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Level sets of infinite length

GEORGE PIRANIAN AND ALLEN WEITSMAN

Dedicated to Professor Albert Pfluger on his seventieth birthday

The geometry of level sets plays an important role in the analysis of various function-theoretic problems. Often, however, the level sets are so complicated that one must either choose sets associated with special levels (see [3], [4]) or resort to the approximation of level sets by shorter curves (see[1, pp. 550–553]).

In [3] and [4], there are some weak estimates on the length $\ell(r, R)$ of the sets $\{z : |f(z)| = R,\ |z| < r\}$ associated with a function f meromorphic in the plane. For such an f we do not know whether the quantity $\ell(r, R)$ can be bounded in terms of Nevanlinna's characteristic function T without reference to exceptional levels. But the following is implicit in the results in [3, pp. 121–123] and [4, p. 44]: *If f is meromorphic in $|z| \le 2r$ and $f(0) = 1$, then each subinterval $[\alpha, \beta]$ of $(0, \infty)$ contains a set I of measure $(\beta - \alpha)/2$ such that the inequality*

$$\ell(r, R) \le 2\pi r \sqrt{\frac{\beta T(2r, f)}{(\beta - \alpha)\log 2}}$$

holds for each R in I.

For functions univalent in the disk $D = \{|z| < 1\}$, it follows from the basic inequalities of length and area [2, p. 18] that almost all level sets have finite length; but we do not know of any univalent function in D having even one level set of infinite length. Neither do we know whether there exists a bounded holomorphic function f in D such that $\ell(1, R) = \infty$ for all R in some interval (α, β), or for all R in some set of positive measure. In this paper, we show that bounded holomorphic functions can have level sets of infinite length.

THEOREM. *If M is a finite or countable set in the interval $(0, 1)$, then there exists a holomorphic function f in D such that $|f(z)| < 1$ for all z in D and $\ell(1, R) = \infty$ for each R in M.*

Because for $0 \le r < 1$ and $0 < R < \infty$ the quantity $\ell(r, R)$ is a continuous function of r and R (for each function f holomorphic in D), the set of values R for which $\ell(1, R)=\infty$ is of type G_δ. It follows immediately that if the set M in the theorem is dense in $(0, 1)$, then the set of values R for which $\ell(1, R)=\infty$ is residual in $(0, 1)$.

To prove the theorem, we begin with the case where the set M consists of a single value R, and we construct a nondecreasing continuous function μ such that the level set $\ell(1, R)$ of the function

$$f(z)=\exp\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{z+e^{it}}{z-e^{it}}\,d\mu(t) \tag{1}$$

has infinite length.

We write $z=\rho e^{i\theta}$, and we observe that

$$|f(\rho e^{i\theta})|=\exp\frac{-1}{2\pi}\int_{-\pi}^{\pi}P(\rho, \theta, t)\,d\mu(t),$$

where P denotes the Poisson kernel

$$P(\rho, \theta, t)=\frac{1-\rho^2}{1-2\rho\cos(t-\theta)+\rho^2}.$$

If $\mu'(t)$ exists at the point t, then the radial limit of $|f(z)|$ at e^{it} is $\exp(-\mu'(t))$. We shall now construct a sequence $\{\mu_n\}$ such that on some set of positive measure the sequence $\{\mu_n'(t)+\log R\}$ converges to 0 with alternating signs.

On some subinterval $[t_0, t_1]$ of $(-\pi, \pi)$, let μ_0 be linear, and let $\mu_0'(t)$ be slightly less than $-\log R$. On $[-\pi, t_0]$ and on $[t_1, \pi]$, let μ_0 be constant. Then the function

$$|f_0(\rho e^{i\theta})|=\exp\frac{-1}{2\pi}\int_{-\pi}^{\pi}P(\rho, \theta, t)\,d\mu_0(t)$$

is slightly greater than R, on some arc γ_0 joining e^{it_0} to e^{it_1}.

To construct μ_1, we divide the interval (t_0, t_1) into subintervals in such a way that for each $\varepsilon>0$ both $[t_0, t_0+\varepsilon]$ and $[t_1-\varepsilon, t_1]$ contain infinitely many of the subintervals. On each of the subintervals, we define μ_1 so that

(i) $\mu_1(t)=\mu_0(t)$ at the endpoints,

(ii) μ_1' is constant and slightly greater than $-\log R$, on a portion beginning at the left end of the subinterval,

(iii) $\mu_1'(t)=0$ on the remainder of the subinterval (see Figure 1).

