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## Holomorphic Lipschitz functions in balls

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Fix  $n > 1$ , let  $B$  be the open unit ball of  $\mathcal{C}^n$ , suppose that  $f$  is holomorphic in  $B$  and that  $f$  satisfies a Lipschitz condition of order  $\alpha > 0$ . Stein [3] has observed (actually, for domains much more general than  $B$ ) that  $f$  is then, roughly speaking, twice as smooth in the direction of the complex tangents. The present note adds to this that the same conclusion can even be derived from much weaker hypotheses: it is enough to assume that the slice functions  $f_w$  of  $f$  (see below) form a bounded subset of  $\text{Lip } \alpha$ . In particular, it is not even necessary to assume that  $f$  is continuous on  $\bar{B}$ .

For the sake of simplicity, we confine ourselves to the range  $0 < \alpha < 1$ .

**DEFINITIONS.** On  $\mathcal{C}^n$  there is the inner product  $\langle z, w \rangle = \sum z_j \bar{w}_j$  and the associated norm  $|z| = \langle z, z \rangle^{1/2}$ . Thus  $B = \{z : |z| < 1\}$ .

For  $0 < \alpha < 1$ , we let  $K_\alpha$  be the set of all  $f: \bar{B} \rightarrow \mathcal{C}$  such that

- (i)  $f$  is holomorphic in  $B$ ,
- (ii) for each  $w \in S = \partial B$ , the slice function  $f_w$  defined by  $f_w(\lambda) = f(\lambda w)$  is continuous on the closed unit disc in  $\mathcal{C}$ , and satisfies the Lipschitz condition

$$|f_w(e^{i\theta}) - f_w(e^{i\varphi})| \leq |\theta - \varphi|^\alpha \quad (\theta, \varphi \in \mathbf{R}). \tag{1}$$

We say that a  $C^1$ -curve  $\gamma: \mathbf{R} \rightarrow S$  is *complex-tangential* if  $\langle \gamma'(t), \gamma(t) \rangle = 0$  for every  $t \in \mathbf{R}$ . We say that  $\gamma$  is *normalized* if  $|\gamma'(t)| = 1$ , i.e., if  $\gamma$  is parametrized by arc length.

Here are our main results:

**THEOREM 1.** *If  $0 < \alpha < \frac{1}{2}$ , there is a constant  $A(\alpha) < \infty$  such that the inequality*

$$|f(\gamma(t+h)) - f(\gamma(t))| \leq A(\alpha) |h|^{2\alpha}$$

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holds for all  $f \in K_\alpha$ , for all complex-tangential normalized curves  $\gamma$ , and for all  $t, h \in \mathbf{R}$ .

**THEOREM 2.** *If  $\frac{1}{2} \leq \alpha < 1$ , there is a constant  $A(\alpha) < \infty$  such that the inequality*

$$|f(\gamma(t+h)) + f(\gamma(t-h)) - 2f(\gamma(t))| \leq A(\alpha) \|\gamma''\|_\infty |h|^{2\alpha}$$

holds for all  $f \in K_\alpha$ , for all complex-tangential normalized curves of class  $C^2$ , and for all  $t, h \in \mathbf{R}$ .

Here

$$\|\gamma''\|_\infty = \sup \{|\gamma''(t)| : t \in \mathbf{R}\}.$$

*Remarks.* (i) In the terminology of [2] and [3], the main point of these two theorems can be briefly stated as follows:

If  $\{f_w : w \in S\}$  is a bounded set in  $\Lambda_\alpha$ , then  $f \circ \gamma \in \Lambda_{2\alpha}$ . It follows that  $f \in \Gamma_\alpha$ .

(We recall that  $\Lambda_\alpha = \text{Lip } \alpha$  when  $0 < \alpha < 1$ ; see Ch. V, §4 of [2]; for  $\Gamma_\alpha$ , see [3].)

