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On a class of foliations and the evaluation of their characteristic classes

DANIEL BAKER

Introduction

This paper gives a detailed exposition of the results appearing in [B]. Specifically, we develop an algorithm for analyzing the images of characteristic classes from $H^*(WO_n)$ and $H^*(W_n)$ for a certain class of foliations. The foliations are on spaces of the form $\Gamma\backslash G/K$ where G is a semi-simple Lie group, K is a compact subgroup, and Γ is a discrete subgroup of G such that $\Gamma\backslash G/K$ is a compact orientable manifold. The leaves of the foliation are the left translates of a parabolic subgroup $P\supset K$. When $K=1$, the normal bundle of the foliation is trivialized by left invariant sections and we obtain characteristic classes from $H^*(W_n)$. For K non-trivial, the normal bundle is also non-trivial, and the characteristic classes come from $H^*(WO_n)$.

The examples yield new linear independence relations for the images of these classes in $H^*(B\Gamma_n, \mathbf{R})$ and $H^*(F\Gamma_n, \mathbf{R})$. Specific examples of foliations of this type have been anayzed by others (see, for example, [BR], [KT1], [KT2], [Y]).

The contents of this paper are as follows: In Chapter I we give some basic facts, references for others, and we set our notation.

Chapter II uses the results of Cartan ([CA]) to replace the relative complex of forms on the Lie algebra, $\Lambda^*(\mathcal{G}, \mathscr{k}, \mathbf{C})$, with a finitely generated complex $\bar{A}$ having the same cohomology. This is done by observing that $H^*(\mathcal{G}, \mathscr{k}, \mathbf{C})$ has the same cohomology as a certain homogeneous space which is also a fibre bundle. The complex $\bar{A}$ is basically the E_2 term of the spectral sequence for this fibre bundle.

Chapter III contains the basic classification theorem for parabolic subgroups.

Chapter IV contains the main results for computing the image of the characteristic map for the foliation $\Phi: H^*(WO_n)\to H^*(\Gamma\backslash G/K)$. The idea is to show that there is a commutative diagram

$$\begin{tikzcd} H^*(WO_n) \arrow[r, "\rho"] \arrow[dr, "\Phi"'] & H^*(\mathcal{G}, \mathscr{k}) \arrow[d, "F"] \\ & H^*(\Gamma\backslash G/K) \end{tikzcd}$$

where F is injective. This reduces the problem to studying the image of ρ. This is the same technique that has been used by Kamber and Tondeur (see [KT1]).

We now use the isomorphism $H^*(\mathcal{G}, \mathcal{k}) \approx H^*(\bar{A})$ to replace ρ with a map $\lambda^*: H^*(WO_n) \to H^*(\bar{A})$. The map λ^* is a nice map in the sense that it has a geometric interpretation in terms of characteristic classes for principal fibre bundles, and this fact renders its image computable. We emphasize that the construction of λ^* relies heavily on the structure theorem for parabolic subgroups.

Chapter V contains calculations for specific examples. Sample results are the following – (see corollaries 5.15 and 5.19):

For $n \neq 2$, $r \geq 2$ the set of classes

$$\{c_1^{2n} h_1 h_2 h_{i_1} \cdots h_{i_k}, c_2 c_1^{2n-2} h_1 h_2 h_{i_1} \cdots h_{i_k} \in H^*(W_{2n}, \mathbf{R}) \mid 2 < i_1 < \cdots < i_k \leq n\}$$

is non-vanishing and linearly independent in $H^*(F\Gamma_{2n}^r, \mathbf{R})$ (this includes the classes $c_1^{2n} h_1 h_2$ and $c_2 c_1^{2n-2} h_1 h_2$). Here $F\Gamma_n^r$ is the classifying space for codimension n, C^r foliations with trivialized normal bundles.

For $r \geq 2$ the set of classes

$$\{c_1^{2n} h_1 h_{i_1} \cdots h_{i_k}, c_2 c_1^{2n-2} h_1 h_{i_1} \cdots h_{i_k} \in H^*(WO_{2n}, \mathbf{R}) \mid 1 < i_1 < \cdots < i_k \leq n \quad \text{and} \quad i_j \quad \text{are odd}\}$$

is non-vanishing and linearly independent in $H^*(B\Gamma_{2n}^r, \mathbf{R})$ (this includes the classes $c_1^{2n} h_1$ and $c_2 c_1^{2n-2} h_1$). Here $B\Gamma_n^r$ is the classifying space for co-dimension n, C^r foliations.

I. Some preliminary facts and notation

For more detailed information about the Weil homomorphism and TP forms see [CS] and [KN], Volume II.

Let G be a Lie group with Lie algebra $\mathcal{G}$. $I^\ell(\mathcal{G}, V)$ will be the ℓ-dimensional, V-valued, Weil polynomials ($V = \mathbf{R}$ or $\mathbf{C}$). Let $I(\mathcal{G}, V) = \oplus I^\ell(\mathcal{G}, V)$ and $P \epsilon I^\ell(\mathcal{G}, V)$. If $\pi: E \to M$ is a principle G bundle with $\mathcal{G}$ valued connection θ and curvature Ω then we are led to consider the 2ℓ-dimensional forms $P(\Omega) = P(\Omega, \ldots, \Omega)$. In [CS] the authors construct the forms

$$TP(\theta) = \ell \int_0^1 P(\theta \wedge \phi_t^{\ell-1})\, dt$$

where $\phi_t = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta]$. In the next proposition we collect some of the facts which are proven in [CS] and [KN].

PROPOSITION 1.1 *Let θ be any $\mathcal{G}$ valued 1-form on a manifold E and let $\Omega = d\theta + \frac{1}{2}[\theta, \theta]$. Then:*

(i) $dTP(\theta) = P(\Omega)$

(ii) *If θ_1 and θ_0 are two $\mathcal{G}$ valued 1-forms on E, let*

$$\theta_t = t\theta_1 + (1-t)\theta_0,$$

$$\alpha = \frac{d}{dt}\theta_t = \theta_1 - \theta_0$$

$$\Omega_t = d\theta_t + \frac{1}{2}[\theta_t, \theta_t].$$

Then the form

$$\Delta P(\theta_1, \theta_0) = \ell \int_0^1 P(\alpha, \Omega_t^{\ell-1})\, dt$$

satisfies

$$d(\Delta P(\theta_1, \theta_0)) = P(\Omega_1) - P(\Omega_0)$$

(iii) *When θ_0, θ_1 are $\mathcal{G}$ valued connections on a principle G bundle $\pi: E \to M$, then $\Delta P(\theta_1, \theta_0)$ is in the image of π^* and can be thought of as a form in $\Lambda^{2\ell-1}(M, V)$ (in the future, when it causes no confusion, we will do this).*

(iv) *When θ is a connection on $\pi: E \to M$, $TP(\theta) = \Delta P(\theta, 0) \in \Lambda^{2\ell-1}(E, V)$.*

Remarks. (a) In [CS] and [KN] these facts are proven only in the case where θ is a connection form on a bundle. The more general case where θ is an arbitrary $\mathcal{G}$ valued 1-form on E, which need not commute with the action Ad_G, can be obtained by means of the following construction:

On the trivial bundle $E \times G$; let ω be a connection form defined by $\omega(\dot{x}, \dot{g}) = \dot{g} + Ad_{g^{-1}}(\theta(\dot{x}))$ (Here we identify $\dot{g}$ with the left invariant vector field on G taking the value $\dot{g}$ at $g \in G$.) Then using the section $s: E \to E \times G$, $s(e) = (e, 1)$,

$$s^*(\omega) = \theta.$$

Thus if something from the above proposition holds for ω, it will also hold for θ.

