Cohomological characterisations of saturated formations and homomorphs of finite groups.

Autor(en): **Barnes, D.W. / Schmid, P. / Stammbach, U.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **53 (1978)**

PDF erstellt am: **15.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-40762

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

Comment. Math. Helvetici 53 (1978) 165–173 Birkhäuser Verlag, Basel

Cohomological characterisations of saturated formations and homomorphs of finite groups

DONALD W. BARNES, PETER SCHMID AND URS STAMMBACH

1. Introduction

Various cohomological characterisations of finite p-nilpotent groups can be found in the literature (see for instance [HRZ], [Q], [T]). Recently cohomological characterisations of finite p-soluble groups [St] and of finite p-supersoluble groups [Ba], [St] have been given. In this paper we show that any saturated and hence also any local formation of finite groups can be described by means of cohomology. Furthermore we shall investigate homological properties of p-saturated homomorphs of p-soluble groups.

We first state our main results. Throughout the paper we consider only finite groups and finite modules. For the concept and the basic properties of formations we refer to [Hu], Ch. VI 7. Note in particular that formations need not be soluble here. If M is a G-module, we denote by C_GM the centralizer of M in G. If $\mathfrak{X}$ is any class of groups, then M is called *$\mathfrak{X}$-central* if the semi-direct product $M](G/C_GM)$ is in $\mathfrak{X}$, otherwise M is called *$\mathfrak{X}$-eccentric.* This coincides with the usual terminology in case of local formations, provided one deals with inclusive local definitions.

THEOREM A. *Suppose $\mathfrak{F}$ is a saturated formation and G is a group. The following statements are equivalent:*

(i) $G \in \mathfrak{F}$;
(ii) $H^1(G, M) = 0$ *for any irreducible, $\mathfrak{F}$-eccentric G-module M;*
(iii) $H^n(G, M) = 0$ *for any module M as in (ii) and all $n \geq 1$.*

Since any local formation is saturated it is easy to deduce from Theorem A the following result.

THEOREM B. *Let $\mathfrak{F}$ be locally defined by the formation $\mathfrak{F}(p)$ for a fixed prime p, and all finite groups otherwise. Then the following are equivalent:*

(i) $G \in \mathfrak{F}$;
(ii) $H^1(G, M) = 0$ *for any irreducible* $\mathbf{F}_p G$*—module* M *such that* $G/C_G M \notin \mathfrak{F}(p)$;
(iii) $H^n(G, M) = 0$ *for any module* M *as in* (*ii*) *and for all* $n \geq 1$.

To illustrate Theorem B, we give some examples:

(a) If $\mathfrak{F}(p) = \{1\}$, then $\mathfrak{F}$ is the formation of p-nilpotent groups. We obtain that G is p-nilpotent iff $H^1(G, M) = 0$ for all non-trivial irreducible $\mathbf{F}_p G$-modules M (see [St]).

(b) If $\mathfrak{F}(p)$ is the formation of abelian groups with exponent dividing $p-1$, then $\mathfrak{F}$ is the formation of p-supersoluble groups. We obtain that G is p-supersoluble iff $H^1(G, M) = 0$ for all irreducible $\mathbf{F}_p G$-modules M of dimension ≥ 2 (see [Ba], [St]).

(c) If $\mathfrak{F}(p)$ is the formation of all p'-groups, then $\mathfrak{F}$ is the formation of all p-soluble groups of p-length ≤ 1 (see [Hu], p. 698). We obtain that G is in $\mathfrak{F}$ iff $H^1(G, M) = 0$ for all irreducible $\mathbf{F}_p G$-modules M such that $p \nmid |G/C_G M|$.

We now turn to the discussion of homomorphs, i.e. nonempty classes of groups, closed under epimorphic images. A homomorph $\mathfrak{X}$ is called p'-closed if $N \trianglelefteq G$, $G/N \in \mathfrak{X}$ and $|N|$ prime to p implies $G \in \mathfrak{X}$. A subgroup S of G is called an *$\mathfrak{X}$-projector* of G if $S \in \mathfrak{X}$ and, for all $K \trianglelefteq H \leq G$ such that $S \leq H$ and $H/K \in \mathfrak{X}$, we have $SK = H$. A homomorph $\mathfrak{X}$ is called *p-saturated* if every p-soluble subgroup has an $\mathfrak{X}$-projector. (Compare [Sc], see also Propositions 3 and 4 below.)