(ii) Each  $w \in S$  lies on a circle  $T_w = \{e^{i\theta} w : \theta \in \mathbf{R}\}$ . Our smoothness assumption is imposed on the restrictions of  $f$  to these circles. The complex-tangential curves  $\gamma$  (on which  $f$  turns out to be "twice as smooth") are precisely those that are perpendicular (in the sense of the usual real scalar product in  $\mathbf{R}^{2n} = \mathcal{C}^n$ ) to every  $T_w$  that they intersect.

(iii) Although the research announcement [3] contains no proofs, it does mention a key fact: the complex-tangential partial derivatives of a holomorphic function in  $B$  satisfy more restrictive growth conditions than does the radial derivative. This is also the point of Lemma 2 in the present paper.

**THE RADIAL DERIVATIVE  $Rf$ .** Every  $f$  that is holomorphic in  $B$  has an expansion  $f = \sum F_k$  in which each  $F_k$  is a homogeneous polynomial of degree  $k$ . Define

$$(Rf)(z) = \sum_{k=0}^{\infty} kF_k(z) \quad (z \in B). \quad (2)$$

$Rf$  is related to the derivative of the slice functions  $f_w$  by

$$(Rf)(\lambda w) = \lambda f'_w(\lambda) \quad (w \in S, |\lambda| < 1). \quad (3)$$

For our purposes,  $Rf$  is preferable to  $f'_w$  since (2) shows that  $Rf$  is a holomorphic function in  $B$ .

In the following lemmas,  $\{e_1, \dots, e_n\}$  will be an orthonormal basis for  $\mathcal{C}^n$ , so that  $z = \sum z_j e_j$ , and we shall write  $D_j$  for  $\partial/\partial z_j$ .

LEMMA 1. Suppose  $m_1, \dots, m_n$  are nonnegative integers,  $p = \sum m_j$ ,  $P(D) = D_1^{m_1} \cdots D_n^{m_n}$ , and  $f$  is holomorphic in  $B$ . Then, for  $w \in S$ ,  $0 < r < 1$ ,

$$\int_0^r [P(D)Rf](tw)t^{p-1} dt = r^p [P(D)f](rw).$$

*Proof.* By (2), this is an immediate consequence of the fact that  $P(D)F_k$  is homogeneous of degree  $k - p$  when  $k \geq p$ , and that  $P(D)F_k = 0$  when  $k < p$ .

LEMMA 2. Suppose  $G$  is holomorphic in  $B$ ,  $\beta \geq 0$ , and

$$|G(z)| \leq (1 - |z|)^{-\beta} \quad (z \in B). \tag{4}$$

Then, for  $0 < r < 1$ ,

$$|(D_2 G)(re_1)| \leq c(\beta)(1 - r)^{-\beta-1/2} \tag{5}$$

and

$$|(D_2^2 G)(re_1)| \leq c(\beta)(1 - r)^{\beta-1}, \tag{6}$$

where  $c(\beta) < \infty$ .

*Proof.* Put  $g(\lambda) = G(re_1 + \lambda e_2)$ , if  $|\lambda|^2 < 1 - r^2$ . Put  $p = \{\frac{1}{2}(1 - r^2)\}^{1/2}$ . When  $|\lambda| = p$  then

$$1 - |re_1 + \lambda e_2|^2 = p^2$$

so that  $|g(\lambda)| \leq 2^\beta p^{-2\beta}$ . (Note that (4) implies that  $|G(z)| \leq 2^\beta (1 - |z|^2)^{-\beta}$ .) It follows from the Schwarz lemma that

$$|g'(0)| \leq 2^\beta p^{-2\beta-1}, \quad |g''(0)| \leq 2^{\beta+1} p^{-2\beta-2}. \tag{7}$$

Since  $p^2 > \frac{1}{2}(1 - r)$ , (5) and (6) follow from (7).

