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Zeta Functions and Their Asymptotic Expansions for Compact Symmetric Spaces of Rank One

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§0. Introduction

In this paper we apply É. Cartan's theory of class 1 representations [3] to derive explicit formulae for the ζ -functions of the compact riemannian symmetric spaces of strictly positive curvature. We then combine those formulae with an asymptotic expansion of Mulholland [5] and evaluate the coefficients in the Minakshisundaram asymptotic expansion (see [1]) of the ζ -function.

§1. Generalities on Compact Symmetric Spaces

We assemble the basic facts required to discuss ζ -functions of compact symmetric spaces from the representation-theoretic viewpoint. In principle, everything here in §1 is contained in Garth Warner's book [6], and we refer to Warner [6] and Helgason [4] for the original sources (of which Cartan [3] is the principal one).

Fix a compact riemannian symmetric space M and let G be the largest connected group of isometries. Thus G is a compact connected Lie group with an involutive automorphism σ , and $M = G/K$ where K is an open subgroup of $G^\sigma = \{g \in G : \sigma(g) = g\}$, and the riemannian metric on M derives from a positive definite invariant bilinear form on the Lie algebra of G .

\hat{G} denotes the set of all equivalence classes $[\pi]$ of irreducible unitary representations π of G . Given $[\pi]$, V_π denotes the (finite dimensional complex Hilbert) space on which π represents G . A class $[\pi] \in \hat{G}$ is of *class 1* relative to K if there exists

$$0 \neq v \in V_\pi \quad \text{such that} \quad \pi(k)v = v \quad \text{for all } k \in K,$$

that is if V_π has a nonzero K -fixed vector. Let us write

$$\hat{G}_K = \{[\pi] \in \hat{G} : [\pi] \text{ is of class 1 relative to } K\}. \tag{1.1}$$

G acts on $L_2(M)$ through its left regular representation, that is

$$[l(g)f](x) = f(g^{-1}x) \quad \text{for } f \in L_2(M), \quad g \in G \quad \text{and } x \in M = G/K.$$

This action decomposes over \hat{G}_K as follows.

1.2. THEOREM (É. Cartan [3]). $L_2(M) = \sum_{[\pi] \in \hat{G}_K} V_\pi$ as unitary left G -module.

Proof. $L_2(G) = \sum_G V_\pi \otimes V_{\pi^*}$ according to the Peter-Weyl Theorem. Here $V_\pi \otimes V_{\pi^*}$ is identified with the space of all matrix coefficient functions

$$f_{v,w}(g) = \langle v, \pi(g)w \rangle \quad \text{for } v, w \in V_\pi \quad \text{and } g \in G$$

of $[\pi]$. The left and right actions of G on $L_2(G)$ are

$$[l(g_1) \otimes r(g_2)f](x) = f(g_1^{-1}xg_2),$$

so the action on coefficients of $[\pi]$ is

$$\{l(g_1) \otimes r(g_2)\} f_{v,w} + f_{\pi(g_1)v, \pi(g_2)w},$$

which is $\pi \otimes \pi^*$.

View $L_2(M = G/K)$ as $\{f \in L_2(G) : f(gk) = f(g) \text{ for } g \in G \text{ and } k \in K\}$. Writing superscripts for invariants and 1_K for the trivial 1-dimensional representation of K , now

$$\begin{aligned} L_2(M) &= L_2(G)^{r(K)} = \sum_G V_\pi \otimes V_{\pi^*}^K \\ &= \sum_G \text{mult}(1_K, \pi^* |_K) V_\pi = \sum_G \text{mult}(1_K, \pi |_K) V_\pi \\ &= \sum_{G_K} \text{mult}(1_K, \pi |_K) V_\pi \end{aligned}$$

as unitary left G -module. The latter multiplicities all = 1; for example see Helgason [4, p. 408] for a proof of Gelfand's theorem that a certain algebra $C^*(G)$, which is W^* -dense in the commuting algebra of $l(G)$ on $L_2(M)$, is abelian. *q.e.d.*

Now let \mathfrak{g} denote the Lie algebra of G , \mathfrak{G} the universal enveloping algebra of \mathfrak{g} , and \mathfrak{Z} the center of \mathfrak{G} . Every class $[\pi] \in \hat{G}$ maps every element of \mathfrak{Z} to a scalar, giving an associative algebra homomorphism that we denote

$$\pi: \mathfrak{Z} \rightarrow \mathbb{C}, \quad \text{infinitesimal character of } [\pi].$$

Recall that the riemannian metric on M is derived from an invariant positive definite inner product on \mathfrak{g} . If $\{x_1, \dots, x_n\}$ is an orthonormal basis then $\sum x_i^2 \in \mathfrak{Z}$ and depends

only on the inner product, and as differential operator

$$-\sum x_i^2 = \Delta, \quad \text{the Laplace-Beltrami operator on } M. \quad (1.3)$$

If we use the negative of the Cartan-Killing form of \mathfrak{g} for the inner product, then $-\sum x_i^2 = \Omega$, the Casimir element of \mathfrak{G} , and so $\Delta = l(\Omega)$ on $L_2(M)$.

Define $\zeta_M(t) = \sum_{\lambda} e^{-\lambda t}$ where λ ranges over the eigenvalues (with multiplicity) of the Laplace-Beltrami operator (1.3). This is the trace of the heat kernel. The Minakshisundaram-Pleijel zeta function $\sum_{\lambda} \lambda^{-s}$ is related to ζ_M by a Mellin transform.

1.4. COROLLARY. *If the riemannian metric on M is defined by the negative of the Cartan-Killing form of \mathfrak{g} then M has ζ -function given by*

$$\zeta(t) = \sum_{[\pi] \in \hat{G}_K} (\text{degree of } \pi) e^{-t\pi(\Omega)}$$

where $\Omega \in \mathfrak{Z}$ is the Casimir element of \mathfrak{G} .

To specify the ζ -function of M we now have to describe \hat{G}_K , and specify degree (π) and $\pi(\Omega)$ for every class $[\pi] \in \hat{G}_K$.