(b) To see that $\Delta P(\theta, 0) = TP(\theta)$ note that

$$\Delta P(\theta, 0) = \ell \int_0^1 P(\theta, \Omega_t^{\ell-1})\, dt$$

where

$$\begin{aligned} \Omega_t &= t\, d\theta + \frac{1}{2} t^2[\theta, \theta] \\ &= t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta] = \phi_t \end{aligned}$$

Let $\mathcal{F}$ be a $C^r (r \geq 2)$ codimension n foliation on a manifold M. We will use the construction given in [BT] for the characteristic maps $H^*(WO_n) \to H^*(M, \mathbf{R})$ and (in the case where the normal frame bundle to $\mathcal{F}$ is trivialized by a section) $H^*(W_n) \to H^*(M, \mathbf{R})$. We caution only that, if ∇^0 and ∇^1 are two connections on a real vector bundle with $\mathfrak{gl}(n, \mathbf{R})$ valued connection forms θ_0 and θ_1 on the associated bundle of bases, then the form that is referred to as $\lambda(\nabla^0, \nabla^1)(P)$ in [BT] is just $\Delta P(\theta_1, \theta_0)$ in this paper.

For computational purposes the following lemma is useful (see [G] or [KT1] for proofs).

LEMMA 1.2. (i) *A basis for $H^*(W_n, \mathbf{R})$ consists of the classes determined by the cocycles $c_{i_1} \cdots c_{i_k} h_{j_1} \cdots h_{j_\ell}$ where*

$$\begin{aligned} &i_1 \leq \cdots \leq i_k \leq n, \qquad j_1 < \cdots < j_\ell \leq n, \qquad j_1 \leq i_1 \\ &i_1 + \cdots + i_k + j_1 > n, \qquad i_1 + \cdots + i_k \leq n \end{aligned}$$

(ii) *A basis for $H^*(WO_n, \mathbf{R})$ is given by the cocycles listed above where $j_1, \ldots, j_\ell$ are odd, and by the monomials $c_{2i_1} \cdots c_{2i_k}$ where $2(i_1 + \cdots + i_k) \leq n$.*

Note that if $B\Gamma_n^r$ is the classifying space for C^r codimension n Haefliger structures and $F\Gamma_n^r$ the homotopy theoretic fibre of the map $\nu : B\Gamma_n^r \to B_{GL(n,\mathbf{R})}$ which classifies the normal bundle of $B\Gamma_n^r$, then we obtain a commutative diagram (see [BT] for details).

$$\begin{array}{ccc} H^*(W_n, \mathbf{R}) & \longrightarrow & H^*(F\Gamma_n^r, \mathbf{R}) \\ \uparrow & & \uparrow{\scriptstyle i^*} \\ H^*(WO_n, \mathbf{R}) & \longrightarrow & H^*(B\Gamma_n^r, \mathbf{R}) \end{array}$$

Now if $\mathscr{k}$ is a subalgebra of $\mathscr{G}$ we can consider the Lie algebra cohomology $H^*(\mathscr{G}, V)$ and the relative Lie algebra cohomology $H^*(\mathscr{G}, \mathscr{k}, V)$. For the basic facts and definitions concerning these objects, and a proof of the following theorem see [CE].

THEOREM 1.3. (i) *If* $\mathscr{G}$ *is a semi-simple Lie algebra, then* $H^*(\mathscr{G}, V)$ *is an exterior algebra on a finite number of generators* (*called primitive elements*), *each having odd degree.*

(ii) *If* G *is a compact Lie group with Lie algebra* $\mathscr{G}$, K *a closed connected subgroup with subalgebra* $\mathscr{k} \subset \mathscr{G}$ *then* $H^*(\mathscr{G}, \mathscr{k}, V) \approx H^*(G/K, V)$.

We remark only that, if $\pi : G \to G/K$ is projection, then $\Lambda^*(\mathscr{G}, \mathscr{k}, \mathbf{C})$ is the image under π^* of the left invariant forms on G/K. This induces a map $\tilde{\psi} : \Lambda^*(\mathscr{G}, \mathscr{k}, \mathbf{C}) \to \Lambda^*(G/K, \mathbf{C})$ which induces the above isomorphism.

Assume now that K is a compact, connected subgroup of a (not necessarily compact) Lie group G. If there is a compact $\bar{G}$ with closed subgroup $\bar{K}$ such that

$$\bar{\mathscr{G}} \otimes \mathbf{C} \approx \mathscr{G} \otimes \mathbf{C} \tag{1.4}$$

and this isomorphism restricts to an isomorphism $\bar{k} \otimes \mathbf{C} \approx \mathscr{k} \otimes \mathbf{C}$, then it follows that

$$H^*(\mathscr{G}, \mathscr{k}, \mathbf{C}) \approx H^*(\bar{\mathscr{G}}, \bar{k}, \mathbf{C}) \approx H^*(\bar{G}/\bar{K}, \mathbf{C}).$$

In particular, $H^*(\mathscr{G}, \mathscr{k}, \mathbf{C})$ must satisfy Poincare duality.

PROPOSITION 1.5. (*see also* [KT1], *Lemma* 4.88). *Suppose that* G *and* K *satisfy the above hypotheses, and suppose* $\Gamma \subset G$ *is a discrete subgroup such that* $\Gamma \backslash G/K$ *is a compact manifold. Then there is a cochain map*

$$\psi : \Lambda^*(\mathscr{G}, \mathscr{k}, \mathbf{C}) \to \Lambda^*(\Gamma \backslash G/K, \mathbf{C})$$

which is injective on cohomology.

We remark only that the map ψ is constructed in the same fashion as the map $\tilde{\psi}$ above.

In general the isomorphism (1.4) does not restrict to an isomorphism $\bar{\mathscr{G}} \otimes \mathbf{R} \approx \mathscr{G} \otimes \mathbf{R}$. For this reason it will be easier to do all our cohomology computations using complex coefficients.

Finally we note that, when G is semi-simple, a Γ satisfying the hypotheses of Proposition 1.5 always exists (see [BO2]).

II. On the cohomology of homogeneous spaces G/K.

In this section we construct several different complexes which can be used to compute the cohomology $H^*(\mathcal{G}, \mathcal{k}, \mathbf{C})$ using the isomorphism of Theorem 1.3. This material is well known and can be found in, for example, the text [GHV]. Unless otherwise specified, all Lie groups in this chapter will be compact, connected. Cohomology is always taken with complex coefficients.

We first fix some notation. For G a Lie group, let $g_1, \ldots, g_n \in H^*(G)$ denote the primitive generators for the cohomology algebra. Write $S_G = H^*(B_G)$, where B_G is the classifying space for principle G bundles. S_G is a polynomial algebra on generators $\bar{g}_1, \ldots, \bar{g}_n$ where g_i transgresses to $\bar{g}_i$ in the classifying bundle $E_G \to B_G$.