LEMMA 3. If  $0 < \alpha < 1$ , there are constants  $c_i(\alpha) < \infty$ ,  $1 \leq i \leq 4$ , such that every

$f \in K_\alpha$  satisfies the following inequalities:

$$|(Rf)(z)| \leq c_1(\alpha)(1-|z|)^{\alpha-1} \quad (z \in B). \quad (8)$$

$$|f(w) - f(rw)| \leq c_2(\alpha)(1-r)^\alpha \quad (w \in S, 0 < r < 1). \quad (9)$$

$$|(D_2f)(re_1)| \leq c_3(\alpha)(1-r)^{\alpha-1/2} \quad (0 < \alpha < \frac{1}{2}). \quad (10)$$

$$|(D_2^2f)(re_1)| \leq c_4(\alpha)(1-r)^{\alpha-1} \quad (11)$$

*Proof.* (8) follows from (3) and a classical theorem of Hardy and Littlewood; see, for instance, p. 74 of [1]. It is clear that (8) implies (9). With  $\beta = 1 - \alpha$ , (8) and Lemma 2 give estimates of  $D_2Rf$  and  $D_2^2Rf$ ; when these are integrated, Lemma 1 yields (10) and (11).

**PROOF OF THEOREMS 1 AND 2.** Suppose  $f$  and  $\gamma$  are as in the hypotheses. Fix  $h \in (0, 1)$ , put  $r = 1 - h^2$ , and define

$$g(t) = f(r\gamma(t)) \quad (t \in \mathbf{R}). \quad (12)$$

By (9), it is enough to show that

$$|g(t+h) - g(t)| \leq A(\alpha)h^{2\alpha} \quad (0 < \alpha < \frac{1}{2}) \quad (13)$$

and

$$|g(t+h) + g(t-h) - 2g(t)| \leq A(\alpha)\|\gamma''\|_\infty h^{2\alpha} \quad (0 < \alpha < 1). \quad (14)$$

For any  $t_0 \in \mathbf{R}$ , our assumptions on  $\gamma$  show that there is a unitary change of variables which makes  $\gamma(t_0) = e_1$ ,  $\gamma'(t_0) = e_2$ . Then (12) and (10) give

$$|g'(t_0)| = |r(D_2f)(re_1)| \leq c_3(\alpha)h^{2\alpha-1} \quad (15)$$

if  $\alpha < \frac{1}{2}$ . This proves (13) and hence Theorem 1.

The left side of (14) is at most  $h^2\|g''\|_\infty$ . Hence (14) will follow from

$$\|g''\|_\infty \leq A(\alpha)\|\gamma''\|_\infty h^{2\alpha-2}. \quad (16)$$

For any  $t_0 \in \mathbf{R}$ , our preceding change of variables shows that (12) leads to

$$g''(t_0) = r^2(D_2^2f)(re_1) + r \sum_{j=1}^n (D_jf)(re_1)\gamma''(t_0). \quad (17)$$

By (8) and (10), each derivative of  $f$  that occurs in (17) is dominated by  $c(\alpha)(1-r)^{\alpha-1} = c(\alpha)h^{2\alpha-2}$ . [Note that the right side of (10) can be replaced by  $C \log(1/1-r)$  if  $\alpha \geq \frac{1}{2}$ .] Since differentiation of  $\langle \gamma, \gamma \rangle = 1$  gives  $\operatorname{Re} \langle \gamma', \gamma \rangle = 0$ , hence

$$\operatorname{Re} \langle \gamma'', \gamma \rangle = -\langle \gamma', \gamma' \rangle = -1, \quad (18)$$

we see that  $|\gamma''(t_0)| \geq 1$ . Hence (17) gives (16). This completes the proof of Theorem 2.

**EXAMPLE.** Take  $n = 2$ , define

$$f(z_1, z_2) = \frac{z_2}{z_1} \log \frac{1}{1-z_1}. \quad (19)$$

The singularity at  $z_1 = 0$  is removable, and

$$(Rf)(z_1, z_2) = z_2/(1-z_1). \quad (20)$$

Since  $|z_2|^2 < 1 - |z_1|^2$ ,  $|(Rf)(z)| < 2^{1/2}(1-|z|)^{-1/2}$ . This implies that  $Cf \in K_{1/2}$  for some  $C > 0$ .

Since  $t \rightarrow f(\cos t, \sin t)$  is not in Lip 1, we see that Theorem 1 fails when  $\alpha = \frac{1}{2}$ .

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