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Growth of leaves

JOHN CANTWELL AND LAWRENCE CONLON

Introduction

We study relations between the growth type and the intrinsic topology of leaves of codimension one. Aside from standard trivialities and a few results on 3-manifolds [C-C2], no such relations have been known. Examples show that leaves with exponential growth can be topologically very simple, such as planes and cylinders, or very wild, but we will prove that leaves with polynomial growth cannot be too complicated topologically. Indeed, our main result (Theorem 4) implies the existence, for each $n \geq 3$, of an uncountable infinity of topologically distinct $(n-1)$-manifolds that cannot occur as leaves with polynomial growth in C^2 foliations of any closed n-manifold.

In order to be more precise, we must use the concept of *ends* of an open manifold [A-S], [Ri], [Ni]. The technical definition will be reviewed in Section 1, but a few examples here may be intuitively useful. The real line $\mathbb{R}$ has exactly two ends, $\pm\infty$, as does the cylinder $S^1 \times \mathbb{R}$. In Figure 1, the surface N_1 has a sequence of isolated ends "converging" to one limit end, and the surface N_2 has countably many sequences of isolated ends approaching limit ends, and a sequence of limit ends approaching an ultimate limit end. In general, N_k is constructed inductively by setting $N_0 \cong \mathbb{R}^2$ and defining N_k to be the infinite connected sum $N_{k-1} \# N_{k-1} \# \cdots$, $k \geq 1$.

If N is a manifold, the set $\mathscr{E}(N)$ of ends of N has a topology, as suggested in the above examples, in which it is compact, totally disconnected, and separable. It is natural to consider the first derived subset $\mathscr{E}^1(N)$, consisting of the cluster points of $\mathscr{E}(N)$, the second derived subset $\mathscr{E}^2(N)$, consisting of the cluster points of $\mathscr{E}^1(N)$, etc. By convention, $\mathscr{E}^0(N) = \mathscr{E}(N)$. In the above examples, $\mathscr{E}^k(N_k)$ is a single point.

DEFINITION. An open manifold N is of type k if $\mathscr{E}^k(N)$ is a finite, nonempty set. A closed manifold is said to be of type -1.

We will see in Section 1 that, for each $n \geq 2$, there are uncountably many

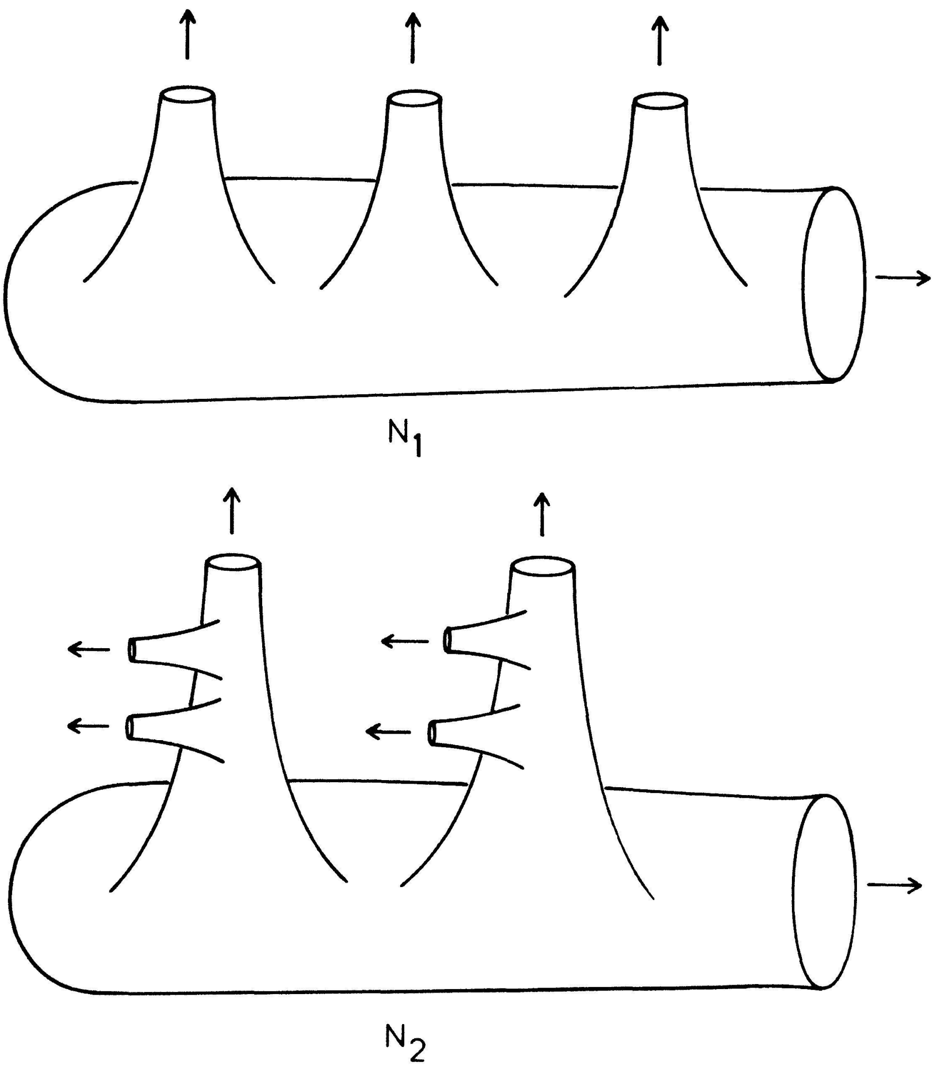

Figure 1

homeomorphism classes of n-manifolds that do not have type k for any integer $k \geq -1$.

Let M be a closed manifold equipped with a foliation $\mathcal{F}$. Let L be a leaf of $\mathcal{F}$, $x \in L$. By relativizing a Riemannian metric from M to L, we can define the growth function $g_x(t)$ of L to be the Riemannian volume of the open ball in L of radius t centered at x. We say that L has polynomial growth of degree r if there is a polynomial P of degree r such that $g_x(t) \leq P(t)$, $t \geq 0$, and r is the smallest integer for which this is true. At the other extreme, L has exponential growth if there are positive constants A, B, and C such that $g_x(t) + C \geq Ae^{Bt}$, $t \geq 0$. The growth type of L is independent of the choice of metric on M and of $x \in L$ [P2]. The growth can also be defined, without a metric, in terms of the growth at $x \in L$ of the holonomy pseudogroup of $\mathcal{F}$ [P2].

Let $\mathcal{F}$ be of class C^2 and codimension one. For leaves $L \in \mathcal{F}$ with polynomial growth, we will give a detailed structure theory of the closure L^c of L in M. This can be viewed as a generalization to foliations of codimension one of the classical Poincaré-Bendixson theory. This analysis of L^c will make the ends of L visible in terms of the simpler leaves around which these ends are winding, leading thereby to an upper bound on the type of L. On the other hand, it will also give a lower bound to the degree of growth of L.

The principal consequence (Theorem 4) of the above theory will be that leaves with polynomial growth of degree r can have type at most r. If the leaf is proper, its type can be at most $r-1$. This generalizes the trivial fact that leaves with growth of degree 0 are compact.

The case in which L (with polynomial growth) is not compact and does not wind around any compact leaf is of some independent interest. This happens precisely when $\mathcal{F}$ is without holonomy and without compact leaves. In addition to the considerable structure theory already available for such foliations [No], [Sa], [T], we prove (Theorem 3) that each leaf has at most two ends.

In Section 6 we sketch some simple examples of leaves not of finite type with growth properly between polynomial and exponential growth. Independently, G. Hector [H] has produced similar examples exhibiting uncountably many distinct nonexponential growth types in a single foliation.

Unless otherwise specified, $\mathcal{F}$ will denote a transversely oriented C^2 foliation of codimension one on a closed, oriented n-manifold M.