Given a continuous homomorphism of Lie groups, $\rho: G \to H$, denote by $\bar{\rho}: S_H \to S_G$ the map on characteristic classes induced by ρ.

Finally, we use the same notation conventions for all Lie groups, e.g. k_i are the primitive generators in $H^*(K)$, transgressing to $\bar{k}_i \in S_K$, etc.

Suppose K is a closed, connected subgroup of a compact Lie group G. Form the complex $S_K \otimes H^*(G)$ with differential d satisfying

$$d(\bar{k}_i \otimes 1) = 0$$
$$d(1 \otimes g_i) = \bar{\rho}(\bar{g}_i) \otimes 1$$

where $\rho: K \to G$ is inclusion. The differential d is extended as an antiderivation.

THEOREM 2.1. (*see* [GHV], 11.5, *Vol* III) (i) *There is a commutative diagram, where λ is induced by a cochain map $S_K \otimes H^*(G) \to \Lambda^*(\mathcal{G}, \mathcal{k}, \mathbf{C})$*

$$\begin{tikzcd} & H^*(S_K \otimes H^*(G)) \arrow[dd, "\lambda"] \arrow[dr, "\pi_1^*"] & \\ S_K \arrow[ur, "i_1^*"] \arrow[dr, "\sigma^*"'] & & H^*(G) \\ & H^*(G/K) \arrow[ur, "\pi^*"'] & \end{tikzcd}$$

where $i_1: S_K \to S_K \otimes H^*(G)$ is the natural inclusion. The map

$$\pi_1: S_K \otimes H^*(G) \to H^*(G)$$

is obtained by composing the natural projection $S_K \otimes H^*(G) \to H^*(G)$ with the isomorphism $\mu^*: H^*(G) \to H^*(G)$ induced by the homeomorphism $\mu: G \to G$, $\mu(g) = g^{-1}$. The map σ^* is induced by a classifying map $\sigma: G/K \to B_K$ of the bundle $\pi: G \to G/K$.

(ii) *If rank* $K = $ *rank* G, *then* σ^* *is onto and* $H^*(G/K) \approx S_K/I$ *where* I *is the ideal generated by* $\bar{\rho}(\bar{g}_i)$, $i = 1, \ldots,$ *rank* G; $\deg(\bar{g}_i) > 0$.

We now consider nested groups $K \subset U \subset G$, all compact, and construct a complex whose cohomology is that of G/K but which reflects the fibre bundle structure

$$\begin{array}{ccc} U/K & \longrightarrow & G/K \\ & & \downarrow \\ & & G/U. \end{array}$$

Specifically consider the complex $A = S_U \otimes H^*(G) \otimes S_K \otimes H^*(U)$ with

$$d(1 \otimes g_i \otimes 1 \otimes 1) = \bar{\rho}_1(\bar{g}_i) \otimes 1 \otimes 1 \otimes 1$$
$$d(1 \otimes 1 \otimes 1 \otimes u_i) = 1 \otimes 1 \otimes \bar{\rho}_2(\bar{u}_i) \otimes 1 - \bar{u}_i \otimes 1 \otimes 1 \otimes 1$$
$$d(\bar{u}_i \otimes 1 \otimes 1 \otimes 1) = 0 = d(1 \otimes 1 \otimes \bar{k}_i \otimes 1)$$

where $\rho_1 : U \to G$ and $\rho_2 : K \to U$ are inclusions. The differential is extended as a antiderivation.

Note that the complex A is filtered by giving $S_U \otimes H^*(G) \otimes 1 \otimes 1$ filtration equal to its degree and $1 \otimes 1 \otimes S_K \otimes H^*(U)$ filtration 0.

THEOREM 2.2. (i) *There is a cochain map* $\alpha : A \to \Lambda^*(\mathcal{G}, \mathscr{k}, \mathbf{C})$ *which becomes a map of filtered complexes when we filter* $\Lambda^*(\mathcal{G}, \mathscr{k}, \mathbf{C})$ *using the fibre bundle structure* $U/K \to G/K \to G/U$. *The map* α *induces isomorphisms on the* $E_2^{p,q}$ *level, so* α *induces cohomology isomorphisms as well.*

(ii) *If rank* $(U) = $ *rank* (G) *then we have a map*

$$\bar{\psi} : A \to S_U/I \otimes S_K \otimes H^*(U) = \bar{A}$$

which is the identity on $1 \otimes 1 \otimes S_K \otimes H^*(U)$, *sends* $1 \otimes H^*(G) \otimes 1 \otimes 1$ *to* 0, *and sends* $\bar{u}_i \otimes 1 \otimes 1 \otimes 1$ *to its reduction mod* I *in* $S_U/I \otimes 1 \otimes 1$. *Here* I *is the ideal generated by the* $\bar{\rho}_1(\bar{g}_i)$. *The map* $\bar{\psi}$ *induces isomorphisms on cohomology.*

(iii) *The map* $\beta : A \to S_K \otimes H^*(G)$

$$\beta(\bar{u}_i \otimes 1 \otimes 1 \otimes 1) = \rho_2(\bar{u}_i) \otimes 1$$
$$\beta(1 \otimes g_i \otimes 1 \otimes 1) = 1 \otimes g_i$$
$$\beta(1 \otimes 1 \otimes \bar{k}_i \otimes 1) = \bar{k}_i \otimes 1$$
$$\beta(1 \otimes 1 \otimes 1 \otimes u_i) = 0$$

induces isomorphisms on cohomology.

Proof. (i) follows from [GHV], 12.10, Vol III for the complex $\Lambda^*(\mathscr{G}, \mathscr{U}, \mathbf{C})\otimes S_K\otimes H^*(U)$ where the differential is the usual one on $\Lambda^*(\mathscr{G}, \mathscr{U}, \mathbf{C})\otimes 1\otimes 1$, $d(1\otimes \bar{k}_i\otimes 1)=0$ and $d(1\otimes 1\otimes u_i)=1\otimes\bar{\rho}_2(\bar{u}_i)\otimes 1-P_{u_i}(\Omega)\otimes 1\otimes 1$. Here $P_{u_i}(\Omega)$ is the Weil form for $\bar{u}_i$ evaluated on the curvature Ω of a left invariant connection $\theta:\mathscr{G}\to\mathscr{U}$ for the bundle $G\to G/U$.

Using Theorem 2.1 we can construct a map $A\to\Lambda^*(\mathscr{G}, \mathscr{U}, \mathbf{C})\otimes S_K\otimes H^*(U)$ again inducing isomorphisms on $E_2^{p,q}$. The composition of these two maps gives α.

(ii) follows as well because $\bar{\psi}$ induces isomorphisms on $E_2^{p,q}$ by Theorem 2.1 (ii).

To prove (iii) filter A by giving $1\otimes H^*(G)\otimes 1\otimes 1$ filtration 0 and $S_U\otimes 1\otimes S_K\otimes H^*(U)$ filtration equal to its degree. Filter $S_K\otimes H^*(G)$ by giving $1\otimes H^*(G)$ filtration 0, $S_K\otimes 1$ filtration equal to its degree. One then verifies that β induces isomorphisms on $E_2^{p,q}$.