The Lie algebra \mathfrak{g} decomposes under the automorphism σ as $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ where \mathfrak{k} is the (+1)-eigenspace and \mathfrak{s} is the (-1)-eigenspace. Of course, \mathfrak{k} is the Lie algebra of K . Choose

$$\mathfrak{a}: \text{maximal abelian subspace of } \mathfrak{s}, \quad (1.5a)$$

and

$$\Sigma_{\mathfrak{a}}^+ \text{ positive } \mathfrak{a}_{\mathbb{C}}\text{-root system on } \mathfrak{g}_{\mathbb{C}}. \quad (1.5b)$$

Define $\mathfrak{m}' = \{x \in \mathfrak{k}: [x, \mathfrak{a}] = 0\}$ and let \mathfrak{t} be a Cartan subalgebra of \mathfrak{m} . Then

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \text{ is a Cartan subalgebra of } \mathfrak{g}. \quad (1.6a)$$

Any choice of positive $\mathfrak{t}_{\mathbb{C}}$ -root system on $\mathfrak{m}_{\mathbb{C}}$ specifies a choice of

$$\begin{aligned} \Sigma^+ &: \text{positive } \mathfrak{h}_{\mathbb{C}}\text{-root system on } \mathfrak{g}_{\mathbb{C}} \text{ such that} \\ \Sigma_{\mathfrak{a}}^+ &= \{\phi \mid_{\mathfrak{a}}: \phi \in \Sigma^+ \text{ and } \phi \mid_{\mathfrak{a}} \neq 0\}. \end{aligned} \quad (1.6b)$$

Each class $[\pi] \in \hat{G}$ is specified by its highest weight relative to (\mathfrak{h}, Σ^+) , and the class 1 representations have a certain remarkable property.

1.7. THEOREM (É. Cartan [3]). *If $[\pi] \in \hat{G}_K$ has highest weight λ relative to (\mathfrak{h}, Σ^+) , then $\lambda(\mathfrak{t})=0$, that is $\lambda \in i\mathfrak{a}^*$.*

Proof. The noncompact dual $\tilde{\mathfrak{g}} = \mathfrak{k} + i\mathfrak{s}$ of \mathfrak{g} has Iwasawa decomposition $\tilde{\mathfrak{g}} = \mathfrak{n} + i\mathfrak{a} + \mathfrak{k}$ where \mathfrak{n} is the sum of its $\Sigma_{\mathfrak{a}}^+$ -negative ($i\mathfrak{a}$)-root spaces. Writing capital German letters for universal enveloping algebras of complexifications, now $\mathfrak{G} = \mathfrak{N}\mathfrak{A}\mathfrak{K}$. Decompose

$$V_{\pi} = \sum V_{\pi, \nu} \text{ sum of weight spaces.}$$

Let w be a nonzero K -fixed vector and decompose

$$w = \sum w_{\nu} \text{ where } w_{\nu} \in V_{\pi, \nu}.$$

Then

$$\begin{aligned} V_{\pi} &= \pi(\mathfrak{G}) w = \pi(\mathfrak{N}) \pi(\mathfrak{A}) \pi(\mathfrak{K}) w = \pi(\mathfrak{N}) \pi(\mathfrak{A}) w \\ &\subset \pi(\mathfrak{N}) \sum_{w_{\nu} \neq 0} V_{\pi, \nu} \subset \sum_{\mu \leq \nu} \sum_{w_{\nu} \neq 0} V_{\pi, \mu}. \end{aligned}$$

We conclude that $w_{\lambda} \neq 0$. As $\pi(\mathfrak{t}) w = 0$ now $\lambda(\mathfrak{t}) = 0$. As the weights are in $i\mathfrak{h}^*$ now $\lambda \in i\mathfrak{a}^*$. *q.e.d.*

1.8 THEOREM (É. Cartan [3]; S. Helgason [4], [7]). *Define*

$$\Lambda^+ = \{ \lambda \in i\mathfrak{a}^* : \langle \lambda, \psi \rangle / \langle \psi, \psi \rangle \text{ integer } \geq 0 \text{ for all } \psi \in \Sigma_{\mathfrak{a}}^+ \} \quad (1.9)$$

where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form. Then

$$[\pi] \rightarrow \text{highest weight relative to } (\mathfrak{h}, \Sigma_{\mathfrak{a}}^+)$$

is an injective map from \hat{G}_K into Λ^+ . If K is connected and G is simply connected then it is a bijection.

The proof is technical and we refer to Chapter III of Warner [6].

1.10. COROLLARY. *If M is simply connected and if its riemannian metric derives from the negative of the Cartan-Killing form of \mathfrak{g} , then M has ζ -function given by*

$$\zeta_M(t) = \sum_{\lambda \in \Lambda^+} P(\lambda) e^{-tq(\lambda)} \quad (1.11)$$

where

$$q = \frac{1}{2} \sum_{\phi \in \Sigma^+} \phi \text{ and } q_{\mathfrak{a}} = q|_{\mathfrak{a}}; \quad (1.12a)$$

$$P(\lambda) = \prod_{\phi \in \Sigma^+} \frac{\langle \lambda + \varrho, \phi \rangle}{\langle \varrho, \phi \rangle}; \quad (1.12b)$$

and

$$q(\lambda) = \|\lambda + \varrho\|^2 - \|\varrho\|^2 = \|\lambda + \varrho_\alpha\|^2 - \|\varrho_\alpha\|^2. \quad (1.12c)$$

Proof. Write $[\pi_\lambda]$ for the class with highest weight λ . Simple connectivity and Theorem 1.8 insure that

$$\hat{G}_K \ni [\pi_\lambda] \rightarrow \lambda \in \Lambda^+$$

is bijective. The Hermann Weyl Degree Formula says that $[\pi_\lambda]$ has degree $P(\lambda)$ as in (1.12), and it is standard that π_λ acts on the Casimir element by

$$\pi_\lambda(\Omega) = \|\lambda + \varrho\|^2 - \|\varrho\|^2 \quad \text{for all } [\pi_\lambda] \in \hat{G}.$$

Here $\varrho = \varrho_\alpha + \varrho_t$ with $\langle \lambda, \varrho_t \rangle = 0$ by Theorem 1.7, so also

$$\pi_\lambda(\Omega) = \|\lambda + \varrho_\alpha\|^2 - \|\varrho_\alpha\|^2.$$

Now $\pi_\lambda(\Omega) = q(\lambda)$ as in (1.12) and our formula for $\zeta(t)$ follows from Corollary 1.4. *q.e.d.*

In the sequel we will explicitly calculate the ingredients (1.12) for symmetric spaces of rank 1 (that is, where $\dim \alpha = 1$), obtaining explicit formulae for their ζ -functions, and then study the asymptotic behaviour of these ζ -functions.

