Q$ is transverse to ∂P and that $f\,|\,f^{-1}(\partial P): f^{-1}(\partial P)\to\partial P$ is a (simple) homology equivalence [CS1] over $Z\pi_1P$. In [CS1, §8] we studied the codimension two splitting problem: when is f homotopic to f_1, relative ∂Q, transverse to P, with $f_1\,|\,f_1^{-1}P: f_1^{-1}P\to P$ a simple homotopy equivalence. If the answer is affirmative, f was said to be (*simply*) *splittable along* P (relative boundary.)

Let $\sum_e(f)$, $e=s, h$, be the *abstract surgery obstruction* of f, as defined on page 322 of [CS1] (see also [B1], [W, §11]); i.e. $\sum_e(f)$ is defined as the surgery obstruction of the normal map f determines on a transverse inverse image of P. In 8.2 of [CS1], we saw that for $n\geq 5$ odd, f is simply splittable (splittable) if and only if $\sum_s(f)=0$ (*resp.* $\sum_h(f)=0$.) In Thm 8.3 of [CS2], we proved the same result in even dimensions, under some additional hypotheses. These results were applied to the study of the existence and uniqueness of invariant spheres of group actions. In 8.5 of [CS2], we gave a general result on the even dimensional case; there is a further obstruction, in a quotient of a suitable Γ-group, defined if $\sum_e(f)=0$, whose vanishing is necessary and sufficient for f to be (simply) splittable. The constructions of [CS3] imply that all these obstructions actually arise in codimension two splitting problems.

If M is closed, $\phi_*[M]$, $[M]\in H_n(M, Z)$ the orientation, can be considered[(3)] as an element of $H_n(W, A)$, $A=Z\otimes_{Z\pi_1M}Z\pi_1W$. Let $\tau_P=\tau_{\phi(M)}: H_2(W, Z\pi)\to A$, $\pi=\pi_1W$, be given by evaluation (Kronecker product) of the image in $H^2(W, A)$ of the Poincaré dual in $H^2(W, \partial W; A)$ of $\phi_*[M]$. If $\partial M\neq\phi$, $\phi_*[M]$ is in $H_n(W, \partial W; A)$; again let τ_P be the evaluation of its Poincaré dual.

THEOREM 2. *Suppose $n\geq 6$ is even and τ_P is surjective. Suppose also that*

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[2] By Poincaré duality, it therefore induces a (simple) homology equivalence of $(Q, \partial Q)$ and $(W, \partial W)$, with coefficients in $Z\pi_1W$.

[3] This assumes base points chosen, so that a lift of P to the covering space of W with fundamental group $\phi_*\pi_1M$ is determined.

$K = \ker(\pi_1(W-P) \to \pi_1 W)$ *is finitely generated. Then* f *is (simply) splittable if and only if* $\sum_e(f) = 0$, $e = s$ or h *as appropriate.*

Note that if $\pi_1 W$ is finite, K is automatically finitely presented. (For example, consider the covering space with $\pi_1 = K$.)

For example, the hypotheses of theorem 2 are satisfied in the following special cases:

2a. ϕ_* surjective on fundamental groups, $H_2(\pi_1 W) = 0$, K finitely presented, and $\phi_*(M) \in H_n(W, Z)$ a primitive class. (e.g. $\pi_1 W = \{e\}$ and $\phi_*[M]$ primitive. Recall that primitive means of infinite order and indivisible.)

2b. $\pi_1 M = \{e\}$, K finitely presented, there exists $\xi \in H^n(W; Z\pi)$ or $H^n(W, \partial W; Z\pi)$ if $\partial M \neq \phi$, with $\langle \xi, \phi_*[M] \rangle = 1 \in Z\pi$.

To prove theorem 2, a result of independent interest on Γ-groups is also needed. Let $G \to \pi$ be a surjective homomorphism with kernel K, and let $\mathscr{F}: ZG \to Z\pi$ be the map induced on integral group rings. Let $\Gamma_n^e(\mathscr{F})$, $e = s, h$ be the homology surgery groups of [CS1], and let

$$j_*: \Gamma_n^e(\mathscr{F}) \to L_n^e(\pi)$$

always denote the natural map. The next result can be found in a paper of Hausmann [H1].