1. Technicalities about ends

Let N be an open, connected manifold and select a nest $K_1 \subset K_2 \subset \cdots \subset K_i \subset \cdots \subset N$ of compact subsets such that $N = \cup K_i$. For each i, suppose that U_i is a component of $N - K_i$ such that $U_1 \supset U_2 \supset \cdots \supset U_j \supset \cdots$. Then $\{U_i\}$ is said to define an end e of N and to be a fundamental neighborhood system of e. Given another nest $\{K_i'\}$ and a corresponding system $\{U_i'\}$ defining an end e', we will say that $e = e'$ if and only if each U_i contains some U_j' and each U_i' contains some U_j. A sequence $\{e_n\}$ of ends is said to converge to an end e if every fundamental neighborhood of e is also a fundamental neighborhood of all but a finite subset of $\{e_n\}$. Similarly, a sequence $\{x_n\}$ of points of N converges to an end e if each fundamental neighborhood of e contains all but a finite subset of $\{x_n\}$. There results a compactification $N \cup \mathscr{E}(N)$ of N, and $\mathscr{E}(N)$ is compact, totally disconnected, and separable [A–S].

By transfinite induction, we define derived subsets $\mathscr{E}^\alpha(N)$ for all ordinals α. As before, $\mathscr{E}^0(N) = \mathscr{E}(N)$ and, if $\mathscr{E}^\alpha(N)$ has been defined, then $\mathscr{E}^{\alpha+1}(N)$ is the set of cluster points of $\mathscr{E}^\alpha(N)$. If α is a limit ordinal and $\mathscr{E}^\beta(N)$ has been defined for all

$\beta<\alpha$, then $\mathscr{E}^{\alpha}(N)=\bigcap_{\beta<\alpha}\mathscr{E}^{\beta}(N)$. In this last case, $\mathscr{E}^{\alpha}(N)$ is not empty unless some $\mathscr{E}^{\beta}(N)$ is empty, $\beta<\alpha$.

An elementary argument shows that, if Ω denotes the first uncountable ordinal, then $\mathscr{E}^{\Omega}(N)$ is either empty or is a Cantor set. Thus, one only considers $\mathscr{E}^{\alpha}(N)$ for $0\leq\alpha\leq\Omega$. Evidently, $\mathscr{E}^{\Omega}(N)$ is a Cantor set if and only if $\mathscr{E}(N)$ is uncountable.

DEFINITION. The open manifold N is of type $\alpha<\Omega$ if $\mathscr{E}^{\alpha}(N)$ is a finite, nonempty set. If no such α exists, N is of type Ω.

Examples may be helpful. If $\{N_k\}$ denotes the sequence of surfaces constructed in the introduction, we can form a limit surface $N_\omega=\lim_{k\to\omega}N_k$ (where ω denotes the first infinite ordinal). Indeed, connected sum has the "absorption" property $N_k\#N_{k+1}\cong N_{k+1}$ so we set $N_\omega=N_1\#N_2\#\cdots\#N_k\#\cdots$. Then $\mathscr{E}^{\omega}(N_\omega)$ is a single point, so this surface has type ω.

In Figure 2 we depict a surface with a Cantor set of ends. It is worth remarking that the complete Riemannian metric implicit in Figure 2 gives an example of exponential growth. Indeed, surface area grows roughly like the powers of 2.

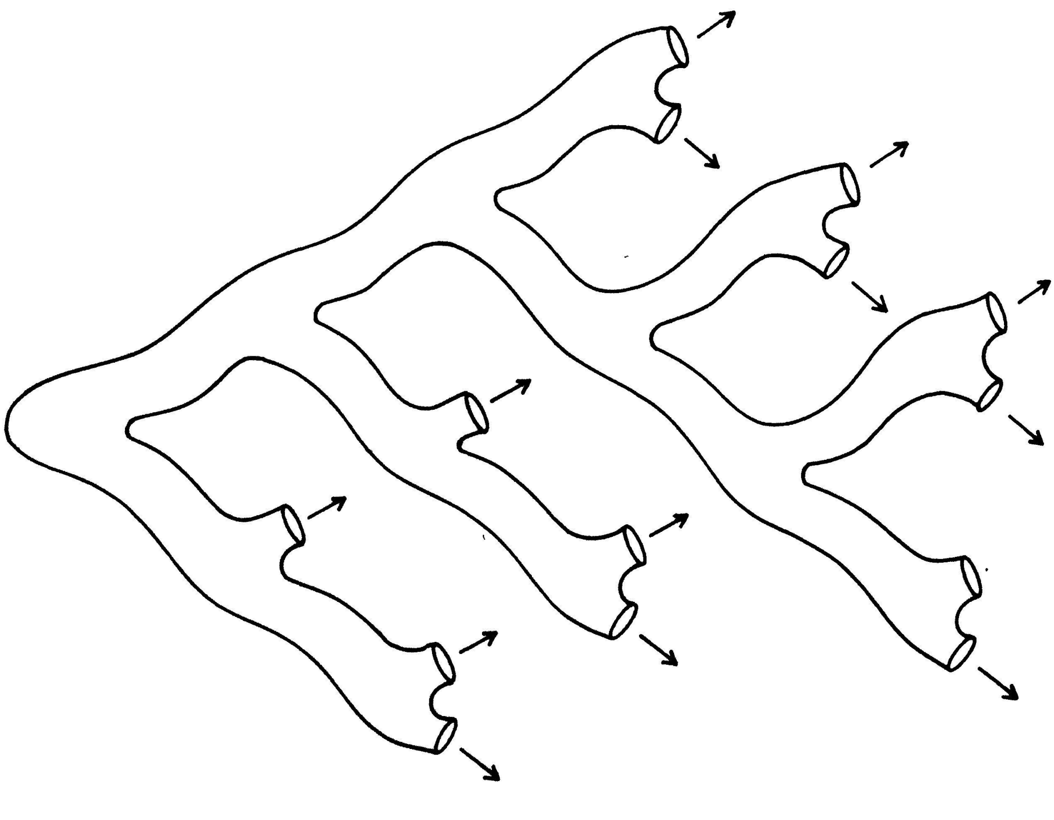

Figure 2

Given $\alpha<\Omega$, there are compact, totally disconnected subsets $E\subset[0,1]$ such that the α-th derived set E^α is finite and nonempty. The Cantor set gives an example of $E\subset[0,1]$ with $E^\Omega\neq\emptyset$. Thus, for $0\leq\alpha\leq\Omega$ and $n\geq 2$, one can imbed suitable $E\subset S^n$ and obtain an n-manifold $N=S^n-E$ of type α. In particular, there are uncountably many topologically distinct n-manifolds that are not of finite type.

Ends play a role in foliation theory as follows. Given an noncompact leaf L of $\mathcal{F}$ and $e\in\mathcal{E}(L)$, let $\{U_i\}$ be a neighborhood system for e, denote the closure of U_i in the ambient manifold M by U_i^c, and define the *asymptote* of e to be $A_e=\bigcap U_i^c$. This is a compact, $\mathcal{F}$-saturated set. In classical Poincaré-Bendixson theory, the asymptotes of the two ends of a noncompact line of flow L are called the limit sets of L.

2. Existence of contracting holonomy

We describe two situations in which the phenomenon of contracting holonomy will arise in this paper.

Let L be a leaf of $\mathcal{F}$ such that L^c is transversely a Cantor set. That is, L is nowhere dense and is not proper. Let R be a finite disjoint union of compact arcs transverse to $\mathcal{F}$ such that each leaf in L^c meets the interior of R. By [Pl, 246–247], the holonomy pseudogroup Γ defined by $\mathcal{F}$ on R contains a finitely generated sub-pseudogroup Γ_0 whose restriction to $L^c\cap R$ coincides with that of Γ. Consequently, [Sa, Theorem 1] implies that, if each leaf of L^c is nonproper, then arbitrarily near any leaf approached only from one side by L (i.e., a leaf corresponding to an endpoint of the Cantor set) there passes a leaf of L^c with an element of 2-sided contracting holonomy. Actually, the proof in [Sa] shows that the requirement that every leaf of L^c be nonproper can be relaxed substantially, and this is necessary for our purposes since we do not intend that L^c be a minimal set.

THEOREM 1. (*Sacksteder*) *Suppose that L is a nowhere dense, nonproper leaf of $\mathcal{F}$, and that some leaf L_0 approached by L only from one side is also nonproper. Then, arbitrarily near L_0, a leaf L_1 of L^c can be found which has an element of 2-sided contracting holonomy.*

This result, together with methods of J. Plante [P2], has a corollary that will be needed.