Remarks. (i) Theorem 2.2 is valid when K is the trivial group $\{1\}$. (ii) An analysis of the map α shows that α maps the subcomplex $S_U\otimes 1\otimes S_K\otimes H^*(U)$ into forms which are polynomials in the connection θ and its curvature Ω in $\Lambda^*(\mathscr{G}, \mathbf{C})$. This fact is important and will be used in Chapter IV.

In Chapter IV we shall be interested in the following application of this theorem. Let G, K, Γ satisfy the hypotheses of Proposition 1.5. In particular assume that G is a non-compact semi-simple Lie group and $\bar{G}$ is the compact form. Then, by Proposition 1.5, we have the injective map γ which is the composition

$$H^*(A)\underset{\approx}{\xrightarrow{\alpha^*}} H^*(\bar{\mathscr{G}}, \bar{k}, \mathbf{C})\approx H^*(\mathscr{G}, \mathscr{k}, \mathbf{C})\xrightarrow{\psi^*} H^*(\Gamma\backslash G/K, \mathbf{C}).$$

As has already been noted, we use complex coefficients because the isomorphism $\mathscr{G}\otimes\mathbf{C}\approx\bar{\mathscr{G}}\otimes\mathbf{C}$ does not restrict to an isomorphism $\mathscr{G}\otimes\mathbf{R}\approx\bar{\mathscr{G}}\otimes\mathbf{R}$.

III. Some facts about parabolic subgroups

The basic reference for this is [S]. Throughout this chapter, $G^{\mathbf{C}}$ denotes a connected semi-simple algebraic group defined over $\mathbf{C}$. All subgroups of $G^{\mathbf{C}}$ are assumed algebraic unless otherwise specified.

DEFINITION 3.1. A Borel subgroup of $G^{\mathbf{C}}$ is a maximal connected solvable subgroup of $G^{\mathbf{C}}$. Any subgroup of $G^{\mathbf{C}}$ which contains a Borel subgroup is called a parabolic subgroup of $G^{\mathbf{C}}$.

Now let $T^{\mathbf{C}}$ be a maximal torus in $G^{\mathbf{C}}$, V the set of roots for $T^{\mathbf{C}}$, V^+ is the set of all positive roots with respect to some linear order, $\Pi \subset V^+$ is the set of simple roots. For every root $\alpha \in V$ there is a unique 1-dimensional subgroup of $G^{\mathbf{C}}$, $P_\alpha^{\mathbf{C}}$, whose Lie algebra, $\mathscr{P}_\alpha^{\mathbf{C}}$, is the root space for the root α.

THEOREM 3.2. (i) *The semi-direct product* $B^{\mathbf{C}} = T^{\mathbf{C}} \cdot (\prod_{\alpha \in V^+} P_\alpha^{\mathbf{C}})$ *is a Borel subgroup of* $G^{\mathbf{C}}$ *and all other Borel subgroups are conjugate to it.*

(ii) *There is a* 1-1 *correspondence between parabolic subgroups containing* $B^{\mathbf{C}}$ *and subsets of vertices (i.e. simple roots) in the Dynkin diagram for* $G^{\mathbf{C}}$. *Let* $\Delta \subset \Pi$ *be such a subset,* $T_1^{\mathbf{C}} \subset T^{\mathbf{C}}$ *is the subtorus annihilated by* Δ, $V_1 \subset V$ *is the subset of roots generated by* Δ. *Then* V_1 *is the set of roots for a unique semi-simple subgroup* $G_1^{\mathbf{C}}$ *of* $G^{\mathbf{C}}$ *and the centralizer of* $T_1^{\mathbf{C}}$, $Z(T_1^{\mathbf{C}}) = G_1^{\mathbf{C}} \times T_1^{\mathbf{C}}$. *Then the* 1-1 *correspondence is given by associating to each* Δ *the semi-direct product*

$$P_\Delta^{\mathbf{C}} = (G_1^{\mathbf{C}} \times T_1^{\mathbf{C}}) \cdot \left(\prod_{\alpha \in V^+ - V_1} P_\alpha^{\mathbf{C}} \right).$$

(iii) *From* (i) *it follows that this classifies all parabolic subgroups of* $G^{\mathbf{C}}$ *up to conjugacy.*

For the proof see [S]. We remark that $\prod_{\alpha \in V^+ - V_1} P_\alpha^{\mathbf{C}} = N(\Delta)$ is a nilpotent subgroup and that a given parabolic $P_\Delta^{\mathbf{C}}$ decomposes the Lie algebra $\mathscr{G}^{\mathbf{C}}$ into $\mathscr{G}_1^{\mathbf{C}} \oplus \mathscr{T}_1^{\mathbf{C}} \oplus \mathscr{N}^{\mathbf{C}} \oplus \mathscr{N}^{-\mathbf{C}}$. Here $\mathscr{G}_1^{\mathbf{C}}$ is the Lie algebra of $G_1^{\mathbf{C}}$, $\mathscr{T}_1^{\mathbf{C}}$ for $T_1^{\mathbf{C}}$, $\mathscr{N}^{\mathbf{C}}$ is the Lie algebra of $N(\Delta)^{\mathbf{C}}$ and $\mathscr{N}^{-\mathbf{C}}$ is the Lie algebra of the group $\prod_{\alpha \in V^- - V_1} P_\alpha^{\mathbf{C}}$. ($V^-$ is the set of negative roots.) The following relations holds:

$$[\mathscr{G}_1^{\mathbf{C}} \oplus \mathscr{T}_1^{\mathbf{C}}, \mathscr{N}^{\mathbf{C}}] \subset \mathscr{N}^{\mathbf{C}}$$
$$[\mathscr{G}_1^{\mathbf{C}} \oplus \mathscr{T}_1^{\mathbf{C}}, \mathscr{N}^{-\mathbf{C}}] \subset \mathscr{N}^{-\mathbf{C}}.$$

Finally we will be interested in real forms G of $G^{\mathbf{C}}$ where the Lie algebra $\mathscr{G} \subset \mathscr{G}^{\mathbf{C}}$ has the form

$$\mathscr{G} = \mathscr{G}_1 \oplus \mathscr{T}_1 \oplus \mathscr{N} \oplus \mathscr{N}^-$$

with

$$\mathscr{N} = \mathscr{N}^{\mathbf{C}} \cap \mathscr{G}, \qquad \mathscr{G}_1 = \mathscr{G}_1^{\mathbf{C}} \cap \mathscr{G}, \text{etc.}$$

Then $\mathscr{P} = \mathscr{P}^{\mathbf{C}} \cap \mathscr{G} = \mathscr{G}_1 \oplus \mathscr{T}_1 \oplus \mathscr{N}$ is a subalgebra of $\mathscr{G}$, and the following relations also hold:

$$[\mathscr{G}_1 \oplus \mathscr{T}_1, \mathscr{N}] \subset \mathscr{N}$$
$$[\mathscr{G}_1 \oplus \mathscr{T}_1, \mathscr{N}^-] \subset \mathscr{N}^-$$

IV. Construction of the foliations and evaluation of their characteristic classes

We continue to examine the situation appearing at the end of the last Chapter. Fix a parabolic $P^{\mathbf{C}}$ and a real form G of $G^{\mathbf{C}}$ with $\mathscr{G}=\mathscr{G}_1\oplus\mathscr{T}_1\oplus\mathscr{N}\oplus\mathscr{N}^-$, $P=G\cap P^{\mathbf{C}}$. Choose a compact form $\bar{G}$ of $G^{\mathbf{C}}$ satisfying

(i) There are closed subgroups $\bar{G}_1$ and $\bar{T}_1$ of $\bar{G}$ with the same complexifications as G_1 and T_1, and $G_1\cap\bar{G}_1$ is maximal compact in G_1.