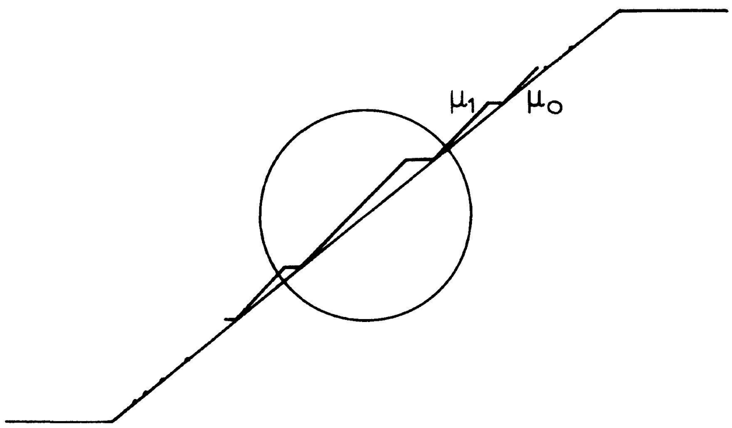

Figure 1

If the subdivision of (t_0, t_1) is fine enough, then the function f_1 determined by μ_1 has modulus greater than R on the previously established arc γ_0. But on a family $\{\gamma_{1k}\}$ of other arcs (corresponding to the intervals on which $\mu_1' > -\log R$) separated from the origin by the arc γ_0, the value of $|f_1(z)|$ is less than R. Moreover, if the positive values of μ_0' and μ_1' are sufficiently close to $-\log R$, the sum of the diameters of the arcs γ_{1k} is greater than $(t_1 - t_0)/2$.

On each segment on which $\mu_1'(t) = 0$, let $\mu_2(t) = \mu_1(t)$. Each of the remaining segments of (t_0, t_1) we divide into infinitely many subsegments. On each of these subsegments we define μ_2 so that $\mu_2'(t)$ is constant and slightly less than $-\log R$ in a large part of the segment, while $\mu_2'(t) = -2 \log R$ in the remainder (see Figure 2).

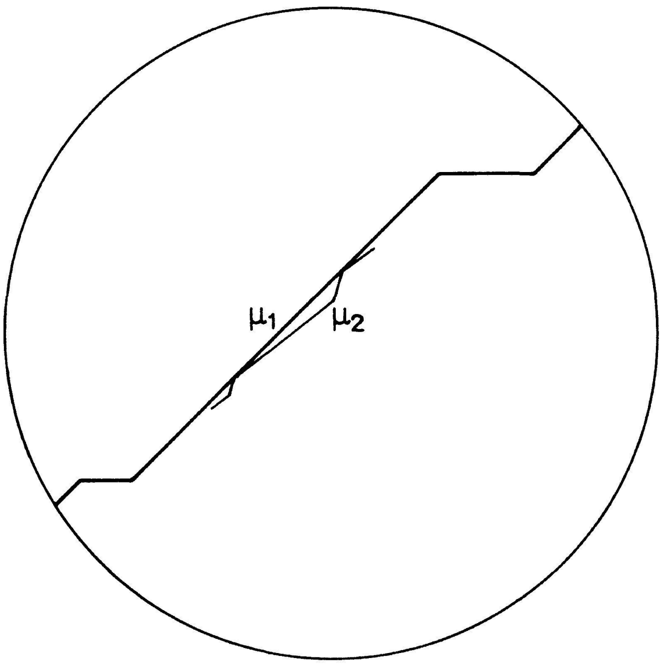

Figure 2

The sequence $\{\mu_n\}$ constructed by the obvious continuation of our process converges to a function μ, and the function (1) has the property that on each of certain arcs γ_{nk} its modulus is greater or less than R according as the index n is even or odd. If the values $\mu_n'(t)$ converge to $-\log R$ rapidly enough in the portions where they are neither 0 nor $-2\log R$, then for each n the sum of the diameters of the arcs γ_{nk} is greater than $(t_1-t_0)/2$. If γ_{nh} separates the origin from $\gamma_{n+1,k}$, in D, then by continuity of f an arc of the level set $\{z:|f(z)|=R\}$ separates γ_{nh} from $\gamma_{n+1,k}$. It follows that the level set has infinite length.

We have proved the theorem for the special case where the set M is a singleton. In the general case, let $M=\{R_j\}$. We divide the interval $(-\pi, \pi)$ into infinitely many disjoint subintervals T_j, and we construct the sequence $\{\mu_n\}$ as in the special case, except that now the limit of $\mu_n'(t)$ is $-\log R_j$, in a substantial part of T_j. A detailed description would require the development of tedious notation; but the underlying principle is simple: if in the passage from μ_{n-1} to μ_n we make the new system of subdivisions fine enough, then on the arcs of the families $\{\gamma_{mk}\}$ $(m=1,2,\ldots,n-1)$ the appropriate differences $|f_n|-R_j$ and $|f_{n-1}|-R_j$ have the same sign.

Because the limit μ of the sequence $\{\mu_n\}$ is absolutely continuous, our function (1) is an outer function. We do not know whether a nonconstant inner function can have a level set of infinite length. Since the radial limit of an inner function has modulus 1 almost everywhere, the construction of an example may well require delicate computations.

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