§2. Odd Dimensional Spheres and Real Projective Spaces

We work out explicit formulae for the ζ -functions of the spheres and real projective spaces of odd dimension $2n - 1$,

$$S^{2n-1} = SO(2n)/SO(2n-1), \quad n \geq 1, \quad (2.1a)$$

and

$$P^{2n-1}(\mathbb{R}) = S^{2n-1}/\{\pm I\} = SO(2n)/O(2n-1). \quad (2.1b)$$

If $n = 1$, both are circles $S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \text{ real}\}$. $L_2(S^1) = \sum_{-\infty}^{\infty} V_m$ where V_m is the 1-dimensional span of

$$f_m(e^{i\theta}) = e^{im\theta}, \quad m \text{ integer.}$$

Normalize the riemannian metric so that the circle has length l . Then the metric is $ds^2 = (l/2\pi)^2 d\theta^2$, so the circle has Laplace-Beltrami operator

$$\Delta = -(2\pi/l)^2 \frac{\partial^2}{\partial \theta^2} : f_m \mapsto (2\pi/l)^2 m^2 f_m.$$

We conclude that the circle of length l has ζ -function

$$\zeta_{S^1}(t) = 1 + 2 \sum_{m=1}^{\infty} e^{-t(2\pi m/l)^2} \quad (2.2)$$

If $n=2$ then $G=SO(4)$ has Dynkin diagram $D_2: \underset{\alpha_1}{\circ} \underset{\alpha_2}{\circ}$. Then $\Sigma^+ = \{\alpha_1, \alpha_2\}$ and $\Sigma_{\mathfrak{a}}^+ = \{\alpha\}$ where $\varrho = \varrho_{\mathfrak{a}} = \alpha = \frac{1}{2}(\alpha_1 + \alpha_2)$, so $\Lambda^+ = \{m\alpha : m \geq 0 \text{ integer}\}$ and $q(m\alpha) = \|m\alpha + \varrho_{\mathfrak{a}}\|^2 - \|\varrho_{\mathfrak{a}}\|^2 = (m^2 + 2m) \|\varrho_{\mathfrak{a}}\|^2$ and we calculate

$$P(m\alpha) = \frac{\langle m\alpha + \varrho, \alpha_1 \rangle}{\langle \varrho, \alpha_1 \rangle} \cdot \frac{\langle m\alpha + \varrho, \alpha_2 \rangle}{\langle \varrho, \alpha_2 \rangle} = (m+1)^2.$$

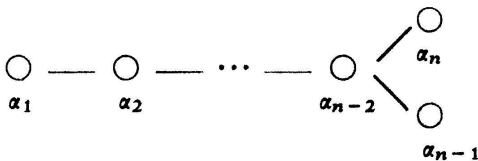
Using the negative of the Killing form to specify the riemannian metrics of S^3 and $P^3(\mathbb{R})$, the tables at the end of Bourbaki [2] show $\langle \alpha_i, \alpha_i \rangle = \frac{1}{2}$, so $\|\varrho_{\mathfrak{a}}\|^2 = \frac{1}{4} \langle \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 \rangle = \frac{1}{4}$, and

$$\zeta_{S^3}(t) = \sum_{m=0}^{\infty} (m+1)^2 e^{-t(m^2+2m)/4}. \quad (2.3a)$$

It is classical that $\pi_{m\alpha}(-I) = 1$ just when m is even, so also

$$\zeta_{P^3(\mathbb{R})}(t) = \sum_{r=0}^{\infty} (2r+1)^2 e^{-t(r^2+r)}. \quad (2.3b)$$

Now we assume $n \geq 3$ in (2.1) so that $G=SO(2n)$ is a simple group of type D_n , and denote its Dynkin diagram



with

$$\Sigma_{\mathfrak{a}}^+ = \{\alpha\}, \quad \alpha_i|_{\mathfrak{a}} = \alpha \quad \text{and} \quad \alpha_i|_{\mathfrak{a}} = 0 \quad \text{for } i > 1.$$

Relative to an appropriate positive multiple of the Cartan-Killing form, $i\mathfrak{h}^*$ has orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ such that

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } 1 \leq i < n \quad \text{and} \quad \alpha_n = \varepsilon_{n-1} + \varepsilon_n.$$

Thus Σ^+ consists of the roots $\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq n$, and so $\alpha = \varepsilon_1$ and $\varepsilon_1 \pm \varepsilon_j$ ($1 < j \leq n$) are the roots that restrict to α .

Now

$$\Lambda^+ = \{m\varepsilon_1 : m \geq 0 \text{ integer}\}, \quad \varrho_\alpha = (n-1)\varepsilon_1 \quad \text{and} \quad \varrho = \sum_{j=1}^{n-1} (n-j)\varepsilon_j.$$

If $1 \leq i < j \leq n$ then $\langle \varrho, \varepsilon_i \pm \varepsilon_j \rangle = \{(n-i) \pm (n-j)\} \|\varepsilon_1\|^2$, so

$$\frac{\langle m\varepsilon_1 + \varrho, \varepsilon_i \pm \varepsilon_j \rangle}{\langle \varrho, \varepsilon_i \pm \varepsilon_j \rangle} = 1 \quad \text{if } i > 1, = \frac{m + (n-1) \pm (n-j)}{(n-1) \pm (n-j)} \quad \text{if } i = 1.$$

That gives us

$$P(m\varepsilon_1) = \prod_{j=2}^n \frac{m + 2n - j - 1}{2n - j - 1} \cdot \frac{m - 1 + j}{j - 1} = \frac{m + n - 1}{n - 1} \prod_{k=1}^{2n-3} \frac{m + k}{k}.$$

Recall from the tables at the end of Bourbaki [2] that $\|\varepsilon_1\|^2 = 1/4(n-1)$. Now

$$q(m\varepsilon_1) = \|m\varepsilon_1 + \varrho_\alpha\|^2 - \|\varrho_\alpha\|^2 = \{m^2 + 2m(n-1)\}/4(n-1).$$

Now Corollary 1.10 gives us

2.4. THEOREM. *Let S^{2n-1} denote the sphere of odd dimension $2n-1$, $n \geq 3$, with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n)$. It has ζ -function*

$$\zeta_{S^{2n-1}}(t) = \sum_{m=0}^{\infty} \left\{ \frac{m + n - 1}{n - 1} \cdot \prod_{k=1}^{2n-3} \frac{m + k}{k} \right\} e^{-t \{m^2 + 2m(n-1)\}/4(n-1)}. \quad (2.5)$$

The real projective space $P^{2n-1}(R) = S^{2n-1}/\{\pm I\}$ has ζ -function given by summing the summands of (2.5) whose representations $[\pi_{m\varepsilon_1}]$ occur in $L_2(P^{2n-1}(R))$, that is the ones with a vector fixed under the subgroup $SO(2n-1) \cup (-I_{2n}) \cdot SO(2n-1)$. These are the $[\pi_{m\varepsilon_1}]$ whose kernel contains $-I_{2n}$, which are easily seen to be the ones for which m is even.

