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An inequality involving the absolute value of an entire function and the counting function of its zeros*

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Dedicated to Professor A. Pfluger on his seventieth birthday

The purpose of this paper is to give a proof of

THEOREM 1. *Let $f(z)$ be an entire function of finite, non-integral order λ. If γ satisfies*

$$\max(\tfrac{1}{2}, \lambda) < \gamma < [\lambda]+1; \qquad \beta = \pi/2\gamma,$$

then

$$\liminf_{|z|\to\infty} \frac{\log|f(ze^{i\beta})| + \log|f(ze^{-i\beta})|}{N(|z|, 1/f)} \le \frac{2\pi\lambda \cos(\pi\lambda - \beta\lambda)}{\sin \pi\lambda}.$$

The canonical product $g_\lambda(z)$ with negative zeros and $N(r, 1/g_\lambda) \sim r^\lambda/\lambda$ has the asymptotic behavior [4, p. 232]

$$\log|g_\lambda(re^{i\theta})| = (1+o(1)) \frac{\pi \cos \lambda\theta}{\sin \pi\lambda} r^\lambda$$

as $r \to \infty$, uniformly in $|\theta| < \pi - \epsilon < \pi$. Therefore

$$\{\log|g_\lambda(re^{i(\pi-\beta)})| + \log|g_\lambda(re^{-i(\pi-\beta)})|\}/N(r, 1/g_\lambda)$$
$$\to \frac{2\pi\lambda \cos(\pi\lambda - \beta\lambda)}{\sin \pi\lambda}$$

as $r \to \infty$. This shows that Theorem 1 is best-possible.

It may well be that the assertion of Theorem 1 holds for a wider range of γ, but the method of proof does not give this result.

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We may assume without loss of generality that

$$f(0)=1.$$

We shall always use the standard notations

$$z=re^{i\theta}, \qquad N(r)=N(r,1/f).$$

Proof of Theorem 1. Theorem 1 is a consequence of

$$\log|f(te^{i(\theta+\beta)})|+\log|f(te^{i(\theta-\beta)})| < \left(\frac{2\pi\lambda\cos(\pi\lambda-\beta\lambda)}{\sin\pi\lambda}+\eta\right)N(t) \qquad (0\le\theta<2\pi,\ \eta>0) \tag{1}$$

where t is a Pólya peak of $N(r)$ satisfying $t>t_0(\eta;N(r))$. It will suffice to prove (1) for $\theta=0$, the general case then follows by considering $f(ze^{i\varphi})$, φ constant. The Pólya peaks are a sequence of positive numbers t_n tending to ∞ such that for given $\epsilon>0$ the inequalities

$$N(u)\le(u/t_n)^{\lambda-\epsilon}N(t_n) \qquad (1<u\le t_n) \tag{2}$$

$$N(u)\le(u/t_n)^{\lambda+\epsilon}N(t_n) \qquad (t_n<u) \tag{3}$$

hold for all $n>n_0(\epsilon)$. It is well known that Pólya peaks exist [2, p. 103]. Since for a function of non-integral order λ $\limsup N(r)r^{-\lambda+\epsilon}=\infty$ $(\epsilon>0)$, (2) implies that

$$\lim_{n\to\infty} N(t_n)t_n^{-\lambda+\epsilon}=\infty. \tag{4}$$

Consequently (2) can be replaced by

$$N(u)\le(u/t_n)^{\lambda-\epsilon}N(t_n) \qquad (0\le u\le t_n) \tag{2'}$$

for all large t_n, since $N(u)\equiv 0$ near $u=0$.