THEOREM C. *Let* $\mathfrak{H}$ *be a* p'*-closed* p*-saturated homomorph of* p*-soluble groups, and let* G *be a finite group. Then* $G \in \mathfrak{H}$ *if and only if* $H^1(G, M) = 0$ *for every* $\mathfrak{H}$*-eccentric irreducible* $\mathbf{F}_p G$*-module* M.

To illustrate Theorem C we give the following example:

(d) Let $\mathfrak{H}$ be the class of all p-soluble groups with no proper p-factor group. It is clear that $\mathfrak{H}$ is a p'-closed homomorph. It is also p-saturated, the $\mathfrak{H}$-projector of a p-soluble group G being the minimal normal subgroup $N \leq G$ with G/N a p-group. It is known that $\mathfrak{H}$ is not a formation (see [Sc]). By Theorem C a group G is in $\mathfrak{H}$ if and only if $H^1(G, M) = 0$ for every irreducible $\mathbf{F}_p G$-module M such that $G/C_G M$ has a non-trivial p-factor group.

We observe that Theorem C does not extend to homomorphs containing groups which are not p-soluble. For an example consider the class $\mathfrak{K}$ of groups consisting of the groups in $\mathfrak{H}$ (example (d)) and of all groups that are extensions of a group in $\mathfrak{H}$ by a fixed non-abelian simple group S with $p \nmid |S|$. It is clear that $\mathfrak{K}$ is a p'-closed p-saturated homomorph. By Lemma 1 there exists an irreducible

$\mathbf{F}_p S$-module M such that $H^1(S, M) \neq 0$, but of course $M]S \notin \mathfrak{K}$, so that M is $\mathfrak{K}$-eccentric.

On might ask whether a statement analogous to (iii) in Theorems A and B holds for homomorphs too. This is answered and explained by

THEOREM D. *Let $\mathfrak{H}$ be a p'-closed p-saturated homomorph of p-soluble groups. Suppose that for all $G \in \mathfrak{H}$ and all $\mathfrak{H}$-eccentric irreducible $\mathbf{F}_p G$-modules M, we have $H^2(G, M) = 0$. Then $\mathfrak{H}$ is a formation.*

2. On the block structure of groups in a saturated formation

Our proofs of Theorem A and B use some information about the block structure of $\mathbf{F}_p G$. In fact, the conclusion (i)$\Rightarrow$(iii) will be achieved by observing that the eccentric modules in question do not belong to the principal block. For the theory of blocks we refer to [CR], Chapters VIII, IX.

We need the following well-known lemma. It is an easy consequence of Shapiro's lemma. For convenience we include the simple argument.

LEMMA 1. *Let G be a group whose order is divisible by the prime p. Then, for every $n \geq 1$, there exists an irreducible $\mathbf{F}_p G$-module M for which $H^n(G, M) \neq 0$.*

Proof. It is obviously enough to find an $\mathbf{F}_p G$-module A such that $H^n(G, A) \neq 0$, for then A has a composition factor M with $H^n(G, M) \neq 0$. Also, it is enough to find A such that $H^1(G, A) \neq 0$, for then dimension shifting proves the more general statement. Now let P be a p-Sylow subgroup of G. Then $H^1(P, \mathbf{F}_p) = \mathrm{Hom}\,(P, \mathbf{F}_p) \neq 0$. Thus if we set $A = \mathrm{Hom}_{\mathbf{F}_p P}(\mathbf{F}_p G, \mathbf{F}_p)$ we have $H^1(G, A) = H^1(P, \mathbf{F}_p) \neq 0$. This proves Lemma 1.

PROPOSITION 1. *Suppose $\mathfrak{F}$ is a saturated formation and $G \in \mathfrak{F}$. Let p be a prime and M an irreducible $\mathbf{F}_p G$-module belonging to some block B of $\mathbf{F}_p G$. If the semi-direct product $M]G$ is in $\mathfrak{F}$, then $A]G \in \mathfrak{F}$ for any $\mathbf{F}_p G$-module A all of whose composition factors belong to B.*

Proof. Let P be the indecomposable projective $\mathbf{F}_p G$-module having M in the head (and in the socle); i.e. $M \cong P/JP$ where J denotes the Jacobson radical of $\mathbf{F}_p G$. By embedding P in $P]G$ we see that JP is contained in the Frattini subgroup $Fr(P]G)$ (see [Gr], p. 98). Since $M]G$ is in $\mathfrak{F}$ and $\mathfrak{F}$ is saturated, it follows that $P]G \in \mathfrak{F}$.