(ii) The Lie algebra $\bar{\mathscr{G}}$ splits as $\bar{\mathscr{G}}=\bar{\mathscr{G}}_1\oplus\bar{\mathscr{T}}_1\oplus\mathscr{k}$ where the complexification of $\mathscr{k}$ equals $\mathscr{N}^{\mathbf{C}}\oplus\mathscr{N}^{-\mathbf{C}}$.

Such a compact form can be constructed using, for example, the contents of Chapter III of Helgason [H].

In what follows K will either be the trivial group 1 or $K=G_1\cap\bar{G}_1$. The left translates of P determine a foliation on $\Gamma\backslash G/K$. When $K=1$ this foliation has a normal bundle trivialized by left invariant sections, and we obtain characteristic classes from $H^*(W_n)$. For $K\neq 1$ we obtain characteristic classes from $H^*(WO_n)$, and we do this case first.

There are the complexes

$$\bar{A}=S_{\bar{G}_1\times\bar{T}_1}/I\otimes S_K\otimes H^*(\bar{G}_1\times\bar{T}_1)$$

and

$$A=S_{\bar{G}_1\times\bar{T}_1}\otimes H^*(\bar{G})\otimes S_K\otimes H^*(\bar{G}_1\times\bar{T}_1)$$

defined in Chapter II. Let $u_i\in H^*(\bar{G}_1\times\bar{T}_1)$ (resp. $g_i\in H^*(G)$, $k_i\in H^*(K)$) be the primitive generators transgressing to $\bar{u}_i$ (resp. $\bar{g}_i$, $\bar{k}_i$). As noted at the end of Chapter II, there is an injective map $\gamma:H^*(A,\mathbf{C})\to H^*(\Gamma\backslash G/K,\mathbf{C})$, and by Theorem 2.2 there is the isomorphism $\bar{\psi}^*:H^*(A,\mathbf{C})\to H^*(\bar{A},\mathbf{C})$.

We will define a map $\lambda:WO_n\to\bar{A}$ where n is the codimension of P in G, and the composition $\gamma\circ(\bar{\psi}^*)^{-1}\circ\lambda^*:H^*(WO_n)\to H^*(\Gamma\backslash G/K)$ will be the characteristic map for the foliation on $\Gamma\backslash G/K$. First note that there is a homomorphism $\sigma:P^{\mathbf{C}}\to GL(\mathscr{G}^{\mathbf{C}}/\mathscr{P}^{\mathbf{C}})$ where $GL(\mathscr{G}^{\mathbf{C}}/\mathscr{P}^{\mathbf{C}})$ is the general linear group on the complex vector space $\mathscr{G}^{\mathbf{C}}/\mathscr{P}^{\mathbf{C}}$

$$\sigma(p)(x)=Ad_p(x)$$

Choose an $Ad_{\bar{G}_1\times\bar{T}_1}$ invariant Hermitian metric on $\mathscr{G}^{\mathbf{C}}/\mathscr{P}^{\mathbf{C}}$ and an orthonormal basis $x_1,\ldots,x_n$ which is also a basis for $\mathscr{G}/\mathscr{P}$ when the scalar field is restricted to $\mathbf{R}$. By means of this basis we can identify $GL(\mathscr{G}^{\mathbf{C}}/\mathscr{P}^{\mathbf{C}})$ with $GL(n,\mathbf{C})$, $GL(\mathscr{G}/\mathscr{P})$ with $GL(n,\mathbf{R})$, etc. In particular, because $K=G_1\cap\bar{G}_1$, and $\sigma(\bar{G}_1)\subset U(n)$, $\sigma(K)\subset U(n)\cap GL(n,\mathbf{R})=O(n)$. Let $\sigma_1=\sigma|_{\bar{G}_1\times\bar{T}_1}$ and $\sigma_2=\sigma|_K$. Then $\bar{\sigma}_2(C_{2i-1})=0$ where $C_{2i-1}\in S_{GL(n,\mathbf{C})}$ is the $(2i-1)^{\text{th}}$ Chern class.

Consider the subcomplex $B = S_{\bar{G}_1\times\bar{T}_1}\otimes 1\otimes S_K\otimes H^*(\bar{G}_1\times\bar{T}_1)\subset A$, and the maps $\lambda_i : S_{\bar{G}_1\times\bar{T}_1}\to B$, $i=1,2$, given by $\lambda_1(\bar{u}_i)=\bar{u}_i\otimes 1\otimes 1\otimes 1$, $\lambda_2(\bar{u}_i)=1\otimes 1\otimes\bar{\rho}_2(\bar{u}_i)\otimes 1$. The λ_i clearly induces the same maps on cohomology. It follows that

$$\bar{\sigma}_1(C_{2i-1})\otimes 1\otimes 1\otimes 1=\bar{\sigma}_1(C_{2i-1})\otimes 1\otimes 1\otimes 1-1\otimes 1\otimes\bar{\sigma}_2(C_{2i-1})\otimes 1=d\xi_{2i-1}$$

for some $\xi_{2i-1}\in S_{\bar{G}_1\times\bar{T}_1}\otimes 1\otimes S_K\otimes H^*(\bar{G}_1\times\bar{T}_1)$, Let $\bar{\xi}_{2i-1}$ denote the image of ξ_{2i-1} in $\bar{A}$.

Now define $\lambda : WO_n\to\bar{A}$ as follows: $\lambda(C_k)$ is the mod I reduction of

$$(\sqrt{-1})^k\bar{\sigma}_1(C_k)\otimes 1\otimes 1$$
$$\lambda(h_{2k-1})=(\sqrt{-1})^{2k-1}\bar{\xi}_{2k-1}.$$

It is easily verified that λ commutes with the differentials and is a cochain map.

Remarks. The reason for the coefficient $(\sqrt{-1})^k$ is that the Weil polynomials for Chern Classes are given by the formulae

$$\det\left(\lambda I-\frac{1}{2\pi i}\Omega\right)=\sum_{k=0}^{n}C_k(\Omega)\lambda^{n-k}.$$

These polynomials are real valued on the Lie algebra for $U(n)$, but complex valued on the Lie algebra for $GL(n,\mathbf{R})$. To avoid complex numbers, one uses the definition

$$\det\left(\lambda I-\frac{1}{2\pi}\Omega\right)=\sum_{k=0}^{n}C_k(\Omega)\lambda^{n-k}$$

when defining characteristic classes for foliations. These polynomials are real valued on the Lie algebra for $GL(n,\mathbf{R})$ and differ from the first definition by a factor of $(\sqrt{-1})^k$.

Note also that, because the dimension of $\bar{G}/\bar{G}_1\times\bar{T}_1$ is $2n$, the algebra $S_{\bar{G}_1\times\bar{T}_1}/I$ is 0 above dimension $2n$. This is necessary if λ is to be a cochain map, since products of dimension greater than $2n$ in the Chern classes are zero in WO_n. It is for this reason that we cannot map WO_n into A instead of $\bar{A}$.