2.6. COROLLARY. *Let $P^{2n-1}(R)$ denote the real projective space of odd dimension $2n-1$, $n \geq 3$, with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n)$. It has ζ -function*

$$\zeta_{P^{2n-1}(R)}(t) = \sum_{r=0}^{\infty} \left\{ \frac{2r+n-1}{n-1} \cdot \prod_{k=1}^{2n-3} \frac{2r+k}{k} \right\} e^{-t \{r^2+r(n-1)\}/(n-1)}. \quad (2.7)$$

§3. Even Dimensional Spheres and Real Projective Spaces

We work out explicit formulae for the ζ -functions of the spheres and real projective spaces of even dimension $2n$,

$$S^{2n} = SO(2n+1)/SO(2n), \quad n \geq 1, \quad (3.1a)$$

and

$$P^{2n}(R) = S^{2n}/\{\pm I\} = SO(2n+1)/SO(2n) \times O(1). \quad (3.1b)$$

$G = SO(2n+1)$ has Dynkin diagram

$$B_n: \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{=} & \bullet \\ \alpha_1 & & \alpha_2 & & & & \alpha_{n-1} & & \alpha_n \end{array}$$

with

$$\Sigma_{\mathfrak{a}}^+ = \{\alpha\}, \quad \alpha_1|_{\mathfrak{a}} = \alpha \quad \text{and} \quad \alpha_i|_{\mathfrak{a}} = 0 \quad \text{for } i > 1.$$

Arguing as in §2 one proves

3.2. THEOREM. *Let S^{2n} denote the sphere of even dimension $2n$ with riemannian metric of constant positive curvature induced by the negative of the Cartan-Killing form of $SO(2n+1)$. It has ζ -function*

$$\zeta_{S^{2n}}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{m+k}{k} \right\} e^{-t \{m^2+m(2n-1)\}/(4n-2)}. \quad (3.3)$$

and

3.4. COROLLARY. *Let $P^{2n}(R)$ denote the real projective space of even dimension $2n$ with riemannian metric of constant positive curvature induced by the negative of the*

Cartan-Killing form $SO(2n+1)$. It has ζ -function

$$\zeta_{P^{2n}(\mathbb{R})}(t) = \sum_{r=0}^{\infty} \left\{ \frac{4r+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{2r+k}{k} \right\} e^{-t \{2r^2+r(2n-1)\}/(2n-1)}. \quad (3.5)$$

§4. Complex Projective Spaces

We state the formula for the ζ -function of the complex projective spaces

$$P^n(\mathbb{C}) = U(n+1)/U(n) \times U(1) = SU(n+1)/S(U(n) \times U(1)) \quad (4.1)$$

of complex dimension n , real dimension $2n$. Since $P^1(\mathbb{C})$ is the sphere S^2 , already considered in §3, we will work under the hypothesis $n > 1$. Then a glance at the case $n=1$ of (3.3) will show our conclusion valid in general.

$G = SU(n+1)/\{e^{2\pi ik/(n+1)}I\}$ has Dynkin diagram

$$A_n: \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ \\ \alpha_1 & & \alpha_2 & & & & \alpha_n \end{array}$$

with

$$\Sigma^+ = \{\alpha, 2\alpha\}, \quad \alpha_1|_{\alpha} = \alpha = \alpha_n|_{\alpha}, \quad \alpha_i|_{\alpha} = 0 \quad \text{for } 1 < i < n.$$

Arguing as before and using the case $n=1$ of Theorem 3.2,

4.2. THEOREM. *Let $P^n(\mathbb{C})$ denote the complex projective n -space with riemannian metric induced by the negative of the Cartan-Killing form of $SU(n+1)$. It has ζ -function*

$$\zeta_{P^n(\mathbb{C})}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+n}{n} \prod_{k=0}^{n-1} \left(\frac{m+k}{k} \right)^2 \right\} e^{-t \{m^2+mn\}/(n+1)} \quad (4.3)$$

§5. Quaternionic Projective Spaces

Here is the formula for the ζ -functions of the quaternionic projective spaces

$$P^{n-1}(\mathbb{Q}) = Sp(n)/Sp(n-1) \times Sp(1), \quad n \geq 2, \quad (5.1)$$

of real dimension $4(n-1)$. Here note that $P^1(\mathbb{Q}) = S^4$.

$G = Sp(n)/\{\pm I\}$ has Dynkin diagram

$$C_n: \begin{array}{ccccccc} \bullet & \text{---} & \bullet & \text{---} & \cdots & \text{---} & \bullet \\ \alpha_1 & & \alpha_2 & & & & \alpha_{n-1} \end{array} \quad \text{---} \quad \circ \quad \alpha_n$$

with

$$\Sigma_{\mathfrak{a}}^+ = \{\alpha, 2\alpha\}, \quad \alpha_2|_{\mathfrak{a}} = \alpha, \alpha_i|_{\mathfrak{a}} = 0 \quad \text{for } i \neq 2.$$

An argument similar to that of §2 gives

5.2. THEOREM. *Let $P^{n-1}(Q)$ denote the quaternionic projective $n-1$ space, with riemannian metric induced by the negative of the Cartan-Killing form of $Sp(n)$. It has ζ -function*

$$\zeta_{P^{n-1}(Q)}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+2n-1}{2n-1} \cdot \prod_{r=2}^{2n-2} \frac{m+r}{r} \cdot \prod_{s=1}^{2n-3} \frac{m+s}{s} \right\} e^{-t(m^2+2mn-m)/2(n+1)}.$$

Notice that the case $n=2$ is $P^1(Q) = S^4$, where both (3.3) and (5.3) provide the same ζ -function

$$\zeta_{S^4}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+3}{3} \cdot \frac{m+2}{2} \cdot \frac{m+1}{1} \right\} e^{-t(m^2+3m)/6} \quad (5.4)$$

§6. The Cayley Projective Plane

Finally, we work out the ζ -function for the Cayley projective plane

$$P^2(\text{Cay}) = F_4/\text{Spin}(9), \quad \text{real dimension } 16. \quad (6.1)$$

$G = F_4$ has Dynkin diagram



with $\Sigma^+ = \{\alpha, 2\alpha\}$ where $\alpha_4|_{\mathfrak{a}} = \alpha$ and the other three $\alpha_i|_{\mathfrak{a}} = 0$. Relative to an appropriate multiple of the Cartan-Killing form, $i\mathfrak{h}^*$ has orthonormal basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ with

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4 \quad \text{and} \quad \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4).$$

Thus Σ^+ consists of the roots

$$\varepsilon_i (1 \leq i \leq 4), \varepsilon_i \pm \varepsilon_j (1 \leq i < j \leq 4), \quad \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4).$$

Now $\alpha = \frac{1}{2}\varepsilon_1$ and the $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ are the roots restricting to α , $2\alpha = \varepsilon_1$ and the roots restricting to it are $\varepsilon_1, \varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4$.