Apply Green's Formula

$$\iint_B (u\Delta v-v\Delta u)\,dA=\int_{\partial B}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds$$

to

$$u=\log|f(z)|, \qquad v=\log\left|\frac{z^\gamma+t^\gamma}{z^\gamma-t^\gamma}\right|-\frac{2t^\gamma r^\gamma\cos\gamma\theta}{r^{2\gamma}+t^{2\gamma}} \qquad (t>0)$$

in the domain B obtained from the sector

$$D: \rho < r < R, \qquad |\theta| < \beta$$

by removing disks of radius δ centered at t and at the zeros a_ν of $f(z)$. Here ρ and R are chosen so that $|z| = \rho$ and $|z| = R$ do not contain zeros of $f(z)$ and

$$\rho < t < R.$$

On letting $\delta \to 0$ we obtain

$$\begin{aligned}\iint_D \log|f(z)|\, \Delta v(z)\, dA &+ \int_{-\beta}^{\beta} \Big\{ v(Re^{i\theta}) \frac{\partial}{\partial r} \log|f(Re^{i\theta})| \\ &- \log|f(Re^{i\theta})| \frac{\partial v}{\partial r}(Re^{i\theta}) \Big\} R\, d\theta \\ &+ \int_{-\beta}^{\beta} \Big\{ \log|f(\rho e^{i\theta})| \frac{\partial v}{\partial r}(\rho e^{i\theta}) - v(\rho e^{i\theta}) \frac{\partial}{\partial r} \log|f(\rho e^{i\theta})| \Big\} \rho\, d\theta \\ &= 2\pi \log|f(t)| + 2\pi \sum_{a_\nu \in D} v(a_\nu).\end{aligned} \tag{5}$$

The definition of $v(z)$ shows that for $R \to \infty$, uniformly in $|\theta| \le \beta$

$$v(Re^{i\theta}) = O((t/R)^{3\gamma}); \qquad \frac{\partial v}{\partial r}(Re^{i\theta}) = O(t^{3\gamma} R^{-3\gamma-1}). \tag{6}$$

It is known that there are arbitrarily large values of R such that [1]

$$\int_0^{2\pi} \left| \frac{\partial}{\partial r} \log|f(Re^{i\theta})| \right| d\theta \le \int_0^{2\pi} \left| \frac{f'}{f}(Re^{i\theta}) \right| d\theta$$
$$< A(\lambda) T(2R, f)/R < A_2(\lambda) R^{\lambda - 1 + \epsilon}. \tag{7}$$

Also in $0 \leqslant \theta < \beta$, $\rho < r < R$, $z \neq t$,

$$\frac{\partial v}{\partial \theta}(re^{i\theta}) = -\frac{\partial v}{\partial \theta}(re^{-i\theta}) = \frac{2\gamma r^\gamma t^\gamma \sin \gamma\theta}{r^{2\gamma} + t^{2\gamma}} \left\{ 1 - \frac{(r^{2\gamma} + t^{2\gamma})^2}{|z^{2\gamma} - t^{2\gamma}|^2} \right\} \le 0,$$

and $v(re^{i\beta}) = v(re^{-i\beta}) = 0$ $(\beta = \pi/2\gamma)$.

Therefore $v(z)>0$ $(z\in D)$. Using these observations and the explicit value

$$\Delta v=\frac{16\gamma^2t^{3\gamma}r^{3\gamma-2}\cos\gamma\theta}{(r^{2\gamma}+t^{2\gamma})^3},$$

we let $R\to\infty$ and $\rho\to 0$ in (5) to obtain

$$\log|f(t)|\le\frac{8\gamma^2t^{3\gamma}}{\pi}\int_0^\infty\frac{r^{3\gamma-1}\,dr}{(r^{2\gamma}+t^{2\gamma})^3}\int_{-\beta}^{\beta}\cos\gamma\theta\log|f(re^{i\theta})|\,d\theta. \tag{8}$$

Let $h(\theta)=h(\theta+2\pi)$ be defined by

$$h(\theta)=\begin{cases}\cos\gamma\theta & (|\theta|\le\beta)\\ 0 & (\beta<|\theta|\le\pi).\end{cases}$$