THEOREM 3. *Suppose* K *is normally generated in* G *by a finitely generated subgroup* N *with* $N_{ab} = 0$. *Then*

$$j_*: \Gamma_n^e(\mathscr{F}) \to L_n^e(\pi)$$

is an isomorphism.

A purely algebraic proof of this result has been given recently by Justin Smith.

Using similar ideas, we also derive a new result on geometrically realizing codimension two Poincaré embeddings by P.L. embeddings (not necessarily locally flat.) Let Θ be a Poincaré or h-Poincaré embedding (see [CS2, §5] for the definitions) of the compact oriented closed P.L. manifold M^n in the compact P.L. manifold W^{n+2}. In [CS2] it was shown that if $n \geq 3$ is odd or if $\pi_1 W = \{e\}$ and Θ is cyclic, then Θ can be realized by a P.L. embedding. Let

$$A_\Theta = Z \otimes_{Z\pi_1 M} Z\pi_1 W / \{x \cdot [M] / x \in \pi_2 W\}.$$

THEOREM 4. *Suppose that $A_\Theta = 0$. Let $n \geq 4$ be even. Assume that the kernel of the natural map $\pi_1 E_\Theta \to \pi_1 W$, E_Θ the Poincaré complement* [CS2], *is finitely generated. Then Θ can be realized by a P.L. embedding of M^n in W^{n+2}.*

Under the hypotheses of Theorem 4, the theory of [CS2] applies to study the singularities of embeddings realizing Θ. In particular, the singularities will carry the Poincaré duals of the total L-class $L(\Theta)$ (see [CS2, §6]).

The same assumptions as in special case 2a and 2b of theorem 2 also guarantee that the hypotheses of theorem 4 are satisfied. For example, essentially as a special case of Theorem 4, we have

THEOREM 5. (*Compare* [*CS*3], [*CS*1, §6]). *Let $h: M^n \to W^{n+2}$, $n \geq 4$ even, be a homotopy equivalence, M^n a closed oriented P.L. manifold, W a compact oriented P.L. manifold. Suppose $H_2(\pi_1 W; Z) = 0$. Assume that the Euler class $\chi(W) \in H^2(M)$ is a primitive generator. Then h is homotopic to a P.L. embedding.*

Recall that $\chi(W)$ is defined in [CS2] as follows: restrict the Poincaré dual of $h_*[M]$ to $H^2(W) \cong_{h^*} H^2(M)$.

In an appendix, we use Theorem 1 to give the proof of Theorem 5.1 of [CS2].

1. Proof of Theorem 1.

Let R be a regular neighborhood of $\phi(M)$ that meets the boundary regularly [H3]. Let $\pi = \pi_1 W$. Then

$$K_{ab} = H_1(W - R; Z\pi) = H_1(\widetilde{W - R})$$

$\widetilde{W-R}$ the covering space associated to K. The following sequence is exact:

$\pi_2 W = H_2(W; Z\pi) \to H_2(W, W-R; Z\pi) \to H_1(W-R; Z\pi) \to H_1(W; Z\pi)$, and $H_1(W; Z\pi) = H_1(\hat{W}) = 0$, $\hat{W}$ the universal cover of W.

By excision $H_2(W, W-R; Z\pi) \cong H_2(R, \dot{R}; Z\pi)$, $\dot{R} = \partial R - \operatorname{Int}(R \cap \partial W)$. Here $Z\pi$ is a $\pi_1 R = \pi_1 M$ module via the inclusion induced map. By Poincaré duality

$$H_2(R, \dot{R}; Z\pi) \cong H^n(R, R \cap \partial W; Z\pi).$$

But the inclusion $(M, \partial M) \subset (R, \dot{R} \cap \partial W)$ is a (simple) homotopy equivalence. Therefore, applying Poincaré duality of M, we finally get an isomorphism $H_2(W, W-R; Z\pi) \cong H_0(M, Z\pi)$.