Let $\bar{P}$ be an indecomposable projective having a composition factor N with P in common. Choose a submodule L of $\bar{P}$ minimal with respect to having N as epimorphic image. Then $N \cong L/JL$. Now $P]G \in \mathfrak{F}$ implies $N]G \in \mathfrak{F}$ by [Hu], Hilfssatz VI.7.21. This yields $L]G \in \mathfrak{F}$ since $\mathfrak{F}$ is saturated. As socle and head of $\bar{P}$ are isomorphic, we may conclude $\bar{P}]G \in \mathfrak{F}$ as well.

Proceeding inductively we see that the split extension by G of any indecomposable projective $\mathbf{F}_pG$-module belonging to B is in $\mathfrak{F}$. Now A is an epimorphic image of a direct sum of some of these indecomposable projectives. Since $\mathfrak{F}$ is a formation, we obtain $A]G \in \mathfrak{F}$, as desired.

PROPOSITION 2. *Let $\mathfrak{F}$ be a saturated formation, and let G be a group in $\mathfrak{F}$ whose order is divisible by the prime p. If A is an $\mathbf{F}_pG$-module such that every composition factor belongs to the principal block of $\mathbf{F}_pG$, then $A]G \in \mathfrak{F}$.*

Proof. By Lemma 1, there is an irreducible $\mathbf{F}_pG$-module M with $H^2(G, M) \neq 0$. Consequently there is a non-split group extension $M \rightarrowtail E \twoheadrightarrow G$ of M by G. It is immediate that $M \subseteq Fr(E)$ where $Fr(E)$ denotes the Frattini subgroup of E and thus $E \in \mathfrak{F}$ since $\mathfrak{F}$ is saturated. By [Hu], Hilfssatz VI.7.21, the semi-direct product $M]G$ is in $\mathfrak{F}$ too. It is well known that for any irreducible $\mathbf{F}_pG$-module which does not belong to the principal block all cohomology groups vanish (see [Gr], p. 178). Hence M is in the principal block. Now apply Proposition 1.

Form Proposition 2 it follows immediately that if the saturated formation $\mathfrak{F}$ contains a group G whose order is divisible by p, then $\mathfrak{F}$ contains all p-groups. Concerning local formations we have the following

COROLLARY. *Let $\mathfrak{F}$ be a formation which is locally defined by $\mathfrak{F}(p)$, and let $G \in \mathfrak{F}$. If $\mathfrak{F}(p) \neq \emptyset$ and if M is an irreducible $\mathbf{F}_pG$-module in the principal block, then $G/C_GM \in \mathfrak{F}(p)$.*

We note that together with the fact that p-chief factors of a group G belong to the principal block of $\mathbf{F}_pG$ the above Corollary immediately yields the following well-known results:

(i) A group G is p-nilpotent if and only if the principal block of $\mathbf{F}_pG$ consists of $\mathbf{F}_p$ only;
(ii) A group G is p-supersoluble if and only if the irreducible modules in the principal block of $\mathbf{F}_pG$ have dimension 1.

Remark. Let $\mathfrak{F}$ be a locally defined formation with $\mathfrak{F}(p) \neq \emptyset$. Then $G \in \mathfrak{F}$ if and only if $G/O_{p'p}(G) \in \mathfrak{F}(p)$ where $O_{p'p}(G)$ denotes the greatest p-nilpotent normal subgroup of G (see [Hu], Hilfssatz VI.7.4). By a theorem of Brauer

$O_{p'p}(G)$ centralizes every irreducible module belonging to the principal block of $\mathbf{F}_pG$ (see [Br], Theorem 1). This gives an alternative proof of the above Corollary.

3. Proof of Theorems A and B

We begin by stating three lemmas. Lemma 2 has been used implicitly in [St], Lemmas 3 and 4 are due to Gaschütz (see also [Ba]).

LEMMA 2. *Suppose N is a normal subgroup of G and A is an irreducible $\mathbf{F}_pN$-module with $H^n(N, A) \neq 0$ for some $n \geqslant 1$. Then there exists an irreducible $\mathbf{F}_pG$-module M such that $H^n(G, M) \neq 0$ and such that the centralizer $C_N M$ is the intersection of some conjugates of $C_N A$ in G.*

Proof. There is a G-composition factor M of the coinduced module $M' = \mathrm{Hom}_{\mathbf{F}_pN}(\mathbf{F}_pG, A)$ for which $H^n(G, M) \neq 0$. By Clifford's theory M', and hence M, are completely reducible as $\mathbf{F}_pN$-modules, the direct summands being $\mathbf{F}_pN$-modules conjugate to A. This proves the lemma.