COROLLARY. *If L is a leaf of $\mathcal{F}$ having nonexponential growth, then L^c does not contain an exceptional minimal set.*

Proof. Suppose $X \subset L^c$ is an exceptional minimal set. By Theorem 1, there is a leaf $L_1 \subset X$ having an element γ of 2-sided contracting holonomy at (say) $x_0 \in L_1$. Let J be a compact transverse interval through x_0, small enough that $J \subset \operatorname{dom}(\gamma)$. Since L has nonexponential growth and $L \cap J \neq \emptyset$, there is a holonomy invariant measure μ defined on J such that $\mu(J)=1$ [P2, 3.1]. But $\{x_0\}=\bigcap_{n=0}^{\infty}\gamma^n(J)$, hence $\mu\{x_0\}=1$. It follows that each point of the infinite set $L_1 \cap J$ has measure 1, a contradiction. ■

Let L and L' be leaves of $\mathcal{F}$ such that L is proper and is in the limit set of L'. Fix a transverse arc T through L and an identification $T=[-1,1]$ such that $\{0\}=T \cap L$ and $L' \cap (0,1]$ accumulates on 0. Let Γ_L be the pseudogroup on T defined by the holonomy along L.

The following is what is actually established in the proof of [S-S, Theorem 1].

LEMMA 1. (*Sacksteder and Schwartz*) *Under the above hypotheses, there is an* $\varepsilon>0$ *such that, for each* $t_0 \in (0,\varepsilon) \subset T$, *there is an element* $\gamma \in \Gamma_L$ *with* $\gamma(t)<t_0$, $0<t<\varepsilon$.

THEOREM 2. *If L' has polynomial growth and L is as above, then L has an element of contracting holonomy on whatever side is approached by L'.*

Proof. Indeed, suppose L does not have an element of contracting holonomy. Choose $t_1 \in L' \cap (0,\varepsilon)$ and $\gamma_1 \in \Gamma_L$ such that $\gamma_1(t)<t_1$, $0<t<\varepsilon$. By assumption, $\lim \gamma_1^n(t_1)=t_2>0$. Choose $\gamma_2 \in \Gamma_L$ such that $\gamma_2(t)<t_2$, $0<t<\varepsilon$, and set $t_3=\lim \gamma_2^n(t_2)>0$. In this way choose infinite sequences $\{\gamma_i\} \subset \Gamma_L$ and $\{t_i\} \subseteq (0,\varepsilon)$ such that $\gamma_i(t)<t_i$, $0<t<\varepsilon$, and $\lim_{n\to\infty}\gamma_1^n(t_1)=t_{i+1}$. Let Γ_k be the pseudogroup generated by $\{\gamma_1,\dots,\gamma_k\}$, and let g_k denote the growth function of Γ_k at t_1 [P2]. Let $g_k^+(n)$ denote the number of distinct points of $L' \cap T$ of the form $\gamma_k^{n(k)}\gamma_{k-1}^{n(k-1)}\cdots\gamma_1^{n(1)}(t_1)$ where all $n(i)\geq 0$ and $\sum_{i=1}^k n(i)\leq n$. These points are necessarily in (t_{k+1},ε). Evidently, $g_1^+(n)=n+1$, and $g_{k+1}^+(n)=g_k^+(n)+g_k^+(n-1)+\cdots+1$. It is well known that, if $g_k^+(n)$ is a polynomial in n of degree k, then the above summation defines a polynomial in n of degree $k+1$. Indeed, $\sum_{i=1}^n i^k$ is such a polynomial [B], [D], [W]. Furthermore, $g_{k+1}(n)$ dominates $g_{k+1}^+(n)$. As in [P2], the growth type of L' can be computed from the growth function at t_1 of a suitable holonomy pseudogroup Γ relative to a finite generating set Γ^1. Augmenting Γ^1 by the elements $\{\gamma_1,\dots,\gamma_k\}$ and their inverses gives a larger pseudogroup and a new growth function at t_1, but the type of growth is unchanged. By the above, it follows that, for each $k\geq 1$, L' has growth type greater than that of some polynomial of degree k. This contradiction completes the proof. ■

Remark. While Theorem 1 holds only for C^2 foliations, Lemma 1 and, consequently, Theorem 2 are true for "continuously C^1" foliations, that is, for foliations integral to a C^0 $(n-1)$-plane field. For C^2 foliations, we conjecture that the condition of polynomial growth on L' can be relaxed to nonexponential growth, but an example in Section 6 shows that this is not true for continuously C^1 foliations.

3. Leaf preserving flows

Let $X \subset M$ be a compact $\mathcal{F}$-saturated set, let U be an open saturated subset of $\operatorname{int}(X)$, and let U be dense in X. Finally, let $\varphi: \mathbb{R} \times X \to X$ be a C^0 flow with the following properties.

(a) The homeomorphism $\varphi_t: X \to X$ maps each leaf diffeomorphically onto a leaf, $\forall t$.

(b) The flow is stationary on $X - U$.

(c) The flow is nonsingular on U and transverse to $\mathcal{F}|U$, the flow lines coinciding pointwise with the orthogonal trajectories to $\mathcal{F}|U$ relative to a Riemannian metric on M.

(d) If E denotes the tangent bundle to $\mathcal{F}|U$, the Jacobian $\varphi_{t*}: E \to E$ is a bundle map varying continuously with t.

The above situation has arisen in [C–C2], in [Sa], and in [P1] with X a manifold, possibly with boundary. In the present paper we cannot require that X be a manifold.

For the following proof, we remark that the endset can be defined via a nest $K_1 \subset K_2 \subset \cdots \subset L$ where each K_i is a compact manifold with boundary. Thus $L - K_i$ has only a finite number of components.

An end e of L will be called *nonproper* if the asymptote A_e contains L.

PROPOSITION 1. *If L is a leaf in U that is dense in X, then L has either one or two nonproper ends.*

Proof. If L is dense in X, then at least one of the finitely many components of $L - K_1$ is also dense in X. Call this dense component U_1 and, inductively, choose components U_{j+1} of $U_j - K_{j+1}$, each dense in X. Then $\{U_j\}$ is a neighborhood system for a nonproper end of L.

We show that there are at most three nonproper ends. If e_i, $1 \leq i \leq 4$, are distinct nonproper ends of L, one can find a compact, connected manifold $K \subset L$ such that one component W of $L - K$ is a neighborhood both of e_1 and e_2, but not of e_3 nor e_4. Let $S \subset \delta K$ be the boundary of W in L. Since K is connected, $L-S$ has exactly two components, W itself and a neighborhood V of e_3 and e_4.

Fix $x_0 \in S$ and choose sequences $\{x_{i,n}\}_{n=1}^{\infty}$, $1 \leq i \leq 4$, of points of L converging to e_i and such that $x_{i,n} \to x_0$ in M. We can suppose that there are $t_{i,n} \in \mathbb{R}$ such that $t_{i,n} \to 0$ and $\varphi_{t_{i,n}}(x_0) = x_{i,n}$. By (a) $\varphi_{t_{i,n}}$ maps L to itself and, by (d), the diameters of the sets $\varphi_{t_{i,n}}(S)$ are bounded, so for each $i = 1, 2, 3, 4$ we can choose $t_i = t_{i,n}$ so that the manifolds $S_i = \varphi_{t_i}(S)$ are disjoint and $S_1 \cup S_2 \subset W$, $S_3 \cup S_4 \subset V$. Furthermore, assuming that each $x_{i,n}$ is sufficiently "near" e_i, we obtain a component W' of $W - S_1 - S_2$ bounded by $S_1 \cup S_2 \cup S$, and a component V' of $V - S_3 - S_4$ bounded by $S_3 \cup S_4 \cup S$.

By (d), the orientation of L is preserved by $\psi_i = \varphi_{t_i}$, $1 \leq i \leq 4$. Orient S so that W' lies to the left and V' to the right. Orient S_i by carrying the orientation of S to S_i via ψ_i.