Now, $H_0(M; Z\pi) = H_0(\tilde{M})$, where $\tilde{M}$ is the covering space induced via ϕ from the universal covering space of W. From this, it is not hard to see that $H_0(M; Z\pi) \cong Z \otimes_{Z\pi_1 M} Z\pi_1 W$, as $Z\pi$ modules. Further, well known standard arguments show that if $\phi(M)$ is a smooth or P.L. locally flat submanifold, (so that R is the total space of a bundle over M), then the map $\pi_2 W \to H_2(W, W-R; Z\pi) \cong Z \otimes_{Z\pi_1 M} Z\pi$ is given by taking the (geometric) intersection number with $\phi_*[M]$. In general, we take this as the definition of $x \circ \phi_*[M]$, $x \in \pi_2 W$. Theorem 1 now follows.

Clearly a similar argument works for a Poincaré embedding Θ of M in W. More generally, suppose $M = \bigcup_{i=1}^{t} M_i$, M_i components of M, but W still connected. Dropping orientability hypotheses, let Z_{Θ_i} be the $Z\pi_1 M_i$ module consisting of the integers with

$$g \cdot t = \omega_W(\phi_* g)\omega_{M_i}(g)t,$$

where $g \in \pi_1 M_i$, $t \in Z$, and $\omega_W : \pi_1 W \to \{\pm 1\}$, $\omega_{M_i} : \pi_1 M_1 \to \{\pm 1\}$ are the orientation characters. Let K be as in Theorem 1. Then one easily extends the above argument to show

PROPOSITION 1. $K_{ab} = \bigoplus_{i=1}^{t} (Z_{\Theta_i} \otimes_{Z\pi_1 M_i} Z\pi_1 W / (\pi_2 W) \circ \phi_*[M_i])$.

Some examples: 1. M a point in a 2-manifold W other than S^2 or P^2. Then $\pi_1(W-M)$ is a free group F, and $\pi_2 W = 0$. Hence we have the presentation

$$1 \to K \to F \to \pi \to 1,$$

with $K_{ab} \cong Z\pi$.

2. If W is a prime orientable 3-manifold (so that $\pi_2 W = 0$ by the sphere theorem), and if $S^1 \subset W$ represents $\alpha \in \pi_1 W$, then

$$K_{ab} \cong Z \otimes_{Z[\alpha]} Z\pi_1 W,$$

K the kernel of $\pi_1(W - S^1) \to \pi_1 W$. In particular, if $\alpha = 0$, $K_{ab} \cong Z\pi_1 W$.

3. Let k be a smooth knotted circle in S^3, and let l be a longitude and m a meridian; e.g. l is obtained by pushing k off itself slightly and links it with linking number zero. Let W be obtained from S^3 by zero-framed surgery on k, i.e. surgery killing l. Let $G = \pi_1(S^3 - k)$, and let $N(l)$ be the normal subgroup of G generated by the class of l. Since l lies on a Seifert surface of k and near its

boundary, $N(l) \subset G' = [G, G]$. Clearly $\pi_1 W = G/N(l)$. Thus if $\pi_1 W = Z$, $N(l) = G'$ and so $G'' = G' = N(l) = N(l)'$.

Conversely suppose $N(l) \subseteqq N(l)'$. Then by Theorem 1, $0 = Z \otimes_{Z[m]} Z\pi_1 W/\pi_2 W \cdot [m]$, $[m] \in \pi_1 W$ represented by m. Applying the sphere theorem, it follows that $W = S^1 \times S^2 \# Q$. Since G, and hence $\pi_1 W$, is normally generated by $[m]$, we must have $\pi_1 Q = \{e\}$. Thus we have the following (possibly known) result

PROPOSITION. *Let $k \subset S^3$ be a knot. Then the following are equivalent*:
1. $\pi_1 W = Z$, *W obtained by framed (necessarily zero-) surgery on k.*
2. $N(l) = N(l)'$, *l a longitude.*
3. $N(l) = G'$, $G = \pi_1(S^3 - k)$.

From this proposition it follows in particular that if $W = S^1 \times S^2$, then the Alexander polynomial of k is identically 1. (See [Mo].)