Thus

$$\Lambda^+ = \{m\varepsilon_1 : m \geq 0 \text{ integer}\}, \quad \varrho_\alpha = 11\alpha = \frac{1}{2}11\varepsilon_1, \quad \varrho = \frac{1}{2}(11\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4).$$

Now calculate

$$P(m\varepsilon_1) = \frac{2m+11}{11} \cdot \prod_{q=1}^3 \frac{m+q}{q} \prod_{r=4}^7 \left(\frac{m+r}{r}\right)^2 \cdot \prod_{s=8}^{16} \frac{m+s}{s}.$$

From the tables at the end of Bourbaki [2], $\|\varepsilon_1\|^2 = 1/18$, so $\|\alpha\|^2 = 1/72$, and thus

$$q(m\varepsilon_1) = \|(2m+11)\alpha\|^2 - \|11\alpha\|^2 = (m^2 + 11m)/18.$$

Now Corollary 1.10 says

6.2. THEOREM. *Let P^2 (Cay) denote the Cayley projective plane with riemannian metric induced by the negative of the Cartan-Killing form of F_4 . It has ζ -function*

$$\zeta_{P^2(\text{Cay})}(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+11}{11} \cdot \prod_{q=1}^3 \frac{m+q}{q} \cdot \prod_{r=4}^7 \left(\frac{m+r}{r}\right)^2 \cdot \prod_{s=8}^{10} \frac{m+s}{s} \right\} e^{-t(m^2+11m)/18}. \quad (6.3)$$

§ 7. The Asymptotic Expansion for Compact Riemannian Manifolds

We wish to give a brief account of the properties of the eigenvalues of the Laplacian, Δ , of a compact riemannian manifold. We will assume (M, g) is a compact riemannian manifold without boundary of dimension d . Then Δ will be a self-adjoint elliptic operator with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$. It is known that these eigenvalues contain a great deal of geometric information about (M, g) and a tool to recover some of this information is the zeta-function, $\zeta_M(t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}$. The interest of the zeta-function comes from the following theorem.

7.1. THEOREM (Minakshisundaram). *If (M, g) is a compact riemannian manifold without boundary of dimension d then there exist constants $a_n, n \geq 0$, such that*

$$\zeta_M(t) = (4\pi t)^{-d/2} (a_0 + a_1 t + \dots + a_k t^k + o(t^{k+1}))$$

as $t \downarrow 0$.

Proof. See Berger [1].

In the remainder of this paper we will compute the coefficients, a_m , of the asymptotic expansion of $\zeta_M(t)$ when M is a symmetric space of rank one.

§8. Summation Lemmas

We wish to analyze the zeta-functions derived in the first part of this paper using certain classical summation formulas. The formulas we need will be gathered together in this section.

8.1. LEMMA. Let $f(t) = \sum_{s \in \mathbf{Z}} e^{-s^2 t}$. Then $f(t) = \pi^{1/2} t^{-1/2} + O(e^{-1/t})$ as $t \downarrow 0$ and $(-1)^k f^{(k)} = \sum_{s \in \mathbf{Z}} s^{2k} e^{-s^2 t} = \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \cdots (2k-1)/2 \pi^{1/2} t^{-(2k+1)/2} + O(e^{-1/t})$ as $t \downarrow 0$.

Proof. If $r(x) = e^{-x^2 t}$ we may apply the Poisson summation formula to derive

$$\sum_{s \in \mathbf{Z}} e^{-s^2 t} = \pi^{1/2} t^{-1/2} \sum_{s \in \mathbf{Z}} e^{-\pi^2 s^2 / t}. \quad (8.2)$$

The first part of the Lemma now follows by noting that $\sum_{s \neq 0} e^{-\pi^2 s^2 / t}$ is $O(e^{-1/t})$. We may now take derivatives with respect to t to derive the second formula. *q.e.d.*

For the remainder of the paper we will define $b_0 = 1$, $b_k = \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \cdots ((2k-1)/2)$, $k \geq 1$. Note $\pi^{1/2} b_k = \Gamma((2k+1)/2)$.

8.3. LEMMA. Let $g(t) = \sum_{j=0}^{\infty} (2j+1) e^{-(j+1/2)^2 t}$. Then

$$g(t) = \frac{1}{t} + c_0 + c_1 t + \frac{c_2}{2!} t^2 + \cdots + \frac{c_n}{n!} t^n + O(t^{n+1})$$

and

$$g^{(k)}(t) = \frac{(-1)^k k!}{t^{k+1}} + c_k + c_{k+1} t + \cdots + \frac{c_{k+n}}{n!} t^n + O(t^{n+1})$$

as $t \downarrow 0$, where

$$c_n = \frac{(-1)^n}{(n+1)} B_{2n+2} (1 - 2^{-2n-1}),$$

B_n is the n th Bernoulli number.

Proof. See Mulholland [5].

Before proceeding we wish to note that $\sum_{n=0}^{\infty} (c_n/n!) t^n$ is not convergent. Therefore Lemma 8.3 gives an asymptotic series.

8.4. LEMMA. Let $g_1(t) = \sum_{j=0}^{\infty} (4j+1) e^{-(2j+1/2)^2 t}$ and $g_2(t) = \sum_{j=0}^{\infty} (4j+3) e^{-(2j+3/2)^2 t}$. Then $g_1(t) + g_2(t) = g(t)$ and

$$g_i^{(k)}(t) = \frac{1}{2} \left(\frac{1}{t} + c_0 + c_1 t + \cdots + \frac{c_n}{n!} t^n + o(t^{n+1}) \right) \quad i=1, 2$$

$$g_i^{(k)}(t) = \frac{1}{2} \left(\frac{(-1)^k k!}{t^{k+1}} + c_k + c_{k+1} t + \cdots + \frac{c_{k+n}}{n!} t^n + o(t^{n+1}) \right) \quad i=1, 2$$

as $t \downarrow 0$.

Proof. See Mulholland [5].

The last lemma of this section is similar to Lemma 8.3.

8.5. LEMMA. Let $h(t) = \sum_{j=0}^{\infty} 2j e^{-j^2 t}$. Then

$$h(t) = \frac{1}{t} + d_0 + d_1 t + \frac{d_2}{2!} t^2 + \cdots + \frac{d_n}{n!} t^n + o(t^{n+1})$$

and

$$h^{(k)}(t) = \frac{(-1)^k k!}{t^{k+1}} + d_k + d_{k+1} t + \cdots + \frac{d_{n+k}}{n!} t^n + o(t^{n+1})$$

as $t \downarrow 0$ with $d_n = [(-1)^n / (n+1)] B_{2n+2}$.

Proof. A slight modification of Mulholland's method gives the desired result.

§9. The Asymptotic Expansion for Odd Dimensional Spheres and Real Projective Spaces

Starting in this section we will analyze the zeta-functions developed in §1 through §6 using the results in the previous section. The goal of this section is to calculate the coefficients, a_n , for the symmetric spaces analyzed in §2.

$M = S^1$. $\zeta_M(t) = f((4\pi^2/l^2)t)$. Consequently

$$\begin{aligned} \zeta_M(t) &= \pi^{1/2} \left(\frac{4\pi^2}{l^2} t \right)^{-1/2} + o(e^{-1/t}) \\ &= l(4\pi t)^{-1/2} + o(e^{-1/t}). \end{aligned}$$

Thus we conclude $a_0 = l$ and $a_m = 0$, $m \geq 1$.

$$M = S^3. \quad \zeta_M(t) = \sum_{m=0}^{\infty} (m+1)^2 e^{-t(m^2+2m)/4}$$

$$\begin{aligned}
&= \sum_{p=0}^{\infty} p^2 e^{-t(p^2-1)/4} \quad (\text{where } p=m+1) \\
&= \frac{1}{2} e^{t/4} \sum_{p=-\infty}^{\infty} p^2 e^{-p^2 t/4} \\
&= \frac{1}{2} e^{t/4} (-1) f' \left(\frac{t}{4} \right) \\
&= 2\pi^{1/2} t^{-3/2} e^{t/4} + ES = \frac{16\pi^2 e^{t/4}}{(4\pi t)^{3/2}} + ES,
\end{aligned}$$

therefore $a_m = 16\pi^2/4^m m!$. ES is an error which is exponentially small as $t \downarrow 0$.