Now $h(\theta)$ has the (convergent) Fourier series

$$h(\theta)=\sum_{-\infty}^{\infty}a_ke^{ik\theta}=\sum(\gamma/\pi)\cos k\beta(\gamma^2-k^2)^{-1}e^{ik\theta}, \tag{9}$$

and if

$$\log|f(re^{i\theta})|=\sum_{-\infty}^{\infty}b_k(r)e^{ik\theta},$$

then, by Parseval's formula,

$$\frac{1}{2\pi}\int_{-\beta}^{\beta}\cos\gamma\theta\log|f(re^{i\theta})|\,d\theta=\sum_{-\infty}^{\infty}a_{-k}b_k(r)\le\sum_{-\infty}^{\infty}|a_{-k}|\,|b_k(r)|. \tag{10}$$

By an important observation of J. Miles and D. F. Shea [3]

$$|b_{-k}(r)|=|b_k(r)|\le A_kr^k+\tfrac{1}{2}k\int_0^r\{(r/u)^k-(u/r)^k\}(N(u)/u)\,du+N(r)\,\|\,(0\le k<\lambda)$$

$$\le\tfrac{1}{2}k\int_0^r(u/r)^k(N(u)/u)\,du+\tfrac{1}{2}k\int_r^\infty(r/u)^k(N(u)/u)\,du-N(r)\qquad(k>\lambda). \tag{11}$$

Now we choose $\epsilon>0$ so small that the interval $[\lambda-\epsilon,\lambda+\epsilon]$ does not contain an integer and we choose for $t=t_n$ a Pólya peak satisfying (3) and (2′) with this ϵ.

Using the inequalities (3) and (2′) for $N(u)$ in (11) leads to the following estimates.

If

$$0\le k<\lambda, \qquad 0\le r\le t,$$

$$|b_k(r)|\le A_k r^k+\frac{k^2}{(\lambda-\epsilon)^2-k^2}N(t)(r/t)^{\lambda-\epsilon}+N(r). \tag{12}$$

If

$$0\le k\le\lambda, \qquad t<r,$$

$$|b_k(r)|\le A_k r^k+\frac{k^2}{(\lambda+\epsilon)^2-k^2}N(t)(r/t)^{\lambda+\epsilon}+\frac{\epsilon k}{(\lambda-k)^2-\epsilon^2}N(t)(r/t)^k+N(r). \tag{13}$$

If

$$\lambda<k, \qquad 0\le r\le t,$$

$$|b_k(r)|\le\frac{k^2}{k^2-(\lambda-\epsilon)^2}N(t)(r/t)^{\lambda-\epsilon}+\frac{\epsilon k}{(k-\lambda)^2-\epsilon^2}N(t)(r/t)^k-N(r). \tag{14}$$

If

$$\lambda<k, \qquad t<r,$$

$$|b_k(r)|\le\frac{k^2}{k^2-(\lambda+\epsilon)^2}N(t)(r/t)^{\lambda+\epsilon}+\frac{\epsilon k}{(k+\lambda)^2-\epsilon^2}N(t)(t/r)^k-N(r). \tag{15}$$

Now replace $f(z)$ by $f_1(z)=f(ze^{i\beta})f(ze^{-i\beta})$. Then

$$\log|f_1(\mathrm{re}^{i\theta})|=2\sum_{-\infty}^{\infty}\cos k\beta b_k(r)e^{ik\theta}$$

and by (9) and (10)

$$J=\frac{1}{2\pi}\int_{-\beta}^{\beta}\cos\gamma\theta\log|f_1(\mathrm{re}^{i\theta})|\,d\theta\le(2\gamma/\pi)\sum_{-\infty}^{\infty}\cos^2 k\beta|\gamma^2-k^2|^{-1}|b_k(r)|.$$

Using (12) and (14) this leads to

$$J\le N(t)(r/t)^{\lambda-\epsilon}(2\gamma/\pi)\sum_{-\infty}^{\infty}\cos^2 k\beta(\gamma^2-k^2)^{-1}((\lambda-\epsilon)^2-k^2)^{-1}k^2$$

$$+C_1(1+r^{[\lambda]})+\epsilon C_2N(t)+(2\gamma/\pi)\sum_{-\infty}^{\infty}\cos^2 k\beta(\gamma^2-k^2)^{-1}N(r). \qquad (0<r<t) \tag{16}$$

(Note the disappearance of the absolute value signs.)