LEMMA 3. *If M is an abelian, complemented chief factor of G, then $H^1(G, M) \neq 0$.*

Proof. Because of the injectivity of the inflation map in 1-cohomology, it sufficies to consider the case where M is a minimal normal subgroup of G. We then have an exact sequence

$$0 \rightarrow H^1(G/M, M) \rightarrow H^1(G, M) \rightarrow \mathrm{Hom}_G\,(M, M) \xrightarrow{\tau} H^2(G/M, M),$$

where τ is the transgression homomorphism. It is known (see [HSt], p. 207) that $\tau(1_M)$ is just the cohomology class of the group extension $M \rightarrowtail G \twoheadrightarrow G/M$. Since this extension splits by hypothesis, we may conclude that $\mathrm{H}^1(\mathrm{G}, \mathrm{M}) \neq 0$.

LEMMA 4. *Let G be p-soluble and let M be an irreducible $\mathbf{F}_pG$-module. If $H^1(G, M) \neq 0$, then M is isomorphic to a complemented p-chief-factor of G.*

Proof. Let $C = C_G M$. Then we consider the exact sequence

$$0 \rightarrow H^1(G/C, M) \rightarrow H^1(G, M) \rightarrow \mathrm{Hom}_G\,(C, M) \rightarrow H^2(G/C, M).$$

Since $H^n(G/C, M) = 0$ for all $n \geqslant 1$ (see for example [St]), we may conclude that there exists $D \lhd G$ such that $C/D \cong M$. But then $H^2(G/C, M) = 0$ yields that $M \cong C/D$ is a complemented chief-factor of G.

Proof of Theorem A.

(i)$\Rightarrow$(iii): Suppose $G\in\mathfrak{F}$. Let M be an irreducible, $\mathfrak{F}$-eccentric $\mathbf{F}_pG$-module for some prime p. If G is a p'-group, we clearly have $H^n(G, M)=0$ for all $n\geq 1$. Thus assume that p divides $|G|$. But then, by Proposition 2, M does not belong to the principal block of $\mathbf{F}_pG$ and so again all cohomology groups vanish.

(iii)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i): Assume G is a group of minimal order satisfying (ii) without being in $\mathfrak{F}$. As $\mathfrak{F}\neq\emptyset$, $G\neq 1$. Let N be a minimal normal subgroup of G, and let $Q=G/N$. Suppose M is an irreducible $\mathfrak{F}$-eccentric Q-module. Then M is, in the obvious way, an irreducible $\mathfrak{F}$-eccentric G-module. Hence $H^1(G, M)=0$, and this implies $H^1(Q, M)=0$ in view of the inflation map. Therefore condition (ii) is inherited by factor groups. By the minimality of G we have that Q is in $\mathfrak{F}$ and that N is the unique minimal normal subgroup of G.

Let p be a prime divisor of $|N|$. Assume first that N is non-abelian. Since N does not have any proper p-factor group, $H^1(N, \mathbf{F}_p)=\mathrm{Hom}\,(N, \mathbf{F}_p)=0$. From Lemma 1 it follows that there is a non-trivial, irreducible $\mathbf{F}_pN$-module A for which $H^1(N, A)\neq 0$. Application of Lemma 2 yields the existence of an irreducible $\mathbf{F}_pG$-module M such that $H^1(G, M)\neq 0$ and $C_NM\subseteq C_NA^x$ for some $x\in G$. It follows that M is not $\mathfrak{F}$-excentric; moreover, as A is a non-trivial N-module and N is the unique minimal normal subgroup of G, we have $C_GM=1$. However, this latter leads to the contradiction that M is $\mathfrak{F}$-eccentric.

Hence N is an irreducible $\mathbf{F}_pG$-module. Since $\mathfrak{F}$ is saturated and $Q=G/N\in\mathfrak{F}$ but $G\notin\mathfrak{F}$, G splits over N. It follows $C_GN=N$, because N is the unique minimal normal subgroup in G. Consequently N is an $\mathfrak{F}$-eccentric G-module and so $H^1(G, N)=0$ by hypothesis (ii). However, this contradicts the statement in Lemma 3.