$M = P^3(\mathbf{R})$.

$$\begin{aligned}
\zeta_M(t) &= \sum_{r=0}^{\infty} (2r+1)^2 e^{-(r^2+r)t} \\
&= \sum_{r=0}^{\infty} (2r+1)^2 e^{-[(2r+1)^2-1]t/4} \\
&= \frac{1}{2} e^{t/4} \left[\sum_{s \in \mathbf{Z}} s^2 e^{-s^2 t/4} - \sum_{s \in 2\mathbf{Z}} s^2 e^{-s^2 t/4} \right] \\
&= \frac{1}{2} e^{t/4} \left[(-1) f' \left(\frac{t}{4} \right) + 4f'(t) \right] \\
&= \frac{\pi^{1/2}}{2} e^{t/4} [4t^{-3/2} - 2t^{-3/2}] + ES \\
&= \pi^{1/2} t^{-3/2} e^{t/4} + ES = \frac{8\pi^2 e^{t/4}}{(4\pi t)^{3/2}} + ES
\end{aligned}$$

so $a_m = 8\pi^2/4^m m!$.

$M = S^{2a-1}$, $n \geq 3$. At this point we wish to make a general comment about the procedure we will employ in this and following sections. Though we have established a bijection between the representations of \hat{G}_K and functionals $\lambda \in \mathcal{A}^+$ the proper variable to use is not λ but $\lambda + \varrho_\alpha$. This is the guiding principle behind all changes of variable which are used in this and following sections.

$$\zeta_M(t) = \sum_{m=0}^{\infty} \left\{ \frac{m+n-1}{n-1} \prod_{k=0}^{2n-3} \frac{m+k}{k} \right\} e^{-t\{m^2+2m(n-1)\}/4(n-1)}.$$

We will let $s = m + n - 1$. Then

$$\begin{aligned}
\zeta_M(t) &= \frac{1}{(n-1)(2n-3)!} \sum_{s=n-1}^{\infty} \left\{ \prod_{j=0}^{n-2} (s^2 - j^2) \right\} e^{-\{s^2 - (n-1)^2\}t/4(n-1)} \\
&= \frac{e^{((n-1)/4)t}}{2(n-1)(2n-3)!} \sum_{s \in \mathbf{Z}} \left\{ \prod_{j=0}^{n-2} (s^2 - j^2) \right\} e^{-s^2 t/4(n-1)}
\end{aligned} \tag{9.1}$$

We now define $\alpha_{k,n}$ by

$$\prod_{j=0}^{n-2} (s^2 - j^2) = \sum_{k=0}^{n-1} \alpha_{k,n} s^{2k}.$$

Then

$$\begin{aligned}
\zeta_M(t) &= \frac{e^{(n-1)t/4}}{(2n-2)!} \sum_{s \in \mathbf{Z}} \sum_{k=0}^{n-1} \alpha_{k,n} s^{2k} e^{-s^2(t/4(n-1))} \\
&= \frac{e^{(n-1)t/4}}{(2n-2)!} \sum_{k=0}^{n-1} \alpha_{k,n} (-1)^k f^{(k)}\left(\frac{t}{4(n-1)}\right) \\
&= \frac{\pi^{1/2} e^{(n-1)t/4}}{(2n-2)!} \sum_{k=0}^{n-1} \alpha_{k,n} b_k \left(\frac{t}{4(n-1)}\right)^{-(2k+1)/2} + ES.
\end{aligned}$$

Now by convolving the series we conclude that

$$a_m = \frac{2^{2n-1} \pi^n}{(2n-2)!} \sum_{k=0}^m \frac{(n-1)^{n-1/2}}{k!} \alpha_{n-1-k,n} b_{n-1-k} 4^{n-1/2-2k} \quad \text{if } m < n$$

and

$$a_m = \frac{2^{2n-1} \pi^n}{(2n-2)!} \sum_{k=m-n}^m \frac{(n-1)^{m-1/2}}{k!} \alpha_{m-k-1,n} b_{m-k-1} 4^{m-1/2-2k}$$

if $m \geq n$.

$M = P^{2n-1}(\mathbf{R})$. To compute the asymptotic expansion for the projective spaces we take $s = r + (n-1)/2$. Then

$$\begin{aligned}
\frac{2r+n-1}{n-1} \prod_{j=1}^{2n-3} \frac{2r+k}{k} &= \frac{2}{(2n-2)!} \prod_{j=0}^{n-2} 4(s^2 - (j/2)^2) = \frac{4^{n-1/2}}{(2n-2)!} \sum_{j=0}^{n-1} \alpha'_{j,n} s^{2j}. \\
\zeta_M(t) &= \frac{4^{n-1}}{(2n-2)!} e^{(n-1)t/4} \sum_{s \in 1/2\mathbf{Z}} \sum_{j=0}^{n-1} \alpha'_{j,n} s^{2j} e^{-s^2 t/n-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{4^{n-1}}{(2n-2)!} e^{(n-1)t/4} \sum_{j=0}^{n-1} \alpha'_{j,n} 4^{-j} f^{(j)}\left(\frac{t}{4(n-1)}\right) \\
&= \frac{4^{n-1} \pi^{1/2}}{(2n-2)!} e^{(n-1)t/4} \sum_{j=0}^{n-1} \alpha'_{j,n} b_j 4^{-j} \left(\frac{t}{4(n-1)}\right)^{-(2j+1)/2} + ES.
\end{aligned}$$

Therefore

$$a_m = \frac{4^{n-1} \pi^n}{(2n-2)!} \sum_{j=0}^m \frac{(n-1)^{n-1/2}}{j!} \alpha'_{n-1-j,n} b_{n-1-j} 4^{-j-1/2} \quad \text{if } m < n$$

and

$$a_m = \frac{4^{n-1} \pi^n}{(2n-2)!} \sum_{j=m-n}^m \frac{(n-1)^{m-1/2}}{j!} \alpha'_{m-j-1,n} b_{m-j-1} 4^{-j-1/2} \quad \text{if } m \geq n.$$

§10. The Asymptotic Expansion for Even Dimensional Spheres and Real Projective Spaces

$M = S^{2n}$. As in the preceding section we will change variables to utilize the lemmas of §8. Recall that

$$\zeta_M(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{m+k}{k} \right\} e^{-t\{m^2+m(2n-1)\}/4n-2}.$$