Similarly, by (13) and (15)

$$J \le N(t)(r/t)^{\lambda+\epsilon}(2\gamma/\pi)\sum_{-\infty}^{\infty}\cos^2 k\beta(\gamma^2-k^2)^{-1}((\lambda+\epsilon)^2-k^2)^{-1}k^2$$
$$+C_1(1+r^{[\lambda]})+\epsilon C_3N(t)+(2\gamma/\pi)\sum_{-\infty}^{\infty}\cos^2 k\beta(\gamma^2-k^2)^{-1}N(r). \qquad (t<r) \qquad (17)$$

By (9)

$$0=h(\beta)+h(-\beta)=2(\gamma/\pi)\sum_{-\infty}^{\infty}\cos^2 k\beta(\gamma^2-k^2)^{-1},$$

so that the terms with $N(r)$ in (16) and (17) disappear.

Using (16) and (17) in (8) (with f_1 in place of f) and dividing by $N(t)$ we have

$$\{\log|f(te^{i\beta})|+\log|f(te^{-i\beta})|\}/N(t)\le 16\gamma^2t^{3\gamma}\int_0^t\frac{r^{3\gamma-1}(r/t)^{\lambda-\epsilon}\,dr}{(r^{2\gamma}+t^{2\gamma})^3}S_1$$
$$+16\gamma^2t^{3\gamma}\int_t^\infty\frac{r^{3\gamma-1}(r/t)^{\lambda+\epsilon}}{(r^{2\gamma}+t^{2\gamma})^3}S_2+\epsilon C_4t^{3\gamma}\int_0^\infty\frac{r^{3\gamma-1}\,dr}{(r^{2\gamma}+t^{2\gamma})^3}$$
$$+(C_5/N(t))t^{3\gamma}\int_0^\infty\frac{r^{3\gamma-1}(1+r^{[\lambda]})}{(r^{2\gamma}+t^{2\gamma})^3}\,dr, \qquad (18)$$

where

$$S_1=(2\gamma/\pi)\sum_{-\infty}^{\infty}k^2\cos^2 k\beta(\gamma^2-k^2)^{-1}((\lambda-\epsilon)^2-k^2)^{-1}$$
$$S_2=(2\gamma/\pi)\sum_{-\infty}^{\infty}k^2\cos^2 k\beta(\gamma^2-k^2)^{-1}((\lambda+\epsilon)^2-k^2)^{-1}.$$

The last two integrals in (18) are less than

$$C_6\epsilon+C_7t^{[\lambda]}/N(t)<C_8\epsilon,$$

in view of (4).

By an easy application of dominated convergence the first two terms on the right hand side of (18) tend to

$$L=16\gamma^2t^{3\gamma}\int_0^\infty\frac{r^{3\gamma-1}(r/t)^\lambda\,dr}{(r^{2\gamma}+t^{2\gamma})^3}\cdot(2\gamma/\pi)\sum_{-\infty}^{\infty}k^2\cos^2 k\beta(\gamma^2-k^2)^{-1}(\lambda^2-k^2)^{-1}. \qquad (19)$$

The sum and the integral can be evaluated explicitly (see below) giving

$$L=\frac{2\pi\lambda\cos(\pi\lambda-\beta\lambda)}{\sin\pi\lambda}. \tag{20}$$

(19), (20) and (18) together prove (1) and the Theorem is proved.

The evaluation of the sum in (19) can be accomplished by integrating $z^2(1+e^{2\beta zi})/\{(\gamma^2-z^2)(\lambda^2-z^2)(e^{2\pi iz}-1)\}$ around a large square and taking real parts. The integral is obtained by differentiating the formula

$$\int_0^\infty \frac{x^{\mu-1}}{x+s}\,dx=\frac{\pi s^{\mu-1}}{\sin\pi\mu} \qquad (0<\mu<1)$$

twice with respect to s and noting that the resulting integral formula remains valid in $0<\mu<2$ by analytic continuation.

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