Proof of Theorem B.

In view of Theorem A we only have to prove that an irreducible $\mathbf{F}_pG$-module is $\mathfrak{F}$-eccentric if and only if $G/C_GM\notin\mathfrak{F}(p)$. But this is clear since $M](G/C_GM)\in\mathfrak{F}$ if and only if $G/C_GM\in\mathfrak{F}(p)$.

Remark. Let M be an irreducible $\mathbf{F}_pG$-module not contained in the principal block. In our proof we have made use of the fact that $H^n(G, M)=0$. If M does belong to the principal block of $\mathbf{F}_pG$, then one might ask whether there is an $n\geq 1$ such that $H^n(G, M)\neq 0$. This question is answered affirmatively if $M=\mathbf{F}_p$ by a theorem of Swan [Sw]. However no general answer seems to be known.

4. Proof of Theorems C and D

In order to justify our definition of p-saturated homomorphs, we first prove the following results.

PROPOSITION 3. *Let $\mathfrak{H}$ be a p'-closed p-saturated homomorph of p-soluble groups. Suppose $G/FrG \in \mathfrak{H}$. Then $G \in \mathfrak{H}$.*

Proof. Suppose $G \notin \mathfrak{H}$. Let N be a minimal normal subgroup of G contained in FrG. By induction, $G/N \in \mathfrak{H}$. Since $G \notin \mathfrak{H}$, N is a p-group and G is p-soluble. Let S be an $\mathfrak{H}$-projector of G. Then $S \neq G$, $SN = G$ and it follows that S is a maximal subgroup of G not containing N. This contradicts $N \subseteq FrG$.

PROPOSITION 4. *Let $\mathfrak{F}$ be a p'-closed formation of p-soluble groups, and suppose that $G \in \mathfrak{F}$ whenever $G/FrG \in \mathfrak{F}$. Then $\mathfrak{F}$ is p-saturated.*

Proof. Let G be a p-soluble group. We have to prove that there exists an $\mathfrak{F}$-projector of G. This is certainly true if $G \in \mathfrak{F}$, so we suppose $G \notin \mathfrak{F}$. Let N be a minimal normal subgroup of G. By induction there exists an $\mathfrak{F}$-projector T/N of G/N. If $T \neq G$, then there exists an $\mathfrak{F}$-projector S of T. It is then easy to see that S is also an $\mathfrak{F}$-projector of G (see [Hu], Hilfssatz VI.7.9(c)). Suppose $T = G$. Then $G/N \in \mathfrak{F}$. Since $G \notin \mathfrak{F}$, $N \nsubseteq FrG$ and N is a p-group. Thus N is complemented in G. But since $\mathfrak{F}$ is a formation, a complement to N in G is easily seen to be an $\mathfrak{F}$-projector of G.

Proof of Theorem C.

(i)$\Rightarrow$(ii): Suppose $G \in \mathfrak{H}$ and M is an $\mathfrak{H}$-eccentric irreducible $\mathbf{F}_p G$-module. Since G is p-soluble we may conclude from Lemmas 3 and 4 that $H^1(G, M) = 0$ if and only if M is *not* a complemented chief-factor of G (see also [Ba], Theorem 1). Suppose it is. Then there exist normal subgroups $H, K \lhd G$, such that $K/H \cong M$ and $G/H = M](G/K)$. Since $C_G M \supseteq K$, it would follow that $M](G/C_G M) \in \mathfrak{H}$. contradicting the hypothesis that M is $\mathfrak{H}$-eccentric.

(ii)$\Rightarrow$(i): Let G be a counterexample of minimal order and let N be a minimal normal subgroup of G. As in the proof of Theorem A we may infer that $Q = G/N$ satisfies (ii). Hence $Q \in \mathfrak{H}$ by the choice of G. Since $\mathfrak{H}$ is p'-closed it follows that p divides the order of N. Assume first that N is non-abelian. We have $H^1(N, \mathbf{F}_p) = 0$, because N does not possess a proper p-factor group. By Lemmas 1 and 2 there exists an irreducible $\mathbf{F}_p G$-module M such that $H^1(G, M) \neq 0$ and $C_N M = 1$. In particular we see that $G/C_G M$ is not p-soluble and thus M is $\mathfrak{H}$-eccentric. This is a contradiction. Hence N is an irreducible $\mathbf{F}_pG$-module and G is p-soluble. Now let S be an $\mathfrak{H}$-projector of G. Then we have $SN = G$. Since $N \cap S = 1$ this implies that $G = N]Q$. We may thus conclude from Lemma 3 that $H^1(G, N) \neq 0$. We next claim that $C_G N = N$. Suppose $C_G N \supset N$. Then there exists a minimal normal subgroup $N_1 \lhd G$ with $1 \subset N_1 \subseteq S$. But then $SN_1 = S \neq G$, contradicting the fact that S is an $\mathfrak{H}$-projector. Since $N](G/C_G N) = N]Q = G \notin \mathfrak{H}$ we conclude that N is $\mathfrak{H}$-eccentric. This is the desired contradiction.