2. Proof of Theorems 2, 4, and 5.

First consider theorem 2. It is known and not hard to prove that if h is (simply) splittable, then $\sum_e (h) = 0$, $e = s$ or h as appropriate. (See e.g. [B1], [CS1]) Conversely suppose $\sum_e (h) = 0$. Let Φ be the diagram

$$\begin{array}{ccc} Z\pi_1(W-P) & \xrightarrow{\mathscr{F}} & Z\pi_1 W \\ \downarrow & & \downarrow \\ Z\pi_1 W & \longrightarrow & Z\pi_1 W, \end{array}$$

consisting of identities and inclusion induced maps. According to Theorem 8.5 of [CS1], h is splittable if and only if an obstruction in a quotient of $\Gamma^e_{n+3}(\Phi)$ vanishes.[4]

We have the exact sequence [CS1, §3]:

$$\Gamma^e_{n+3}(\mathscr{F}) \to L^e_{n+3}(\pi_1 W) \to \Gamma^e_{n+3}(\Phi) \to \Gamma^e_{n+2}(\mathscr{F}) \to L^e_{n+2}(\pi_1 W).$$

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[4] Actually in §8 of [CS1] we supposed h induces a (simple) homotopy equivalence of boundaries ∂Q and ∂W. However, everything in §8 goes through without change if h induces a (simple) $Z\pi_1 W$-homology equivalence of boundaries.

Let K be the kernel of $\pi_1(W-P)\to\pi_1(W)$. By hypothesis, K is finitely presented. By Theorem 1,

$$K_{ab}=Z\otimes_{Z\pi_1 P}Z\pi_1 W/\{x\cdot[P]/x\in\pi_2 W\};\quad i.e.$$

$$K_{ab}=\operatorname{coker}\tau_P.$$

(Here $[P]=\phi_*[M]$.) Thus $K_{ab}=0$. Therefore, by Theorem 3, the two extreme maps in the above sequence are isomorphisms, and hence $\Gamma^e_{n+3}(\Phi)$ vanishes. Therefore, by 8.5 of [CS1], h is (simply) splittable.

The special case 2a. follows easily. For if $\pi_1 P\to\pi_1 W$ is surjective, then $Z\otimes_{Z\pi_1 P}Z\pi_1 W=Z$. If $[P]=\phi_*[M]$ is primitive, then by Poincaré duality and universal coefficients, there is a class $\bar{x}\in H_2 W$ with $\bar{x}\cdot[P]=1$. Since $H_2(\pi_1 W)=0$, the Hurewicz homomorphism $\pi_2 W\to H_2 W$ is surjective. Therefore τ_P is surjective in this case.

For 2b., $Z\otimes_{Z\pi_1 P}Z\pi_1 W=Z\pi_1 W$, and ξ is Poincaré dual to $x\in H_2(W,Z\pi)=\pi_2 W$. By definition,

$$x\cdot\phi_*[M]=\langle\xi,\phi_*[M]\rangle=1;$$

so again τ_P will be surjective.

Theorem 4 is proven using Theorems 1 and 3 and the same method as for Theorem 5.3 of [CS2]. Given a Poincaré embedding Θ, one obtains, as in the section of §5 of [CS2] just preceding lemma 5.5, an element $\Sigma\in\Gamma^s_{n+2}(\mathscr{F}_\Theta)$, where $\mathscr{F}_\Theta$ is the natural map $Z\pi_1 E_\Theta\to Z\pi_1 W$. Using the result from Theorem 3 that $j_*:\Gamma^s_{n+3}(\mathscr{F}_\Theta)\to L^s_{n+3}(\pi_1 W)$ is surjective, one shows that the vanishing of Σ implies the existence of the desired P.L. embedding (actually with any regular neighborhood with the correct normal invariant and associated bundle). This is proven using same argument as for the proof of Lemma 5.5d of [CS2]. Theorem 3 applies in present case because, by Theorem 1 (for a Poincaré embedding) and the hypothesis $A_\Theta=0$, the kernel of $\pi_1 E_\Theta\to\pi_1 W$ has trivial abelianization.