THEOREM 4.1 *There is a commutative diagram*

$$\begin{tikzcd} H^*(\bar{A},\mathbf{C}) \arrow[r, "F"] & H^*(\Gamma\backslash G/K,\mathbf{C}) \\ H^*(WO_n,\mathbf{R}) \arrow[u, "\lambda^*"] \arrow[r, "\Phi"] & H^*(\Gamma\backslash G/K,\mathbf{R}) \arrow[u, "i"'] \end{tikzcd}$$

where i is induced by inclusion $\mathbf{R} \subset \mathbf{C}$, Φ *is the characteristic map of the foliation determined by P and* $F = \gamma \circ (\bar{\psi}^*)^{-1}$.

The point of this theorem is that, because F is injective, linear independence of classes in the image of λ^* implies the linear independence of their image by Φ. The image of λ^* is fairly computable because of its topological interpretation in terms of characteristic classes.

To prove Theorem 4.1, we define the auxiliary notion of a classifying homomorphism. The initial data is a Lie group U and subgroup K, a principle U bundle $\pi: E \to B$ with dimension of B equals $2n$, and a representation $r: U \to GL(n, \mathbf{C})$ with $r(K) \subset O(n, \mathbf{C})$.

Let $\bar{\pi}: EGL \to BGL$ be a finite dimensional smooth approximation to the classifying space for $GL(n, \mathbf{C})$ bundles and choose a bundle map

$$\begin{array}{ccc} E/K & \xrightarrow{\bar{f}} & EGL/r(K) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{\pi}} \\ B & \xrightarrow{f} & BGL \end{array}$$

where $f: B \to BGL$ classifies the bundle $E \times_U GL(n, \mathbf{C}) \to B$ where the action of U on $GL(n, \mathbf{C})$ is given by the representation r.

Choose forms $\alpha_k \in \Lambda^*(BGL, \mathbf{C})$ representing the k^{th} Chern class C_k. Since $r(K) \subset O(n, \mathbf{C})$, the odd Chern classes vanish on $EGL/r(K)$, so choose $\beta_{2k-1} \in \Lambda^*(EGL/r(K), \mathbf{C})$ with $d\beta_{2k-1} = \bar{\pi}^* \alpha_{2k-1}$. Then define a cochain map $\nu: WO_n \to \Lambda^*(E/K, \mathbf{C})$ by

$$\nu(C_k) = (\sqrt{-1})^k \bar{f}^* \circ \bar{\pi}^*(\alpha_k)$$
$$\nu(h_{2k-1}) = (\sqrt{-1})^{2k-1} \bar{f}^*(\beta_{2k-1}).$$

Define the classifying homomorphism Ξ to be the induced map

$$\Xi = \nu^*: H^*(WO_n, \mathbf{R}) \to H^*(E/K, \mathbf{C}).$$

The next Lemma will show that Ξ is actually independent of the various choices made in defining ν.

LEMMA 4.2 *Let* $\pi: E \hookrightarrow B$ *be a U bundle with dimension* $B = 2n$. *If* $\phi_0, \phi_1: WO_n \to \Lambda^*(E, \mathbf{C})$ *are two cochain maps with* $\phi_i(C_k) \in$ *Image* π^*,

$$(\phi_0 - \phi_1)(C_k) = \pi^* d\sigma_k,$$
$$(\phi_0 - \phi_1)(h_{2k-1}) = \pi^* \xi_k + d\eta_k,$$

then ϕ_0 *and* ϕ_1 *induce the same map on cohomology.*

Proof. It suffices to prove that ϕ_0 and ϕ_1 are cohomologous on the basis for $H^*(WO_n, \mathbf{C})$ given in Lemma 1.2 (ii). This is fairly straightforward, keeping in mind that, since dimension of B is $2n$, any form in the image of π^* of dimension greater than $2n$ must be 0. We leave the details to the reader.

Now to see that Ξ is well defined, suppose first that we are given two different classifying maps

$$(\bar{f}, f), (\bar{g}, g): (E/K, B) \to (EGL/r(K), BGL).$$

Then these maps are homotopic, so there are cochain homotopies $\bar{f}^* - \bar{g}^* = d\bar{K} + \bar{K}d$ and $f^* - g^* = dK + Kd$ and one can choose K and $\bar{K}$ to satisfy $\pi^* K = \bar{K}\bar{\pi}^*$. Then $\pi^* f^*(\alpha_k) - \pi^* g^*(\alpha_k) = \pi^*(dK(\alpha_k))$ and $\bar{f}^*(\beta_{2k-1}) - \bar{g}^*(\beta_{2k-1}) = d\bar{K}(\beta_{2k-1}) + \bar{K}\bar{\pi}^*(\alpha_{2k-1}) = d\bar{K}(\beta_{2k-1}) + \pi^* K^*(\alpha_{2k-1})$. So by Lemma 4.2 Ξ is independent of the classifying map. If α_k, β_{2k-1} and $\tilde{\alpha}_k$, $\tilde{\beta}_{2k-1}$ are different choices of forms determining ν, then $\alpha_k - \tilde{\alpha}_k = d\eta_k$ and $d(\beta_{2k-1} - \tilde{\beta}_{2k-1} - \bar{\pi}^* \eta_{2k-1}) = 0$, so, since $\Lambda^*(EGL/r(K), \mathbf{C})$ is acyclic in low odd dimensions, $\beta_{2k-1} - \tilde{\beta}_{2k-1} = \bar{\pi}^* \eta_{2k-1} + d\xi_{2k-1}$ and again Lemma 4.2 shows that Ξ is independent of this choice.

Finally note that we can use the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & EU \arrow[r, "\bar{R}"'] \arrow[d] & EGL \arrow[d] \\ B \arrow[r] & BU \arrow[r, "R"'] & BGL \end{tikzcd}$$

where $EU \to BU$ is a smooth finite dimensional approximation to the classifying space for U bundles, and R is induced by the representation r. Then we can define ν by choosing $\alpha_k \in \Lambda^*(BU, \mathbf{C})$ to represent $R^*(C_k)$ and $\beta_{2k-1} \in \Lambda^*(EU/K, \mathbf{C})$ with $d\beta_{2k-1} = \pi^* \alpha_k$. Again since $H^*(EU/K, \mathbf{C}) = 0$ in odd dimensions, ν will induce the same map Ξ.

Remark. Let $B^{2n}_{GL(n,\mathbf{C})}$ be the $2n$-skeleton of $B_{GL(n,\mathbf{C})}$ using the cell decomposition by Schubert cells. By taking the restriction $E(GL) \to B^{2n}_{GL(n,\mathbf{C})}$ of the classifying bundle to its $2n$-skeleton, we obtain a classifying space for bundles whose base has dimension $\leq 2n$. One can show that $H^*(E(GL)/O(n, \mathbf{C})) \approx H^*(WO_n)$ and that Ξ is induced by a classifying map $\bar{f}: E/K \to E(GL)/O(n, \mathbf{C})$.