Proof of Theorem D.

Suppose $\mathfrak{H}$ is not a formation. Let G be a group of minimal order having normal subgroups N_1, N_2 such that $G/N_i \in \mathfrak{H}$, $i = 1, 2$, but $G/N_1 \cap N_2 \notin \mathfrak{H}$. Then $N_1 \cap N_2 = 1$ and N_1, N_2 are minimal normal p-subgroups of G. In particular G is p-soluble.

Let S be an $\mathfrak{H}$-projector of G. Then clearly $G = SN_1 = SN_2$ and $S \cap N_1 = S \cap N_2 = 1$. Set $D = N_1N_2 \cap S$. We get $DN_1 = N_1N_2 = DN_2$. Hence

$$N_1 \cong N_1N_2/N_2 = DN_2/N_2 \cong D \cong N_2$$

as G-modules. But N_1 is an $\mathfrak{H}$-eccentric G/N_1N_2—module, so that $H^2(G/N_1N_2, N_1) = 0$ by hypothesis. Since $N_1N_2/D \cong N_1$ we conclude that G/D and G/N_1 are both split extensions of N_1 by G/N_1N_2. Thus $G/D \cong G/N_1 \in \mathfrak{H}$. But this contradicts the fact that S is an $\mathfrak{H}$-projector.

REFERENCES

[Ba] Barnes, D. W.: *First cohomology groups of p-soluble groups.* To appear in J. of Algebra.

[Br] Brauer, R.: *Some applications of the theory of blocks of characters of finite groups,* I. J. of Algebra *1* (1964), 152–167.

[CR] Curtis, C. W., Reiner, I.: *Representation theory of finite groups and associative algebras.* Interscience Publishers, 1962.

[Gr] Gruenberg, K. W.: *Cohomological topics in group theory.* Lecture Notes in Mathematics, Vol. *143,* Springer 1970.

[HRZ] Hoechsmann, K., Roquette, P., Zassenhaus, H.: *A cohomological characterization of finite nilpotent groups.* Arch. Math. *19* (1968), 225–244.

[HSt] Hilton, P. J., Stammbach, U.: *A course in homological algebra.* Graduate Texts in Mathematics, vol. *4,* Springer 1971.

[Hu] Huppert, B.: *Endliche Gruppen I.* Springer 1967.

[Q] Quillen, D. G.: *A cohomological criterion for p-nilpotence.* J. of Pure and Appl. Algebra *1,* (1971), 361–372.

[Sc] Schunck, H.: *$\mathfrak{H}$-Untergruppen in endlichen auflösbaren Gruppen.* Math. Z. *97* (1967), 326–330.

[St] STAMMBACH, U.: *Cohomological characterizations of finite solvable and nilpotent groups.* To appear in J. of Pure and Appl. Algebra.

[Sw] SWAN, R. G.: *The non-triviality of the restriction map in the cohomology of groups.* Proc. Amer. Math. Soc. *11* (1960), 885–887.

[T] TATE, J.: *Nilpotent quotient groups.* Topology *3* (1964), 109–111.

Donald W. Barnes
Department of Pure Mathematics
University of Sydney
Sydney, N.S.W. 2006
Australia

Peter Schmid
Mathematisches Institut der Universität
D-7400 Tübingen 1
Auf der Morgenstelle 10

Urs Stammbach
Mathematik
Eidg. Technische Hochschule
CH-8092 Zürich

Received May 3, 1977

Added in proof. Since this paper has been submitted, one of the authors has applied methods similar to the ones used in this paper to prove that every saturated formation of finite groups is locally definable:
PETER SCHMID: *Every Saturated Formation is a Local Formation*, to appear in J. of Algebra.