If W' lies to the right of both S_1 and S_2, we produce a contradiction as follows. Since $\psi_1(W')$ lies to the left of S_1 and $\psi_2(W')$ lies to the left of S_2, the same holds for $W_1 = \psi_1(W)$ and $W_2 = \psi_2(W)$. Thus W_1 is the component of $L - S_1$ not containing W' and W_2 is the component of $L - S_2$ not containing W'. We claim $W_1 \cap W_2 = \emptyset$ since, otherwise, there is a path from W_1 to W' not meeting S_1. Also, W_1 and $W_2 \subset W$. But $W_1 \supset \psi_1(W_2) = \psi_1\psi_2(W) = \psi_2\psi_1(W) \subset W_2$, this being the desired contradiction. Thus we can assume that W' lies to the left of S_1.

A completely similar argument shows that W' cannot lie to the right of S_2 while lying to the left of both S_1 and S. Similarly, V' must lie to the right of both S_3 and S_4. But then, the component of $L - S_1 - S_3 - S_4$ bounded by $S_1 \cup S_3 \cup S_4$ lies to the left of S_1 and to the right of both S_3 and S_4, leading to the same contradiction. Thus L has at most three nonproper ends.

Finally, suppose that e_1, e_2, and e_3 are distinct nonproper ends. One finds a compact, connected $K \subset L$ as usual so that $L - K$ has exactly three components W_i, respective neighborhoods of e_i, $i = 1, 2, 3$. Let $S_i \subset \delta K$ denote the boundary of W_i in L, and by drilling suitable tunnels out of K, if necessary, assume that S_1 is connected. Using a sequence in L converging to e_1 in $L \cup \mathscr{E}(L)$ and to $x_0 \in S_1$ in M, we argue as before to find a real number t such that $\psi = \varphi_t$ maps L to itself and has $\psi(K) \subset W_1$. Remark that the homeomorphism $\psi\colon L \to L$ extends to a homeomorphism $\tilde{\psi}\colon L \cup \mathscr{E}(L) \to L \cup \mathscr{E}(L)$ and that nonproper ends are carried by $\tilde{\psi}$ to nonproper ends. The manifold $\psi(S_1)$ separates L into two components, one of which, $\psi(W_1)$, is a neighborhood of exactly one nonproper end. The component containing S_1 must also contain W_2 and W_3, hence cannot be $\psi(W_1)$. Thus $\psi(W_1) \subset W_1$ and $\tilde{\psi}(e_1) = e_1$. If $\psi(W_2)$ does not contain S_1, then it does not meet S_1 and $\psi(W_2) \subset W_1$, contradicting the fact that W_1 is a neighborhood of only the one nonproper end e_1. Similarly, $\psi(W_3)$ must contain S_1. But $\psi(W_2) \cap \psi(W_3) = \psi(W_2 \cap W_3) = \emptyset$, and this contradiction completes the proof. ■

THEOREM 3. *If $\mathcal{F}$ is a foliation without holonomy (for instance, this is the case if $\mathcal{F}$ has no compact leaves and has at least one leaf with nonexponential growth) then each leaf of $\mathcal{F}$ has at most two ends.*

Proof. By the argument in [P1, 6.3], together with the corollary to Theorem 1, the foliation has no holonomy if it has no compact leaf and at least one leaf has nonexponential growth. By [Sa, Theorem 6], if $\mathcal{F}$ has no holonomy, there is a flow satisfying our hypotheses with $X = U = M$, and either all leaves are compact or each leaf is dense in M. In the case of a dense leaf L, M itself is the only minimal set of $\mathcal{F}$, so $A_e = M$, $\forall e \in \mathscr{E}(L)$. By Proposition 1, L has at most two ends. ■

Remark. We can list all open, orientable surfaces that occur as leaves in C^2 foliations without holonomy of closed 3-manifolds. If T_g denotes the closed, orientable surface of genus g, then standard examples on $T_g \times S^1$ show that $\mathbb{R}^2$ and $\mathbb{R}^2 \# \mathbb{R}^2$ so occur (for $g=1$) and that T_∞ and $T_\infty \# T_\infty$ so occur (for $g>1$), where T_∞ denotes the orientable surface with one end and infinite genus. By Theorem 3 and the classification theory [Ri], the only remaining possibilities are $\mathbb{R}^2 \# T_\infty$, $\mathbb{R}^2 \# T_g$, and $\mathbb{R}^2 \# \mathbb{R}^2 \# T_g$, $g \geq 1$. A noncompact leaf in a foliation without holonomy is dense and, by [Sa, Theorem 6] and [P2, Theorem 6.3], it has polynomial growth. Thus, Theorem 5 of [C–C2] shows that the above possibilities cannot occur.

4. The structure of L^c

Let L be a leaf of $\mathcal{F}$ having polynomial growth of positive degree. Suppose M is not a minimal set. By the corollary to Theorem 1, there must be a compact leaf in L^c, and it is standard that there can only be finitely many such.

DEFINITION. Each compact leaf of L^c is said to be of class 0. A leaf $L' \subset L^c$ is of class $k \geq 1$ if L' is asymptotic only to leaves of class at most $k-1$ and to at least one leaf of class $k-1$.

Let C denote the union of all leaves of finite class in $L^c - L$. We will establish the following two results by a sequence of lemmas.

PROPOSITION 2. *There are only finitely many leaves in C, and each has an element of contracting holonomy on whatever side is approached by L.*

PROPOSITION 3. *If L is proper, then $L^c - C = L$. Otherwise, $U = L^c - C$ is a* *dense subset of $X=L^c$, U is open in M, and there is a topological flow $\varphi:\mathbb{R}\times X\to X$ such that X, U, and φ satisfy properties (a), (b), (c), and (d) of Section 3.*

For proper leaves the analogies with Poincaré-Bendixson theory should be evident.

LEMMA 2. *If $L'\subset L^c-L$ is a leaf of finite class, it is a proper leaf and, consequently, L' has an element of contracting holonomy on whatever side is approached by L.*

This lemma is evident, the element of contracting holonomy being guaranteed by Theorem 2.

Fix a compact transverse 1-manifold R (possibly with boundary) such that, for each leaf F of class 0, $F\cap R$ is a single point in $\operatorname{int}(R)$. It follows that every leaf of L^c meets $\operatorname{int}(R)$. Note that we do not demand that R be connected.

LEMMA 3. *For each $k\geq 0$, there are at most finitely many leaves in L^c of class k.*

Proof. For $k=0$, this has been observed above, so suppose $k\geq 1$. Let $\{L_i\}_{i=1}^{\infty}$ be an infinite set of leaves of class k, and suppose that there are only finitely many leaves of class $\leq k-1$. This will lead to a contradiction and the lemma will follow by induction.

We may suppose that every L_i is asymptotic to a fixed leaf L' of class $k-1$. Let $T\cong[-1,1]$ be a subinterval of R with $L'\cap T=\{0\}$ and such that T meets no other leaf of class $\leq k-1$. Using the element of contracting holonomy on L', we can find a compact interval $J\subset T-\{0\}$ such that each L_i meets J in a point x_i. Let $x_0\in J$ be a cluster point of $\{x_i\}$ and let L_0 be the leaf through x_0. Evidently, $L_0\subset L^c$ and L_0 cannot be of class $\leq k-1$. If L_0 were proper, then its element of contracting holonomy (Theorem 2) would provide the contradiction that some L_i is asymptotic to L_0. If L_0 is asymptotic to a proper leaf L_0' meeting $\operatorname{int}(T-\{0\})$, then the element of contracting holonomy on L_0' gives the same sort of contradiction. If L_0 is locally dense, it is asymptotic to L, hence to every L_j, and any one of the leaves L_j can be chosen to play the role of L_0' above. Thus, L_0 is nowhere dense, is not proper, and is not asymptotic to any proper leaf meeting $\operatorname{int}(T-\{0\})$. Without loss of generality, we assume that the endpoints of R do not lie in L_0^c, hence $L_0^c\cap R=K$ is a Cantor set and the endpoints of the components of $T-K$ (except for 0 and the endpoints of T) correspond to nonproper leaves. Let z_0 be such an endpoint. By Theorem 1, it follows that, arbitrarily near z_0 in R, hence in $T-\{0\}$, there passes a leaf $L_0'\subset L_0^c$ having an element of 2-sided contracting holonomy. Since the sequence $\{L_i\}$ approaches L_0, it approaches L_0', and the contracting holonomy of L_0' again causes some L_i to be asymptotic to L_0', a leaf not of class $\leq k-1$. ■

LEMMA 4. *There is a largest integer $q \geq 0$ for which there exists a leaf in $L^c - L$ of class q. Furthermore, L has growth of degree at least $q+1$.*

Proof. If there were no such integer, then, by Lemma 2 and the argument in the proof of Theorem 2, we could show that the growth type of L dominates polynomials of arbitrarily high degree. The same argument proves the second assertion. Details will be left to the reader. ■

The proof of Proposition 2 is now complete.

