Holomorphic Lipschitz functions in balls.

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Holomorphic Lipschitz functions in balls

WALTER RUDIN

Fix $n>1$, let B be the open unit ball of $\mathbb{C}^n$, suppose that f is holomorphic in B and that f satisfies a Lipschitz condition of order $\alpha>0$. Stein [3] has observed (actually, for domains much more general than B) that f is then, roughly speaking, twice as smooth in the direction of the complex tangents. The present note adds to this that the same conclusion can even be derived from much weaker hypotheses: it is enough to assume that the slice functions f_w of f (see below) form a bounded subset of Lip α. In particular, it is not even necessary to assume that f is continuous on $\bar{B}$.

For the sake of simplicity, we confine ourselves to the range $0<\alpha<1$.

DEFINITIONS. On $\mathbb{C}^n$ there is the inner product $\langle z, w\rangle=\sum z_j\bar{w}_j$ and the associated norm $|z|=\langle z, z\rangle^{1/2}$. Thus $B=\{z:|z|<1\}$.

For $0<\alpha<1$, we let K_α be the set of all $f:\bar{B}\to\mathbb{C}$ such that

(i) f is holomorphic in B,

(ii) for each $w\in S=\partial B$, the slice function f_w defined by $f_w(\lambda)=f(\lambda w)$ is continuous on the closed unit disc in $\mathbb{C}$, and satisfies the Lipschitz condition

$$|f_w(e^{i\theta})-f_w(e^{i\varphi})|\le|\theta-\varphi|^\alpha \qquad (\theta, \varphi\in\mathbf{R}). \tag{1}$$

We say that a C^1-curve $\gamma:\mathbf{R}\to S$ is *complex-tangential* if $\langle\gamma'(t), \gamma(t)\rangle=0$ for every $t\in\mathbf{R}$. We say that γ is *normalized* if $|\gamma'(t)|=1$, i.e., if γ is parametrized by arc length.

Here are our main results:

THEOREM 1. *If $0<\alpha<\frac{1}{2}$, there is a constant $A(\alpha)<\infty$ such that the inequality*

$$|f(\gamma(t+h))-f(\gamma(t))|\le A(\alpha)|h|^{2\alpha}$$

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holds for all $f \in K_\alpha$, *for all complex-tangential normalized curves* γ, *and for all* t, $h \in \mathbf{R}$.

THEOREM 2. *If* $\frac{1}{2} \leq \alpha < 1$, *there is a constant* $A(\alpha) < \infty$ *such that the inequality*

$$|f(\gamma(t+h)) + f(\gamma(t-h)) - 2f(\gamma(t))| \leq A(\alpha) \|\gamma''\|_\infty |h|^{2\alpha}$$

holds for all $f \in K_\alpha$, *for all complex-tangential normalized curves of class* C^2, *and for all* t, $h \in \mathbf{R}$.
Here

$$\|\gamma''\|_\infty = \sup\{|\gamma''(t)| : t \in \mathbf{R}\}.$$

Remarks. (i) In the terminology of [2] and [3], the main point of these two theorems can be briefly stated as follows:

If $\{f_w : w \in S\}$ is a bounded set in Λ_α, then $f \circ \gamma \in \Lambda_{2\alpha}$. It follows that $f \in \Gamma_\alpha$.

(We recall that $\Lambda_\alpha = \text{Lip}\,\alpha$ when $0 < \alpha < 1$; see Ch. V, §4 of [2]; for Γ_α, see [3].)

(ii) Each $w \in S$ lies on a circle $T_w = \{e^{i\theta} w : \theta \in \mathbf{R}\}$. Our smoothness assumption is imposed on the restrictions of f to these circles. The complex-tangential curves γ (on which f turns out to be "twice as smooth") are precisely those that are perpendicular (in the sense of the usual real scalar product in $\mathbf{R}^{2n} = \mathbb{C}^n$) to every T_w that they intersect.

(iii) Although the research announcement [3] contains no proofs, it does mention a key fact: the complex-tangential partial derivatives of a holomorphic function in B satisfy more restrictive growth conditions than does the radial derivative. This is also the point of Lemma 2 in the present paper.

THE RADIAL DERIVATIVE Rf. Every f that is holomorphic in B has an expansion $f = \sum F_k$ in which each F_k is a homogeneous polynomial of degree k. Define

$$(Rf)(z) = \sum_{k=0}^{\infty} kF_k(z) \qquad (z \in B). \tag{2}$$

Rf is related to the derivative of the slice functions f_w by

$$(Rf)(\lambda w) = \lambda f'_w(\lambda) \qquad (w \in S, |\lambda| < 1). \tag{3}$$

For our purposes, Rf is preferable to f'_w since (2) shows that Rf is a holomorphic function in B.

In the following lemmas, $\{e_1, \dots, e_n\}$ will be an orthonormal basis for $\mathbb{C}^n$, so that $z = \sum z_j e_j$, and we shall write D_j for $\partial/\partial z_j$.

LEMMA 1. *Suppose* $m_1, \dots, m_n$ *are nonnegative integers,* $p = \sum m_j$, $P(D) = D_1^{m_1} \cdots D_n^{m_n}$, *and* f *is holomorphic in* B. *Then, for* $w \in S$, $0 < r < 1$,

$$\int_0^r [P(D)Rf](tw)t^{p-1}\,dt = r^p[P(D)f](rw).$$

Proof. By (2), this is an immediate consequence of the fact that $P(D)F_k$ is homogeneous of degree $k-p$ when $k \geq p$, and that $P(D)F_k = 0$ when $k < p$.