We will let $s = m + (n-1/2)$. Then

$$\frac{2m+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{m+k}{k} = \frac{2s}{(2n-1)!} \prod_{j=1/2}^{n-3/2} (s^2 - j^2) = \frac{2s}{(2n-1)!} \sum_{j=0}^{n-1} \beta_{j,n} s^{2j},$$

where the product runs through the half-integers which are not integers. Also $t\{m^2+m(2n-1)\}/4n-2 = t\{s^2 - (n-1/2)^2\}/4n-2$. Therefore

$$\begin{aligned}
\zeta_M(t) &= \frac{e^{(n-1/2)t/4}}{(2n-1)!} \sum_{s \geq 1/2} \sum_{j=0}^{n-1} \beta_{j,n} 2s^{2j+1} e^{-s^2 t/4(n-1/2)} \\
&= \frac{e^{(n-1/2)t/4}}{(2n-1)!} \sum_{j=0}^{n-1} \beta_{j,n} (-1)^j g^{(j)}\left(\frac{t}{4(n-1/2)}\right)
\end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^2}{(2n-1)!} \sum_{k=0}^m \frac{(n-1-m+k)!}{k!} 4^{n-m-1} (n-1/2)^{m-n+2k+1} \beta_{n-1-m+k,n} \quad \text{if } m < n$$

and

$$a_m = \frac{(4\pi)^n}{(2n-1)!} \left(\sum_{k=0}^{n-1} \frac{k!}{(m-n+k+1)!} (n-1/2)^{m-n+2k+2} \beta_{k,n} 4^{n-m} \right. \\ \left. + \sum_{k=0}^{m-n} \sum_{j=0}^{n-1} \frac{(-1)^j c_{j+k} \beta_{j,n} (n-1/2)^{m-n-2k}}{k!(m-n-k)!} 4^{n-m} \right)$$

if $m \geq n$.

$M = P^{2^n}(\mathbf{R})$. Recall that

$$\zeta_M(t) = \sum_{r=0}^{\infty} \left\{ \frac{4r+2n-1}{2n-1} \prod_{k=0}^{2n-2} \frac{2r+k}{k} \right\} e^{-t\{2r^2+r(2n-1)\}/(2n-1)}.$$

We will let $s = r + ((2n-1)/4)$. Then

$$\frac{4r+2n-1}{2n-1} \prod_{k=1}^{2n-2} \frac{2r+k}{k} = \frac{4s}{(2n-1)!} \prod_{j=1/2}^{n-3/2} \left(4s^2 - \frac{j^2}{4} \right) = \frac{4s}{(2n-1)!} \sum_{j=0}^{n-1} \beta'_{j,n} (2s)^{2j}$$

and

$$t\{2r^2+r(2n-1)\}/(2n-1) = t \left(2s^2 - 2 \left(\frac{2n-1}{4} \right)^2 \right) / (2n-1).$$

Then

$$\zeta_M(t) = \frac{e^{t((2n-1)/8)}}{(2n-1)!} \sum_{s \geq 1/2} \sum_{j=0}^{n-1} \beta'_{j,n} 4s (2s)^{2j} e^{-2s^2 t / (2n-1)} \\ = \frac{e^{t(n-1/2)/4}}{(2n-1)!} \sum_{j=0}^{n-1} \beta'_{j,n} (-1)^j g_i^{(j)} \left(\frac{t}{2(2n-1)} \right)$$

where $i=1$ if n is odd and $i=2$ if n is even.

Thus

$$a_m = \frac{(4\pi)^n}{2(2n-1)!} \sum_{k=0}^m \frac{(n-1-m+k)!}{k!} 4^{n-m-1} (n-1/2)^{n-m+2k+1} \beta'_{n-1-m+k,n}$$

if $m < n$ and

$$a_m = \frac{(4\pi)^m}{2(2n-1)!} \left(\sum_{k=0}^{n-1} \frac{k!}{(m-n+k+1)!} (n-1/2)^{m-n+2k+2} \beta_{k,n} 4^{n-m} \right. \\ \left. + \sum_{k=0}^{m-n} \sum_{j=0}^{n-1} \frac{(-1)^j c_{j+k} \beta'_{j,n} (n-1/2)^{m-n-2k}}{k!(m-n-k)!} 4^{n-m} \right) \text{ if } m \geq n.$$

§11. The Asymptotic Expansion for Complex Projective Spaces

We will have to treat $P^n(\mathbf{C})$ with 2 separate arguments according to whether n is odd or even. The reason for this division is that when n is even $\varrho_a \in \Lambda^+$ while when n is odd $\varrho \notin \Lambda^+$. The treatment of the 2 cases will then differ only in that when n is even we will use Lemma 8.3 while for n odd we will use Lemma 8.5.

$M = P^n(\mathbf{C}), n$ odd

$$\zeta_M(t) = \sum_{m=0}^{\infty} \left\{ \frac{2m+n}{n} \prod_{k=1}^{n-1} \left(\frac{m+k}{k} \right)^2 \right\} e^{-t \{m^2 + mn\}/(n+1)}.$$

Let $s = m + (n/2)$. Then

$$\frac{2m+n}{n} \prod_{k=1}^{n-1} \left(\frac{m+k}{k} \right)^2 = \frac{2s}{n!(n-1)!} \prod_{j=1/2}^{n/2-1} (s^2 - j^2)^2 = \frac{2s}{n!(n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} s^{2k}$$

and $t \{m^2 + mn\}/n+1 = t \{s^2 - (n/2)^2\}/n+1$. Therefore

$$\begin{aligned} \zeta_M(t) &= \frac{e^{tn^2/4(n+1)}}{n!(n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} \sum_{s \geq 1/2} 2s^{2k+1} e^{-s^2 t/(n+1)} \\ &= \frac{e^{tn^2/4(n+1)}}{n!(n-1)!} \sum_{k=0}^{n-2} (-1)^k \gamma_{k,n} g^{(k)}(t/(n+1)). \end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^{n-1}}{n!(n-1)!} \sum_{k=0}^m (n+1)^{n-1-m} \left(\frac{n}{2} \right)^{2k} \frac{(n-m+k-2)!}{k!} \gamma_{n-m+k-2,n}$$

if $m < n-1$ and

$$\begin{aligned} a_m &= \frac{(4\pi)^{n-1}}{n!(n-1)!} \left(\sum_{k=0}^{n-2} \frac{k!}{(m-n+2+k)!} \left(\frac{n}{2} \right)^{2(m-n+2+k)} \gamma_{k,n} (n+1)^{n-m-1} \right. \\ &\quad \left. + \sum_{k=0}^{m-n+1} \left(\frac{n^2}{4(n+1)} \right)^k \frac{1}{k!} \sum_{j=0}^{n-2} \frac{(-1)^j \gamma_{j,n} c_{m-n+1-k+j}}{(m-n+1-k)!} (n+1)^{m-n+1-k} \right) \\ &\hspace{15em} \text{if } m \geq n-1. \end{aligned}$$