We will find a noncompact 1-manifold $J \subset R - C$ such that every leaf of $L^c - C$ meets the interior of J, and there will be a holonomy invariant measure μ on J, supported in $L^c \cap J$ and finite on compact sets. This involves a technically fussy application of [P2, Theorem 3.1 and Lemma 3.2].

Let C_j denote the union of leaves in $L^c - L$ of class at most j, $0 \leq j \leq q$. Of course, $C = C_q$.

LEMMA 5. *For each $j = 0, 1, \ldots, q$ there is a finite set of connected components of $R - C_j$ such that every leaf of $L^c - C_j$ meets at least one of these components.*

The easy proof of this lemma (by induction on j, using the elements of contracting holonomy of Lemma 2) will be left to the reader.

Let $\{P_1, \ldots, P_m\}$ be a set of components of $R - C = R - C_q$ satisfying the assertion in Lemma 5, each being met by some leaf of $L^c - C$. If $P_i = (p_i, q_i)$, then both p_i and q_i belong to C. It is also possible that one endpoint of P_i will belong to P_i, in which case it belongs to δR and the other endpoint belongs to C.

We define $J_i \subset P_i$. If P_i contains one of its endpoints, set $J_i = P_i$. If $P_i = (p_i, q_i)$ and $L \cap P_i$ clusters at both p_i and q_i, again set $J_i = P_i$. If $L \cap P_i$ does not cluster at p_i, choose $a_i \in (p_i, q_i)$ such that $(p_i, a_i] \cap L^c = \emptyset$. Similarly, if $L \cap P_i$ does not cluster at q_i, choose $b_i \in (p_i, q_i)$ such that $[b_i, q_i) \cap L^c = \emptyset$. If both situations hold, we can take $a_i < b_i$. In these three cases we define J_i to be respectively $[a_i, q_i)$, $(p_i, b_i]$, or $[a_i, b_i]$. Let J denote the union of all J_i, a noncompact 1-manifold with possibly empty boundary. Every leaf of $L^c - C$ meets the interior of J.

Using the elements of contracting holonomy of Lemma 2, choose closed intervals $A_i \subset J_i$ such that every leaf of $\mathcal{F}$ that meets J_i also meets the interior A_i^0 (taken relative to J_i) of A_i. We emphasize that A_i^0 will contain an endpoint if that point is also in δJ_i. Let A denote the union of all A_i, a compact 1-manifold with boundary, and let A^0 denote the union of all A_i^0, this being the interior of A relative to J.

Let $\{U_1, \ldots, U_s\}$ be a regular open cover of M in the sense of [P2], together with compact arcs $\Delta_j \subset U_j$ transverse to $\mathcal{F}$, such that every leaf of $\mathcal{F}$ meets the interior of some Δ_j. Without loss of generality, suppose $m \leq s$ and $A_i \subset \Delta_i$, $1 \leq i \leq m$. Let Γ be the pseudogroup on $\Delta = \bigcup_{j=1}^{s} \Delta_j$ finitely generated by the transition functions $\gamma_{ij} : \Delta_j \cap U_i \to \Delta_i$ as in [P2]. Let Γ_A be the pseudogroup induced on A by Γ. Let Γ_k denote the sub-pseudogroup of Γ_A finitely generated by compositions of the γ_{ij}'s in chains of length $\leq k$. Then Γ_A is the increasing union $\bigcup \Gamma_k$ and, in the sense of [P2], Γ_A has nonexponential growth at $x \in L \cap A$. By [P2, Theorem 3.1], there is a Γ_A-invariant normalized measure μ_A on A supported in $L^c \cap A$. If Γ_J denotes the pseudogroup induced on J by $\mathcal{F}$, then the fact that each leaf meeting J also meets A^0 implies that μ_A extends to a nontrivial Γ_J-invariant measure μ on J, supported in $L^c \cap J$ and finite on compact subsets of J [P2, Lemma 3.2].

LEMMA 6. *Let L_0 be a leaf meeting* supp (μ) *and different from L. Then L_0 cannot be proper.*

Proof. Suppose L_0 is proper. Let $x_0 \in L_0 \cap J \subset \operatorname{supp}(\mu)$. There is an element of contracting holonomy at x_0 on whatever side is approached by $L \cap J$, hence there is such a contraction on whatever side is approached by supp (μ). Suppose, then, that x_0 is not isolated in supp (μ) and let $x_1 \in \operatorname{supp}(\mu)$ be close enough to x_0 (say, on the right) so that $[x_0, x_1] \subset \operatorname{dom}(\gamma_0)$. Then $\mu(\gamma_0(x_1), x_1] = \mu(\gamma_0^2(x_1), \gamma_0(x_1)]$ and $\mu(\gamma_0^2(x_1), x_1] > 0$ (since the interior point $\gamma_0(x_1)$ belongs to the support), hence $\mu(\gamma_0(x_1), x_1] > 0$. Since $(x_0, x_1] = \bigcup_{n=1}^{\infty} (\gamma_0^n(x_1), \gamma_0^{n-1}(x_1)]$, it follows that $\mu[x_0, x_1] = \infty$, contradicting the fact that μ is finite on compact subsets of J. Thus, x_0 must be isolated in supp (μ), so $\mu\{x_0\} > 0$. If $L_0 \cap J$ accumulates at $y \in J$, a compact neighborhood of y in J will have infinite measure, so L_0 can only be asymptotic to leaves of C. Since $L_0 \not\subset C$, it follows that L_0 is of class $q+1$. Since $L_0 \neq L$, this contradicts Lemma 4. ∎

LEMMA 7. *If L meets* supp (μ) *and is the only leaf of L^c that does so, then $L^c - C = L$ and L is of class $q+1$ (hence L is proper).*

Proof. Since supp (μ) is closed in J, the leaf L cannot be asymptotic to any leaf in c except, perhaps, to itself. In this latter case $L^c \cap J = L \cap J$ is nowhere dense and perfect, hence this set is uncountable. But a finite set of transverse arcs cannot meet a single leaf in an uncountable set of points. All assertions follow. ∎

LEMMA 8. *If* supp (μ) *is met by some leaf* $L_0 \neq L$, *then every leaf of* $L^c - C$ *is locally dense in* M *and* supp $(\mu) = J$.

Proof. Suppose L_0 is not locally dense in M. Without loss of generality, assume that the endpoints of R do not lie in L_0^c. Then, by Lemma 6, $L_0^c \cap R$ must be a Cantor set. Also, by Lemma 6, $L_0^c \cap J$ cannot be met by a proper leaf different from L. Not every endpoint of intervals in the complement of L_0^c in $\text{int}(J)$ can correspond to a proper leaf since, in that case, all would lie on L and L would not be proper after all. Thus, there is $y \in \text{int}(J)$ such that y is an endpoint of a component of $R - L_0^c$ and the leaf through y is nonproper. Arbitrarily near y in R, hence in $\text{int}(J)$, there is $z \in L_0^c \cap R$ corresponding to a leaf having an element of 2-sided contracting holonomy (Theorem 1). This is a cluster point in J of supp (μ), so the argument in the proof of Lemma 6 again applies and contradicts the finiteness of μ on compact sets.