LEMMA 2. *Suppose* G *is holomorphic in* B, $\beta \geq 0$, *and*

$$|G(z)| \leq (1-|z|)^{-\beta} \qquad (z \in B). \tag{4}$$

Then, for $0 < r < 1$,

$$|(D_2G)(re_1)| \leq c(\beta)(1-r)^{-\beta-1/2} \tag{5}$$

and

$$|(D_2^2G)(re_1)| \leq c(\beta)(1-r)^{\beta-1}, \tag{6}$$

where $c(\beta) < \infty$.

Proof. Put $g(\lambda) = G(re_1 + \lambda e_2)$, if $|\lambda|^2 < 1 - r^2$. Put $p = \{\tfrac{1}{2}(1-r^2)\}^{1/2}$. When $|\lambda| = p$ then

$$1 - |re_1 + \lambda e_2|^2 = p^2$$

so that $|g(\lambda)| \leq 2^\beta p^{-2\beta}$. (Note that (4) implies that $|G(z)| \leq 2^\beta(1-|z|^2)^{-\beta}$.) It follows from the Schwarz lemma that

$$|g'(0)| \leq 2^\beta p^{-2\beta-1}, \qquad |g''(0)| \leq 2^{\beta+1} p^{-2\beta-2}. \tag{7}$$

Since $p^2 > \tfrac{1}{2}(1-r)$, (5) and (6) follow from (7).

LEMMA 3. *If* $0 < \alpha < 1$, *there are constants* $c_i(\alpha) < \infty$, $1 \leq i \leq 4$, *such that every*

$f \in K_\alpha$ satisfies the following inequalities:

$$|(Rf)(z)| \le c_1(\alpha)(1-|z|)^{\alpha-1} \qquad (z \in B). \tag{8}$$

$$|f(w)-f(rw)| \le c_2(\alpha)(1-r)^\alpha \qquad (w \in S, 0<r<1). \tag{9}$$

$$|(D_2f)(re_1)| \le c_3(\alpha)(1-r)^{\alpha-1/2} \qquad (0<\alpha<\tfrac{1}{2}). \tag{10}$$

$$|(D_2^2f)(re_1)| \le c_4(\alpha)(1-r)^{\alpha-1} \tag{11}$$

Proof. (8) follows from (3) and a classical theorem of Hardy and Littlewood; see, for instance, p. 74 of [1]. It is clear that (8) implies (9). With $\beta = 1-\alpha$, (8) and Lemma 2 give estimates of D_2Rf and D_2^2Rf; when these are integrated, Lemma 1 yields (10) and (11).

PROOF OF THEOREMS 1 AND 2. Suppose f and γ are as in the hypotheses. Fix $h \in (0, 1)$, put $r = 1-h^2$, and define

$$g(t) = f(r\gamma(t)) \qquad (t \in \mathbf{R}). \tag{12}$$

By (9), it is enough to show that

$$|g(t+h)-g(t)| \le A(\alpha)h^{2\alpha} \qquad (0<\alpha<\tfrac{1}{2}) \tag{13}$$

and

$$|g(t+h)+g(t-h)-2g(t)| \le A(\alpha)\|\gamma''\|_\infty h^{2\alpha} \qquad (0<\alpha<1). \tag{14}$$

For any $t_0 \in \mathbf{R}$, our assumptions on γ show that there is a unitary change of variables which makes $\gamma(t_0) = e_1$, $\gamma'(t_0) = e_2$. Then (12) and (10) give

$$|g'(t_0)| = |r(D_2f)(re_1)| \le c_3(\alpha)h^{2\alpha-1} \tag{15}$$

if $\alpha < \frac{1}{2}$. This proves (13) and hence Theorem 1.

The left side of (14) is at most $h^2\|g''\|_\infty$. Hence (14) will follow from

$$\|g''\|_\infty \le A(\alpha)\|\gamma''\|_\infty h^{2\alpha-2}. \tag{16}$$

For any $t_0 \in \mathbf{R}$, our preceding change of variables shows that (12) leads to

$$g''(t_0) = r^2(D_2^2f)(re_1) + r\sum_{j=1}^{n}(D_jf)(re_1)\gamma''(t_0). \tag{17}$$

By (8) and (10), each derivative of f that occurs in (17) is dominated by $c(\alpha)(1-r)^{\alpha-1}=c(\alpha)h^{2\alpha-2}$. [Note that the right side of (10) can be replaced by $C\log(1/1-r)$ if $\alpha\geq\frac{1}{2}$.] Since differentiation of $\langle\gamma,\gamma\rangle=1$ gives $\operatorname{Re}\langle\gamma',\gamma\rangle=0$, hence

$$\operatorname{Re}\langle\gamma'',\gamma\rangle=-\langle\gamma',\gamma'\rangle=-1, \tag{18}$$

we see that $|\gamma''(t_0)|\geq 1$. Hence (17) gives (16). This completes the proof of Theorem 2.

EXAMPLE. Take $n=2$, define

$$f(z_1,z_2)=\frac{z_2}{z_1}\log\frac{1}{1-z_1}. \tag{19}$$

The singularity at $z_1=0$ is removable, and

$$(Rf)(z_1,z_2)=z_2/(1-z_1). \tag{20}$$

Since $|z_2|^2<1-|z_1|^2$, $|(Rf)(z)|<2^{1/2}(1-|z|)^{-1/2}$. This implies that $Cf\in K_{1/2}$ for some $C>0$.

Since $t\to f(\text{cost},\text{sint})$ is not in Lip 1, we see that Theorem 1 fails when $\alpha=\frac{1}{2}$.

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University of Wisconsin
Madison

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