However, lemma 5.5a of [CS2] is also valid in the present situation, by the same proof. Thus $j_*\Sigma=0$. Hence, by theorem 3 again, $\Sigma=0$. Therefore the desired P.L. embedding exists. If Θ was only a h-Poincaré embedding to begin with, this argument only leads to an embedding of M in a manifold h-cobordant to W. As in [CS2], one then uses the same type argument as in 8.1 of [CS1] to correct the torsion, in the complement of M, to obtain the desired embedding of M in (a manifold s-cobordant to) W.

Finally, to prove Theorem 5, recall that in 6.1 of [CS2] we saw that, under the hypotheses of Theorem 5, there is an h-Poincaré embedding Θ of M in W whose underlying map $M\to W$ is precisely h. Further, the natural map $j_*:\pi_1(E_\Theta)\to\pi_1(W)$ is in this case just the composition with $(h_*)^{-1}$ on π_1 of $p_*:\pi_1(S(\xi))\to\pi_1 M$, p the projection of an orientable circle bundle over M with Euler class

$\chi(W) \in H^2(M)$. In particular, the kernel K of i_* is a cyclic group. Therefore, by Theorem 1,

$$K = K_{ab} = Z/\{x \cdot h_*[M]/x \in \pi_2 W\}.$$

For $\bar{x} \in H_2 W$, the evaluation $\langle \chi(W) m \bar{x} \rangle$ is just the intersection number $\bar{x} \circ h_*[M]$. Hence, if $\chi(W)$ is primitive, it follows by universal coefficients that $\exists \bar{x}$ with $\bar{x} \cdot h_*[M] = 1$. Since $H_2(\pi_1 W) = 0$ by assumption, $\bar{x}$ is the image of $x \in \pi_2 W$ under the Hurewicz map. Thus $K = 0$, and the result now follows from Theorem 5.3 of [CS2].

Note: Under the hypotheses of Theorem 5, one can use the methods of [CS2] to try to find a locally flat spine of W; i.e. a locally flat embedding $h': M' \to W$ that is a homotopy equivalence. It can be shown that it is possible if and only if the normal invariant $\eta(\Theta)$ in $[M; G/PL]$ has trivial surgery obstruction in $L_n(\pi_1 M)$ (Compare [CS2, h. 2]). If $\eta(\Theta)$ has trivial surgery obstruction in the reduced Wall group $\tilde{L}_n(\pi_1 M)$, then there exists an almost locally flat spine (i.e. locally flat except at one point.) This assertions can also be derived from Theorem 2, and should be compared to the results of [CS3] on the existence of totally spinless manifolds, as well as [Ma] and the exposition [Sh2]. For the case $\pi_1 M = \{e\}$, results on the existence of locally flat spines were obtained in [KM], by different methods.

Appendix

The next result was given without proof in [CS2]. It is proven using Theorem 1.

THEOREM (5.1 of [CS2]). *Let Θ be an (oriented) Poincaré or h-Poincaré embedding of $(M^n, \partial M)$ in $(W^{n+2}, \partial W)$, W connected, $M = M_1 \cup \cdots \cup M_t$, M_i connected. Then there is an (oriented) cyclic Poincaré or h-Poincaré embedding of $(M, \partial M)$ in $(W, \partial W)$, Θ' say, and a map $\Theta \to \Theta'$, if and only if*

$$A_\Theta = \bigoplus_{i=1}^{t} (Z_{\Theta_i} \underset{Z\pi_1 M_i}{\otimes} Z\pi_1 W)/\pi_2 W \cdot [M_i])$$

is a cyclic abelian group.

Recall that Θ' is be definition [CS2, §5] cyclic if and only if the kernel of $\pi_1(E_{\Theta'}) \to \pi_1(W)$, $E_{\Theta'}$ the Poincaré "complement", is cyclic. Also, see [CS2, §5] for the definition of a map $\Theta \to \Theta'$.

To prove this result, let K be the kernel of $\pi_1 E\Theta \to \pi_1 W$. A map $\Theta \to \Theta'$, if it exists, restricts to a homology equivalence $E_\Theta \to E_{\Theta'}$ with coefficients in $Z\pi_1 W$. Therefore

$$K_{ab} = H_1(E_\Theta, Z\pi_1 W) = H_1(E_{\Theta'}, Z\pi_1 W) = K'_{ab}.$$

Therefore, Theorem 1, applied to Poincaré embeddings shows that a cyclic Θ' and a map $\Theta \to \Theta'$ exist only if A_Θ is cyclic.