Thus L_0 is locally dense in M, and, necessarily, $L \subset \text{int}(L_0^c)$ and L is dense in that set. It also follows that $L \cap J \subset \text{supp}(\mu)$, hence that $L^c \cap J = \text{supp}(\mu)$. Thus, supp (μ) is open and closed in J and every leaf meeting supp (μ) is dense in that set. ∎

In order to complete the proof of Proposition 3, we consider the case in which L is not proper. That is, every leaf of $L^c - C$ is locally dense in M. Set $X = L^c$ and $U = L^c - C$, and remark that every leaf in U is dense in X. We want to produce a topological flow $\varphi: \mathbb{R} \times X \to X$ satisfying (a), (b), (c), and (d) of Section 3.

In the standard way, choose a smooth transverse circle Σ to $\mathcal{F}$ such that $\Sigma \subset U$ and such that Σ is an integral curve to a unit normal field v to $\mathcal{F}$. Since L meets Σ and has nonexponential growth, one again applies [P2, Theorem 3.1] to produce on Σ a normalized measure ν invariant under holonomy. Each leaf of $\mathcal{F}|U$ meets Σ in a dense subset, so supp $(\nu) = \Sigma$. There results a transverse invariant measure (again denoted by ν) on the saturated set U with supp $(\nu) = U$, hence one obtains local reparametrizations of the integral curves to $v\,|\,U$ so as to define a local flow on U that preserves the local leaves.

If $L_0 \subset C$ is a leaf bordered on at least one side by U, the existence of the element of contracting holonomy implies that the measure is unbounded near L_0 on whatever side is bordered by U. Consequently, since each $L' \subset C$ is either bordered on at least one side by U, or is approached on at least one side by such leaves, the local flow extends to one on all of X, stationary at all points of C and nonsingular on U. By the compactness of X, this defines a global flow and properties (a), (b), and (c) are satisfied. For property (d), proceed as in [Sa, Theorem 6] by changing the differentiable structure on U so as to make the flow smooth and so as not to change the differentiable structures of the leaves nor the smoothness of $\mathscr{F}|U$. The new tangent bundle of U contains E in a natural way as the tangent bundle of $\mathscr{F}|U$ and property (d) follows. The proof of Proposition 3 is complete.

5. The topology of L

We continue with the hypotheses and notations of Section 4.

For $0 \le k \le q$, let E_k be the set of $e \in \mathscr{E}(L)$ such that A_e contains a leaf of class k. Let E_{q+1} be the set of nonproper ends of L. Remark that $\mathscr{E}(L) = E_0 \supset E_1 \supset \cdots \supset E_q \supset E_{q+1}$ and that E_{q+1} is empty if L is proper, and, in any case, is finite by Proposition 1. Recall that the derived set E_k^1 is the set of cluster points of E_k.

PROPOSITION 4. *If* $0 \le k \le q$, *then* $E_k^1 \subset E_{k+1}$.

Proof. Let $e \in E_k^1$ and let $\{e_j\}_{j=1}^\infty$ be a sequence of elements of E_k converging to e in $\mathscr{E}(L)$. Since there are only finitely many leaves of class k, we lose no generality in assuming that there is one such leaf L_0 contained in every A_{e_j}. We can also suppose that the filtration $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset L$ is such that one component U_i of $L - K_i$ is a neighborhood of e and of all e_j with $j \ge i$, $1 \le i < \infty$. Finally, it can be arranged that U_{i+1} is not a neighborhood of e_i, for each i. That is, for each i, $U_i - U_{i+1}$ is a neighborhood of e_i but not of e_j for $j \ne i$. Also, $\{U_i\}_{i=1}^\infty$ is a neighborhood system for e and $L_0 \subset (U_i - U_{i+1})^c$ for all i.

Select $x_0 \in L_0$ and a transverse arc T properly crossing L_0 at x_0 and such that $T - \{x_0\}$ meets no leaf of class $\le k$. We can assume that $(U_i - U_{i+1}) \cap T$ accumulates on x_0 from the right for all i.

Let γ_0 be the element of contracting holonomy defined on $T_0 = (x_0, y] \subset T$ by a loop σ_0 on L_0 based at x_0. For each i, choose $\varepsilon_i > 0$ such that, for every $x \in T_0$ that is ε_i-close to x_0, the holonomy path from x to $\gamma_0(x)$ that is the lift of σ_0 misses the compact set $K_{i+1} \subset L \subset M$. We can assume $\varepsilon_i \downarrow 0$. Choose $x_i \in (U_i - U_{i+1}) \cap T_0$ to be ε_i-close to x_0. Then $\gamma_0^n(x_i) \in U_i - U_{i+1}$ for all $n \ge 0$. If $i \ne j$, it follows that the sets $\{\gamma_0^n(x_i)\}_{n=0}^\infty$ and $\{\gamma_0^n(x_j)\}_{n=0}^\infty$ are disjoint.

For each integer $i > 0$ there exists an integer $n_i \ge 0$ so that $y_i = \gamma_0^{-n_i}(x_i) \in [\gamma_0(y), y]$. We claim that the points y_i are mutually distinct. Otherwise, for some $i \ne j$ and for $n > \max\{n_i, n_j\}$, we would have $\gamma_0^{n-n_i}(x_i) = \gamma_0^{n-n_j}(x_j)$ in contradiction to the above paragraph. Let $y' \in [\gamma_0(y), y]$ be an accumulation point of $\{y_i\}$ and let L' be the leaf through y'. Since $T - \{x_0\}$ meets no leaf of class $\le k$, L' is not such a leaf.

For a fixed but arbitrary number $i+1$, choose m so large that $\gamma_0^m(y)$ is ε_i-close to x_0. Since also x_j is ε_i-close to x_0 for $j \ge i+1$, we see that $\gamma_0^m(y_j) = \gamma_0^{m-n_j}(x_j)$ is a point of U_{i+1}. That is, $\gamma_0^m(y_j)\in[\gamma_0^{m+1}(y), \gamma_0^m(y)]\cap U_{i+1}$, $j\geq i+1$. Thus U_{i+1} accumulates at $\gamma_0^m(y')\in L'$. Since $L'\subset L^c$ is a leaf not of class $\leq k$, it follows that $e\in E_{k+1}$. ■

We are ready to prove the main result. We need not assume M to be orientable nor $\mathcal{F}$ to be transversely orientable.

THEOREM 4. *Let L be a leaf of dimension $n-1$ in a C^2 foliation of a closed n-manifold. If L has polynomial growth of degree r, then L has type at most r. If L is proper, then the type is at most $r-1$.*

Proof. By passing to a finite cover we obtain the situation in which the n-manifold M is orientable and the foliation is transversely orientable, so we can assume this without loss of generality.

If L is compact, then $r=0$ and the type of L is -1. If M is a minimal set, then $r\geq 1$ and, by Theorem 3, the type of L is 0. Thus, we assume that M is not a minimal set and that L is noncompact.

By Lemma 4, $r\geq q+1$. By Proposition 4, the type of L is at most $q+1$. If L is proper, then $E_{q+1}=\emptyset$ and the type is at most q. ■

Elsewhere we have shown [C–C1] that every orientable surface of finite type occurs with polynomial growth as a leaf in suitable C^∞ foliations of suitable closed 3-manifolds. It is not true [C–C2] that every such surface so occurs in all closed 3-manifolds.

We do not know an example of a C^2 foliation with a leaf of type r having growth of degree r, but it is not difficult to construct examples, for all integers r and p with $1\leq p\leq r$, of smooth foliations with leaves having growth of degree r and type $r-p$.

We give an application of our theory to foliations "almost without holonomy." Such a foliation has nontrivial holonomy only along the compact leaves [M].

COROLLARY. *If $\mathcal{F}$ is almost without holonomy, then each leaf is of type at most* 1, *and the proper leaves are of type at most* 0. *The leaves of type* 1 *have at most two limit ends.*

Proof. We assume there is a compact leaf. Otherwise we are reduced to Theorem 3. It is known, and easily proven using [Sa, Theorem 4 and the proof of Theorem 6] and a relative version of [P2, Theorem 6.3], that every leaf of $\mathcal{F}$ has polynomial growth. By Theorem 2, each leaf can only be asymptotic to a compact leaf or to a nonproper leaf, so, for any noncompact leaf L, the integer q of

Lemma 4 is 0. By Proposition 4, if L is proper and noncompact, it is of type 0, while a nonproper leaf L is of type at most 1. For a nonproper leaf, Proposition 1 asserts that E_1 has at most two elements. Since the limit ends are in E_1, all assertions are proven. ■

6. Subexponential growth

We give examples showing how essential it has been to assume polynomial growth as opposed to subexponential growth in the results of this paper.