The point now is that $\lambda^*: H^*(WO_n, \mathbf{R}) \to H^*(\bar{A}, \mathbf{C})$ is the classifying homorphism Ξ when we compose with the isomorphism $\alpha \circ (\bar{\psi}^*)^{-1}: H^*(\bar{A}, \mathbf{C}) \to H^*(\bar{G}/K, \mathbf{C})$ of Chapter II. Here the representation is $\sigma_1: \bar{G}_1 \times \bar{T}_1 \to GL(n, \mathbf{C})$ and $\bar{G} \to \bar{G}/\bar{G}_1 \times \bar{T}_1$ is the principle bundle with dimension $\bar{G}/\bar{G}_1 \times \bar{T}_1 = 2n$ (recall dim $\bar{G}_1/\bar{G}_1 \times \bar{T}_1 = \dim(\mathcal{N} \oplus \mathcal{N}^-)$ in the Lie algebra $\mathcal{G}$, and $\dim \mathcal{N} = \dim \mathcal{N}^- = n$). For the map $\alpha \circ (\bar{\psi}^*)^{-1}$ sends classes in the image of λ^* to Weil polynomials in a

$\bar{\mathscr{G}}_1 \oplus \bar{\mathscr{T}}_1$ valued connection and its curvature (see the remark after Theorem 2.2). We must show that these forms are pull-backs of forms from a classifying space and to do this it suffices to show that the connection is the pull-back of a connection on a finite dimensional approximation to $E_{\bar{G}_1 \times \bar{T}_1} \to B_{\bar{G}_1 \times \bar{T}_1}$. This follows by a theorem of Narasimhan-Ramanan on universal connections (see [NR]).

Thus to prove Theorem 4.1 we must do the following:

(i) Show that $i \circ \Phi : H^*(WO_n, \mathbf{R}) \to H^*(\Gamma\backslash G/K, \mathbf{C})$ is induced by a cochain map $\phi : WO_n \to \Lambda^*(\mathscr{G}, \mathscr{k}, \mathbf{C}) \approx \Lambda^*(\bar{\mathscr{G}}, \mathscr{k}, \mathbf{C})$ and

(ii) Show that ϕ induces the classifying map for the bundle $\bar{G} \to \bar{G}/\bar{G}_1 \times \bar{T}_1$ under the isomorphism $H^*(\bar{\mathscr{G}}, \mathscr{k}, \mathbf{C}) \approx H^*(\bar{G}/K, \mathbf{C})$.

We show (i) first. Let $\theta : \mathscr{G}^{\mathbf{C}} \to \mathscr{P}^{\mathbf{C}}$ be the projection induced by the splitting

$$\mathscr{G}^{\mathbf{C}} = \mathscr{G}_1^{\mathbf{C}} \oplus \mathscr{T}_1^{\mathbf{C}} \oplus \mathscr{N}^{\mathbf{C}} \oplus \mathscr{N}^{-\mathbf{C}} = \mathscr{P}^{\mathbf{C}} \oplus \mathscr{N}^{-\mathbf{C}}$$
$$\mathscr{G} = \mathscr{G}_1 \oplus \mathscr{T}_1 \oplus \mathscr{N} \oplus \mathscr{N}^- = \mathscr{P} \oplus \mathscr{N}^-.$$

LEMMA 4.3. *The normal bundle to the foliation on $\Gamma\backslash G/K$ is just $\Gamma\backslash G \times_K GL(n, \mathbf{R}) \to \Gamma\backslash G/K$ where the action of K on $GL(n, \mathbf{R})$ is given by the representation σ_2. The form*

$$\omega(\dot{g}, \dot{x}) = \dot{x} + Ad_{x^{-1}}(\sigma_*(\theta(\dot{g})))$$

is a basic-connection form on this bundle. (See [BT] *for the definition of a basic connection.)*

Proof. Let T be the $GL(n, \mathbf{R})$ bundle associated to the normal bundle to the foliation. We give a bundle isomorphism $\Gamma\backslash G \times_K GL(n, \mathbf{R}) \to T$ and leave the other details to the reader. Choose an orthonormal basis $x_1, \dots, x_n$ for $\mathscr{G}/\mathscr{P}$ and view each x_i as a left invariant vector field in $\mathscr{N}^-$. Then, given $g \in G$, $x \in GL(n, \mathbf{R})$, x determines a normal frame at g and, by using projection, at $gK \in G/K$. This process is independent of the left action of Γ and the right action of K, and gives the desired map.

We must now define a Riemannian connection on the normal bundle to compute Φ. By choosing this connection carefully we can save some work when it comes time to prove part (ii) of the proof of Theorem 4.1.

LEMMA 4.4. *There is a projection $p : \mathscr{gl}(n, \mathbf{C}) \to \sigma_*(\mathscr{k})^{\mathbf{C}}$ commuting with $Ad_{\sigma(K)^{\mathbf{C}}}$ and a projection $\theta_0 : \mathscr{G}^{\mathbf{C}} \to \mathscr{k}^{\mathbf{C}}$ commuting with Ad_K such that*

(i) $\theta_0(\mathscr{G}) \subset \mathscr{k}$

(ii) $p \circ \sigma_* \circ \theta = \sigma_* \circ \theta_0$.

Proof. Choose an Ad_K stable splitting of $\mathscr{P} = \mathscr{k} \oplus N \oplus M$ where $\mathscr{k} \oplus N = \mathscr{k} + \operatorname{Ker}(\sigma_*)$ and $\sigma_*(N) = 0$. Define $\theta_0 : \mathscr{G} \to \mathscr{k}$ by projection from $\mathscr{G}$ to $\mathscr{P}$ (using $\theta_{|\mathscr{G}} : \mathscr{G} \to \mathscr{P}$) composed with projection from $\mathscr{P}$ onto $\mathscr{k}$ (using the above splitting).

Now choose an $Ad_{\sigma(K)}$ stable decomposition of $\mathfrak{gl}(n, \mathbf{R}) = \sigma_*(\mathscr{k}) \oplus \sigma_*(M) \oplus \tilde{M}$. Define $p : \mathfrak{gl}(n, \mathbf{R}) \to \sigma_*(\mathscr{k})$ by projection via this splitting. It is easily seen that $\sigma_* \circ \theta_{0|_{\mathscr{P}}} = p \circ \sigma_{*|_{\mathscr{P}}}$.

One can now extend θ_0 to all of $\mathscr{G}^{\mathbf{C}}$ and p to all of $\mathfrak{gl}(n, \mathbf{C})$ by complexifying the above maps.

Now define a connection on $\Gamma\backslash G \times_K GL(n, \mathbf{R})$ by $\omega_0(\dot{g}, \dot{x}) = \dot{x} + Ad_{x^{-1}}(\sigma_* \theta_0(\dot{g}))$. Then ω_0 is $\mathfrak{so}(n)$ valued on the $SO(n)$ reduction $\Gamma\backslash G \times_K SO(n)$ by Lemma 4.4 (i), and so is Riemannian.