Suppose that A_Θ is cyclic. Let $G = \pi_1 E_\Theta$. The sequence

$$1 \to K_{ab} \to G/[K, K] \to G/K = \pi_1 W \to 1$$

is exact. (That $G \to \pi_1 W$ is surjective follows either by general position or an argument using Van Kampen's theorem and the exact homotopy sequence of a circle bundle.) Hence, as K_{ab} is cyclic by Theorem 1 and the hypothesis on A_Θ, $G/[K, K]$ is finitely presented. Therefore $[K, K]$ is normally generated by a finite number of elements, $\alpha_1, \ldots, \alpha_r$ say.

Let $Y \supset E_\Theta$ be obtained by attaching 2-cells along circles representing $\alpha_1, \ldots, \alpha_r$. Then we may identify $\pi_1 Y$ with $G/[K, K]$. Let $\pi = G/K$. We have the exact sequence

$$H_i(E_\Theta; Z\pi) \to H_i(Y, Z\pi) \to H_i(Y, E_\Theta; Z\pi) \to H_{i-1}(E_\Theta; Z\pi),$$

and

$$H_i(Y; E_\Theta; Z\pi) = \begin{cases} 0, & i \neq 2 \\ Z\pi \oplus \cdots \oplus Z\pi \ (r \text{ summands}), & i = 2. \end{cases}$$

Further, the connecting homomorphism

$$H_2(Y, E_\Theta; Z\pi) \to H_1(E_\Theta; Z\pi) = K_{ab}$$

is trivial. Hence

$$H_2(Y; Z_\pi) \cong H_2(E_\Theta; Z\pi) + (Z\pi)^r,$$

and the inclusion $E_\Theta \subset Y$ induces isomorphisms of homology groups in other dimensions over $Z\pi$, and is in fact a simple equivalence over $Z\pi$ with respect to the basis of $H_2(Y; E_\Theta; Z\pi)$ determined by the 2-cells attached.

We assert that the Hurewicz homomorphism $\pi_2 Y \to H_2(Y; Z\pi)$ is surjective. In fact $H_2(Y; Z\pi) = H_2\tilde{Y}$, $\tilde{Y}$ the covering space of Y with $\pi_1\tilde{Y} = K_{ab}$, and one has the (Hopf) exact sequence

$$\pi_2 Y = \pi_2\tilde{Y} \to H_2\tilde{Y} \to H_2(K_{ab}, Z) = 0.$$

Now let $E_{\Theta'} \supset Y$ be obtained by attaching 3 cells along two-spheres representing the elements of the obvious basis of $Z\pi \oplus \cdots \oplus Z\pi$. The inclusion $E_\Theta \to E_{\Theta'}$ will be a (simple) homology equivalence over $Z\pi$, $\pi_1 E_\Theta \to \pi_1 W$ is just $G/[K, K] \to \pi_1 W$, with kernel K_{ab}. Let $F_{\Theta'} = F_\Theta \subset E_{\Theta'}$ (this is the Poincaré complement of ∂M in ∂W; see [CS2]) and let $h_{\Theta'}$ be the composite of $h_\Theta: W \to E(\xi_\Theta) \cup_{S(\xi_\Theta)} E_\Theta$ and the union of the identity on $E(\xi_\Theta)$ and the inclusion $E_\Theta \subset E_{\Theta'}$. Let $\xi_{\Theta'} = \xi_\Theta$. Then $h_{\Theta'}$ is a (simple) homotopy equivalence, and it is not hard to see that $\Theta' = (\xi_{\Theta'}, (E_{\Theta'}, F_{\Theta'}), h_{\Theta'})$ is a cyclic Poincaré or h-Poincaré embedding, as appropriate, with a map $\Theta \to \Theta'$ defined by the inclusion $E_\Theta \subset E_{\Theta'}$.

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Courant Institute of Mathematical Sciences,
Rutgers University

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