$M = P^n(\mathbf{C}), n$ even. Since n is even we will let $n = 2n_0$. If $s = m + n_0$ then we will write

$$\prod_{k=1}^{n-1} (m+k)^2 = \prod_{k=0}^{n_0-1} (s^2 - j^2)^2 = \sum_{k=0}^{n-2} \gamma_{k,n} s^{2k}.$$

Then

$$\begin{aligned}\zeta_M(t) &= \frac{e^{tn_0^2/(n+1)}}{n!(n-1)!} \sum_{k=0}^{n-2} \gamma_{k,n} \sum_{s \geq 0} 2s^{2k+1} e^{-s^2t/(n+1)} \\ &= \frac{e^{tn_0^2/(n+1)}}{n!n-1!} \sum_{k=0}^{n-2} (-1)^k \gamma_{k,n} h^{(k)}(t/(n+1)).\end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^{n-1}}{n!(n-1)!} \sum_{k=0}^m (n+1)^{n-1-m} n_0^{2k} \frac{(n-m+k-2)!}{k!} \gamma_{n-m+k-2,n} \quad \text{if } m < n-1$$

and

$$\begin{aligned}a_m &= \frac{(4\pi)^{n-1}}{n!(n-1)!} \left(\sum_{k=0}^{n-2} \frac{k!}{(m-n+2+k)!} n_0^{2(m-n+2+k)} \gamma_{k,n} (n+1)^{n-m-1} \right. \\ &\quad \left. + \sum_{k=0}^{m-n-1} \left(\frac{n_0^2}{n+1} \right)^k \frac{1}{k!} \sum_{j=0}^{n-2} \frac{(-1)^j \gamma_{j,n} d_{m-n+1-k+j}}{(m-n+1-k)!} (n+1)^{m-n+1-k} \right) \quad \text{if } m \geq n-1.\end{aligned}$$

§12. The Asymptotic Expansion of Quaternionic Projective Spaces

If $M = P^{n-1}(Q)$, $n \geq 2$ then recall that

$$\zeta_M(t) = \sum_{m=0}^{\infty} \frac{2m+2n-1}{2n-1} \prod_{r=2}^{2n-2} \frac{m+r}{r} \prod_{p=1}^{2n-2} \frac{m+p}{p} e^{-t(m^2+2mn-m)/2(n+1)}.$$

We will let $s = m + (n-1/2)$. Then

$$\begin{aligned}\frac{2m+2n-1}{2n-1} \prod_{r=2}^{2n-2} \frac{m+r}{r} \prod_{p=1}^{2n-2} \frac{m+p}{p} &= \frac{2s}{(2n-1)!(2n-3)!} \prod_{j=1/2}^{n-3/2} (s^2-j^2) \prod_{j=1/2}^{n-5/2} (s^2-j^2) \\ &= \frac{2s}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \delta_{k,n} s^{2k}\end{aligned}$$

$$t(m^2+2mn-m)/2(n+1) = t(s^2 - (n-1/2)^2)/2(n+1).$$

Therefore

$$\begin{aligned}\zeta_M(t) &= \frac{e^{t(n-1/2)^2/2(n+1)}}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \sum_{s \geq 1/2} \delta_{k,n} 2s^{2k+1} e^{-s^2} \\ &= \frac{e^{t(n-1/2)^2/2(n+1)}}{(2n-1)!(2n-3)!} \sum_{k=0}^{2n-3} \delta_{k,n} (-1)^k g^{(k)}(t)\end{aligned}$$

Thus

$$a_m = \frac{(4\pi)^{2n-2}}{(2n-1)!(2n-3)!} \sum_{k=0}^m \left(\frac{(n-1/2)^2}{2(n+1)} \right)^k \frac{(2n-3-m+k)!}{k!} \delta_{2n-3-m+k, n}$$

if $m < 2n-2$

and

$$a_m = \frac{(4\pi)^{2n-2}}{(2n-1)!(2n-3)!} \left(\sum_{k=0}^{2n-3} \left(\frac{(n-1/2)^2}{2(n+1)} \right)^{2(m+2n-3-k)} \frac{k!}{(m+2n-3-k)!} \delta_{k, n} \right. \\ \left. + \sum_{k=0}^{m-2n+2} \frac{(n-1/2)^{2k}}{2^k (n+1)^k k!} \sum_{j=0}^{2n-3} \frac{(-1)^j \delta_{j, n} c_{j+m-k}}{(m-k)!} \right) \text{ if } m \geq 2n-2.$$

§13. The Asymptotic Expansion of the Cayley Projective Plane

We will deal with the Cayley projective plane by letting $s = m + 11/2$. Then

$$P(m\varepsilon_1) = \frac{3!}{11!7!} 2s (s^2 - (1/2)^2)^2 (s^2 - (3/2)^2)^2 (s^2 - (5/2)^2) (s^2 - (7/2)^2) (s^2 - (9/2)^2) \\ = \frac{3!}{11!7!} 2s \sum_{j=0}^7 \eta_j s^{2j}$$

with

$$\eta_7 = 1, \quad \eta_6 = -\frac{170}{4}, \quad \eta_5 = \frac{10,437}{16}, \quad \eta_4 = -\frac{262,075}{64}, \quad \eta_3 = \frac{2,858,418}{256}, \\ \eta_2 = -\frac{13,020,525}{1024}, \quad \eta_1 = \frac{18,455,239}{4096}, \quad \eta_0 = -\frac{8,037,225}{16,384}.$$

$t(m^2 + 11m)/18 = t(s^2 - (11/2)^2)/18$ so

$$\zeta_{P^2(\text{Cay})} = \frac{3!}{7!11!} e^{(121/72)t} \sum_{j=0}^7 \sum_{s \geq 1/2} \eta_j 2s^{2j+1} e^{-s^2 t} \\ = \frac{3!}{7!11!} e^{(121/72)t} \sum_{j=0}^7 \eta_j (-1)^j g^{(j)}(t).$$

Therefore

$$a_m = \frac{3!}{7!11!} (4\pi)^8 \sum_{k=0}^m \left(\frac{121}{72} \right)^k \eta_{7-m+k} \frac{(7-m+k)!}{k!} \text{ if } m \leq 7 \\ a_m = \frac{3!}{7!11!} (4\pi)^8 \left(\sum_{k=0}^7 \left(\frac{121}{72} \right)^{(m+7-k)} \eta_k \frac{k!}{(m+7-k)!} \right. \\ \left. + \sum_{k=0}^{m-8} \frac{(121/72)^k}{k!} \sum_{j=0}^7 \frac{(-1)^j \eta_j c_{j+m-k}}{(m-k)!} \right), \text{ if } m > 7.$$

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