Let T designate the closed, orientable surface of genus 2, and choose disjoint circles C_1 and C_2 on T that together do not separate T. Let $F, G: I \to I$ be C^r diffeomorphisms, $0 \le r \le \infty$, that are C^r-tangent to the identity at δI. Cut $T \times I$ apart along $C_1 \times I$ and reglue with the identification $(x, t) \equiv (x, F(t))$ and do the same along $C_2 \times I$ with $(x, t) \equiv (x, G(t))$. This converts the product foliation of $T \times I$ to a C^r foliation of $T \times I$, C^r-trivial at the boundary, denoted by $\mathcal{F}(F, G)$. This can be viewed as part of a C^r-foliation of $T \times S^1$. If $r = 0$, the foliation is continuously C^1.

Fix F (with $r = \infty$) so that $F(t) > t$ on $\operatorname{int}(I)$. For notational convenience, identify I with $[-\infty, \infty]$ in such a way that $F(t) = t + 1$, $-\infty < t < \infty$. Also, fix a basepoint $x_0 \in T$.

We will choose $G = G_\alpha$, $0 \le \alpha \le \omega$, and so obtain infinitely many foliations $\mathcal{F}^\alpha = \mathcal{F}(F, G_\alpha)$. The symbol L_t^α will denote the leaf of $\mathcal{F}^\alpha$ containing the point (x_0, t). Remark that $L_t^\alpha = L_{t+1}^\alpha$.

Set $G_0 =$ identity. Every leaf of $\mathcal{F}^0$, except the boundary leaves $T \times \{\pm\infty\}$, will be homeomorphic to the surface of type 0 with two ends as pictured in Figure 3.

Let $0 < N_0 < N_1 < \cdots < N_k < \cdots$ be a sequence of integers. We will define $\{G_k\}$ inductively, $1 \le k < \omega$, so that $G_k - G_{k-1} = \varphi_k$ is a C^∞ bump function vanishing identically outside of $(N_k, N_k + 1)$. If we choose the bump functions so that, for each $r \ge 0$, the sequence $\{\varphi_k^{(r)}\}_{k=1}^\infty$ of r^{th} derivatives converges to 0 uniformly and rapidly enough, we can guarantee that $G_\omega = \lim_{k \to \omega} G_k$ is a C^∞ diffeomorphism and is C^∞-tangent to the identity at $\pm\infty$. In any case, G_ω will be a homeomorphism.

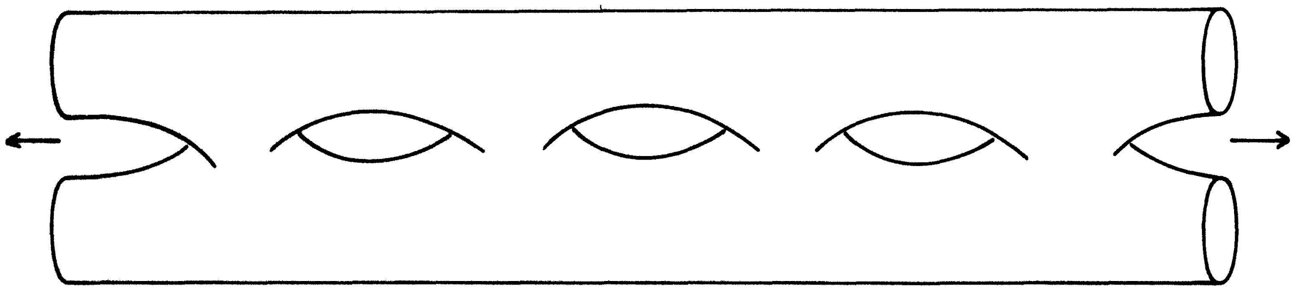

Figure 3

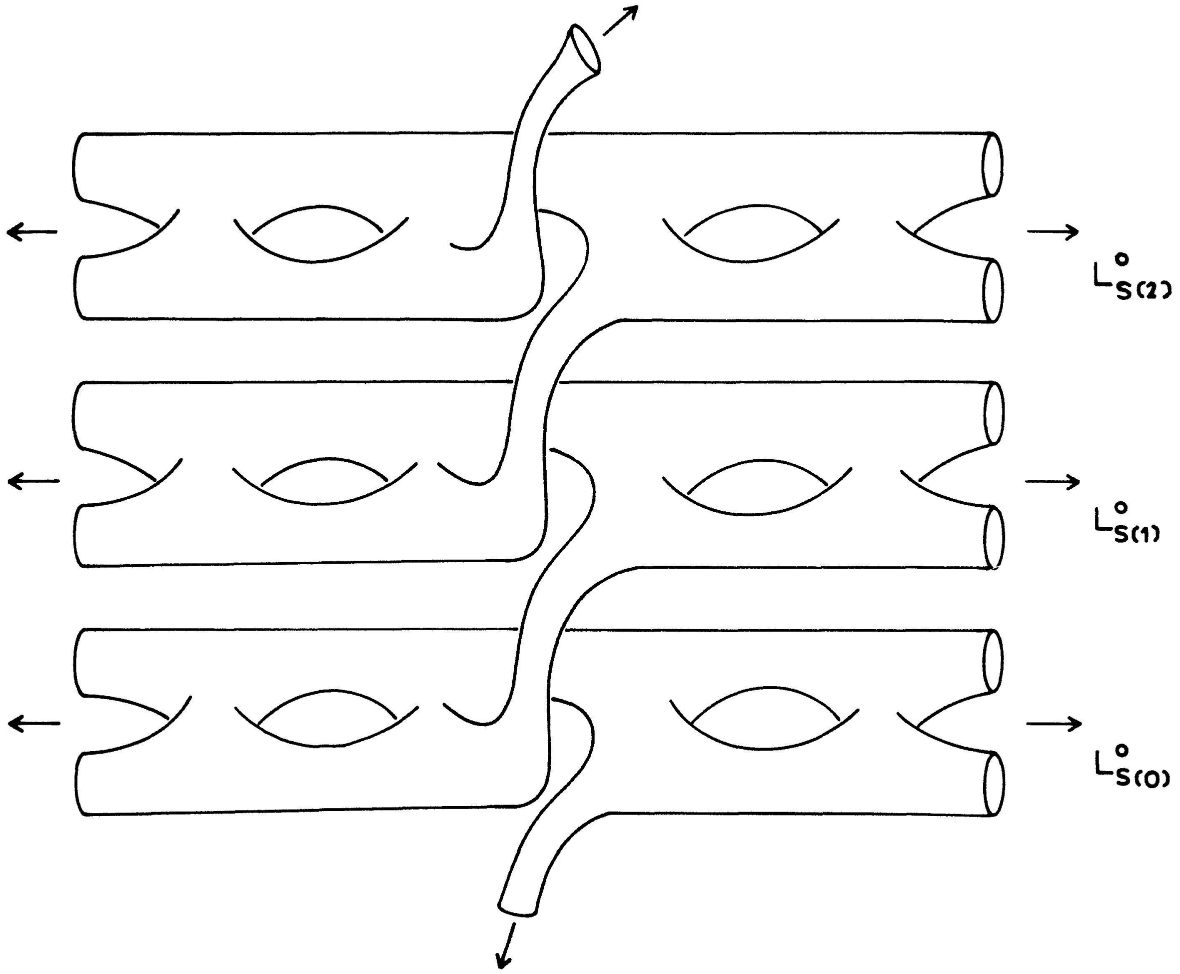

Figure 4

Choose $c_1 = 0 \leq a_1 < b_1 \leq 1 = d_1$ and define $\varphi_1 = G_1 - G_0$ as above so that $\operatorname{supp}(\varphi_1) = [N_1 + a_1, N_1 + b_1]$. Inductively, choose $c_k \leq a_k < b_k \leq d_k$ so that $c_k \in (a_{k-1}, b_{k-1})$ and $G_{k-1}(N_{k-1} + c_k) = N_{k-1} + d_k$, and choose $\varphi_k = G_k - G_{k-1}$ so that $\operatorname{supp}(\varphi_k) = [N_k + a_k, N_k + b_k]$. All of this can be done in such a way that the first k derivatives of φ_k have absolute values uniformly as small as desired. Thus, $G_\omega = \lim_{k \to \omega} G_k$ is a C^∞ diffeomorphism as desired.

LEMMA 9. *If $a_k < t < b_k$, then L_t^k is a surface of type k, and if $t \in [c_k, a_k] \cup [b_k, d_k]$, then $L_t^k = L_t^{k-1}$. Finally, if $t \in \bigcap_{k=1}^\infty [a_k, b_k]$, then L_t^ω is of type ω.*

Proof. For $t \in (a_k, b_k)$, $L_t^k = L_s^k$ where $s = t + N_k$. Then $s(n) = G_k^n(s)$, $n \in \mathbb{Z}$, will define distinct points for distinct values of n, and L_s^k will be the infinite connected sum of the leaves $L_{s(n)}^{k-1}$, $n \in \mathbb{Z}$ (for $k = 1$, cf. Figure 4).

By induction, the leaves $L^{k-1}_{s(n)}$ are of type $k-1$, so L^k_s is of type k. Clearly, $L^k_t = L^{k-1}_t$ if $t \in [c_k, a_k] \cup [b_k, d_k]$. Finally, if $t \in \bigcap [a_k, b_k]$, choose a fundamental system $\{V_i\}$ of open neighborhoods of $T \times \{\infty\}$, let C_i denote the complement of V_i in $T \times [-\infty, \infty]$, and arrange that $\mathcal{F}^\omega | C_i = \mathcal{F}^i | C_i$ and that $L^\omega_t \cap V_i$ has exactly one component of type $> i$, $0 \le i < \infty$. It is rather easy to see that this is possible. If $K \subset L^\omega_t$ is compact, choose N so large that $K \subset C_N$, hence exactly one component of $L^\omega_t - K$ is a neighborhood of ends of type $> N$. It follows that $\mathcal{E}^\omega(L^\omega_t)$ is a single point. ■

Remarks. The closure of L^ω_t contains leaves of class k, all $k \ge 0$. If $c_j = a_j$ and $d_j = b_j$ for $j \ge 1$, then L^ω_t is an everywhere dense leaf, but if $c_j < a_j < b_j < d_j$ for all $j \ge 1$, then L^ω_t is a nowhere dense, nonproper leaf. These situations are in contrast with the behavior of leaves with polynomial growth.

LEMMA 10. *If $N_k \ge (k+1)^2$ for all k and if $t \in \bigcap_{k=1}^\infty [a_k, b_k]$, then L^ω_t has neither exponential nor polynomial growth.*

Proof. Since L^ω_t is of infinite type, it cannot have polynomial growth (Theorem 4). For $0 \le \alpha \le \omega$, let $g_\alpha(m)$ denote the number of distinct points in $[-\infty, \infty]$ that can be reached by applying to t a word in F and G_α of length at most m. By standard theory, g_α has the same growth type as the leaf L^α_t, $0 \le \alpha \le \omega$. We will show that $\lim_{m\to\infty} (1/m) \log (g_\omega(m)) = 0$, thus proving that L^ω_t has nonexponential growth. The proof of this is due to Hector [H] and substantially simplifies an earlier argument of the authors. If $k^2 \le m \le (k+1)^2$, then $m \le (k+1)^2 \le N_k$ and so $g_\omega(m) = g_{k-1}(m) \le (2m+1)^k$ (a very generous inequality). Since $k \le \sqrt{m}$, it follows that $(1/m) \log (g_\omega(m)) \le (1/\sqrt{m}) \log (2m+1)$ for all $m > 0$. By L'Hôpital's rule, $\lim_{m\to\infty} (1/\sqrt{m}) \log (2m+1) = 0$. ■

A variation on the above theme produces an example showing that, for continuously C^1 foliations (as defined in Section 2), the statement of Theorem 2 becomes false when the assumption of polynomial growth is replaced by that of nonexponential growth.

Again take $G_0 =$ identity and require that $\varphi_k = G_k - G_{k-1}$ be a bump function with support in $[N_k, N_k+1]$. We let $\alpha_k = 1/(k+3)$, $\beta_k = (k+2)/(k+3)$, and require that

$$G_k(t) = t, \qquad t \in [N_k, N_k + \alpha_k] \cup [N_k + \beta_k, N_k + 1]$$
$$G_k(t) > t, \qquad t \in (N_k + \alpha_k, N_k + \beta_k)$$
$$G_k(N_k + \alpha_{k-1}) = N_k + \beta_{k-1}.$$

This time, $G_\omega = \lim_{k\to\omega} G_k$ is only asserted to be a homeomorphism and the foliation $\mathcal{F}^\omega$ is continuously C^1. Every leaf of $\mathcal{F}^\omega$ is proper.

For $0 \le \alpha \le \omega$, the leaf L_0^α is independent of α and will be denoted by L. The limit set of $L_{1/2}^\omega$ contains L and, as before, a suitable choice of $\{N_k\}$ will guarantee that $L_{1/2}^\omega$ has nonexponential growth.

Again we can choose a fundamental system of open neighborhoods $\{V_i\}$ of $T\times\{\infty\}$ such that $\mathcal{F}^\omega|C_i = \mathcal{F}^i|C_i$, C_i the complement of V_i in $T\times[-\infty,\infty]$. Since L has trivial germinal holonomy in each $\mathcal{F}^i$, it follows that the same is true for L in $\mathcal{F}^\omega$. In particular, L cannot have an element of contracting holonomy in $\mathcal{F}^\omega$.

We remark that one can demonstrate the impossibility of carrying out this construction in such a way that F is of class C^2 and G_ω of class C^1. This requires a generalized version of [K, Lemma 1].

REFERENCES

[A-S] Ahlors, L. V. and Sario, L., *Riemann Surfaces*, Chapter I, Section 6, Princeton Univ. Press, Princeton, N.J., 1960.

[B] Bernoulli, J., *Ars Conjectandi*, Basle, 1713.

[C-C1] Cantwell, J. and Conlon, L., *Leaf prescriptions for closed* 3-*manifolds*, Trans. A.M.S., to appear.

[C-C2] Cantwell, J. and Conlon, L., *Leaves with isolated ends in foliated* 3-*manifolds*, Topology, to appear.

[D] Dorrie, H., 100 *Great Problems of Elementary Mathematics, Their History and Solution*, Dover Publications, Inc., New York, 1965.

[H] Hector, G., *Subexponential leaves*, preprint.

[K] Kopell, N., *Commuting diffeomorphisms*, Proceedings of Symposia in Pure Mathematics, XIV, Amer. Math. Soc, Providence, R. I., 1970, 165–184.

[M] Moussu, R., *Sur les feuilletages de codimension un*, Thesis, University Paris, 1971.

[Ni] Nishimori, T., *Isolated ends of open leaves of codimension-one foliations*, Quart. J. Math. Oxford (3) *26* (1975), 159–167.

[No] Novikov, S. P., *Topology of foliations*, (Russian) Trudy. Moskov. Mat. Obsc. *14* (1965), 248–278 = Trans. Moscow Math. Soc. (AMS Translation) 1965, 268–304.

[P1] Plante, J. F., *Measure preserving pseudogroups and a theorem of Sacksteder*, Ann. de l'Institute Fourier, *25* (1) (1975), 237–249.

[P2] Plante, J. F., *Foliations with measure preserving holonomy*, Ann. Math., *102* (1975), 327–361.

[Ri] Richards, I., *On the classification of noncompact surfaces*, Trans. AMS, *106* (1963), 259–269.

[Sa] Sacksteder, R., *Foliations and pseudogroups*, Amer. J. Math., *87* (1965), 79–102.

[S-S] Sacksteder, R. and Schwartz, A. J., *Limit sets of foliations*, Ann. Institute Fourier, *15* (2) (1965), 201–214.

[T] Tischler, D., *On fibering certain foliated manifolds over* S^1, Topology, *9* (1970), 153–154.

[W] Widder, D., *Advanced Calculus*, Prentice-Hall, New York, 1947, p. 130.

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