Thus we have a cochain map, inducing Φ

$$\tilde{\phi} : WO_n \to \Lambda^*(\Gamma\backslash G \times_K GL(n, \mathbf{R}), \mathbf{R})$$

given by

$$\tilde{\phi}(C_k) = (\sqrt{-1})^k C_k(\bar{\Omega}), \qquad \bar{\Omega} = d\omega + \tfrac{1}{2}[\omega, \omega]$$

$$\tilde{\phi}(h_{2k-1}) = (\sqrt{-1})^{2k-1} \Delta C_{2k-1}(\omega, \omega_0).$$

It follows that the image $(\tilde{\phi}) \subset \pi^*(\Lambda^*(\Gamma\backslash G/K, \mathbf{R}))$. We have a bundle map, where $s(g) = (g, 1)$,

$$\begin{array}{ccc} \Gamma\backslash G & \xrightarrow{s} & \Gamma\backslash G \times_K GL(n, \mathbf{R}) \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \Gamma\backslash G/K & \xrightarrow[id]{} & \Gamma\backslash G/K \end{array}$$

and from this it follows that $s^* \circ \tilde{\phi}$ maps into the image of $\tilde{\pi}^*$ and also induces Φ. Note that a left invariant form in the image of $\tilde{\pi}^*$ must factor through $\Lambda^*(\mathscr{G}, \mathscr{k}, \mathbf{R})$. Since $s^*\omega = \sigma_*\theta$ and $s^*\omega_0 = \sigma_*\theta_0$, we have a map $\phi : WO_n \to \Lambda^*(\mathscr{G}, \mathscr{k}, \mathbf{R})$

$$\phi(C_k) = C_k(\sigma_*\Omega), \qquad \Omega = d\theta + \tfrac{1}{2}[\theta, \theta]$$

$$\phi(h_{2k-1}) = \Delta C_{2k-1}(\sigma_*\theta, \sigma_*\theta_0)$$

and ϕ induces Φ. This proves (i).

As before, we move to complex coefficients and view ϕ as a map $\phi: WO_n \to \Lambda^*(\mathscr{G}, \mathscr{k}, \mathbf{C})$. Now consider the bundle $\bar{G} \times_{\bar{G}_1 \times \bar{T}_1} GL(n, \mathbf{C}) \to \bar{G}/\bar{G}_1 \times \bar{T}_1$ and let $\bar{\omega}(\dot{g}, \dot{x}) = \dot{x} + Ad_{x^{-1}}(\sigma_*\theta(\dot{g}))$ be a connection on this bundle. Note that $p \circ \bar{\omega}$ is a $\sigma_*(\mathscr{k})^{\mathbf{C}}$ valued connection on the bundle $\bar{G} \times_{\bar{G}_1 \times \bar{T}_1} GL(n, \mathbf{C}) \to \bar{G} \times_{\bar{G}_1 \times \bar{T}_1} GL(n, \mathbf{C})/\sigma(K)^{\mathbf{C}}$. Now, in the diagram

$$\begin{array}{ccc} \bar{G} & \xrightarrow{\bar{s}} & \bar{G} \times_{\bar{G}_1 \times \bar{T}_1} GL(n, \mathbf{C}) \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} \\ \bar{G}/\bar{G}_1 \times \bar{T}_1 & \xrightarrow[id]{} & \bar{G}/\bar{G}_1 \times \bar{T}_1 \end{array}$$

$\bar{s}(g) = (g, 1)$, we have

$$\bar{s}^* \bar{\omega} = \sigma_* \theta$$
$$\bar{s}^*(p \circ \bar{\omega}) = p \circ \sigma_* \circ \theta = \sigma_* \theta_0$$

by Lemma 4.4 (ii). It then follows that the classifying map determined by $\bar{\omega}$

$$C_k \to C_k(\sigma_* \Omega)$$
$$h_{2k-1} \to \Delta C_{2k-1}(\sigma_* \theta, \sigma_* \theta_0) = \bar{s}^* \Delta C_{2k-1}(\bar{\omega}, p \circ \bar{\omega})$$

is the same map as ϕ, and this proves (ii) and completes the proof of Theorem 4.1.

When $K = 1$, the complexes A and $\bar{A}$ are given by

$$A = S_{\bar{G}_1 \times \bar{T}_1} \otimes H^*(\bar{G}) \otimes H^*(\bar{G}_1 \times \bar{T}_1)$$
$$\bar{A} = S_{\bar{G}_1 \times \bar{T}_1}/I \otimes H^*(\bar{G}_1 \times \bar{T}_1).$$

In A the class $\bar{\sigma}_1(C_k) \otimes 1 \otimes 1 = d\xi_k$ for $\xi_k \in S_{\bar{G}_1 \times \bar{T}_1} \otimes 1 \otimes H^*(\bar{G}_1 \times \bar{T}_1)$ by a similar argument to the one used when $K \neq 1$. Let $\bar{\xi}_k \in \bar{A}$ be the image of $\xi_k \in A$ and define $\lambda: W_n \to \bar{A}$ as follows: $\lambda(C_k)$ is the mod I reduction of

$$(\sqrt{-1})^k \bar{\sigma}_1(C_k) \otimes 1$$
$$\lambda(h_k) = (\sqrt{-1})^k \bar{\xi}_k.$$

THEOREM 4.5. *There is a commutative diagram*

$$\begin{array}{ccc} H^*(\bar{A}, \mathbf{C}) & \xrightarrow{F} & H^*(\Gamma \backslash G, \mathbf{C}) \\ \uparrow{\scriptstyle \lambda^*} & & \uparrow{\scriptstyle i} \\ H^*(W_n, \mathbf{R}) & \xrightarrow{\Phi} & H^*(\Gamma \backslash G, \mathbf{R}) \end{array}$$

where i is induced by inclusion $\mathbf{R} \subset \mathbf{C}$, Φ is the characteristic map for the foliation determined by P and $F = \gamma \circ (\bar{\psi}^*)^{-1}$.

The proof is basically the same as that for Theorem 4.1. We omit the details.

V. Examples

Note. In the computations in this chapter, the scale factor $(\sqrt{-1})^i$ appearing in $\lambda(c_i)$ and $\lambda(h_i)$ will be ignored. Because we are interested only in questions of linear independence, this will make no difference, and to include it would unnecessarily clutter the notation.

In the examples which follow it is necessary to evaluate the maps on characteristic classes induced by the representation $\sigma_1 : \bar{G}_1 \times \bar{T}_1 \to GL(n, \mathbf{C})$ and by the inclusions $\bar{G}_1 \times \bar{T}_1 \to \bar{G}$ and $K \to \bar{G}_1 \times \bar{T}_1$. We do this by evaluating the associated maps on Weil polynomials. Since the domains of our homomorphisms are always compact, there is a Weil polynomial representing each characteristic class associated to these groups. To evaluate these maps on Weil polynomials we usually restrict to a Cartan subalgebra. This technique is standard (see [KN], Volume II, for an excellent exposition).

The following proposition will be useful:

PROPOSITION 5.1. *Consider a parabolic subgroup obtained by removing one vertex from the Dynkin diagram for $G^{\mathbf{C}}$. Then $H^2(\bar{G}/\bar{G}_1 \times \bar{T}_1, \mathbf{C})$ is one dimensional and, for $0 \neq x \in H^2(\bar{G}/\bar{G}_1 \times \bar{T}_1, \mathbf{C})$, a power of x is a non-zero top dimensional class in $H^*(\bar{G}/\bar{G}_1 \times \bar{T}_1, \mathbf{C})$.*

Proof. Since $\bar{G}_1$ has rank one less than $\bar{G}$, $\bar{T}_1$ must be a circle. Since $\bar{G}_1$ is semi-simple, $H^1(\bar{G}_1, \mathbf{C}) = 0$ (see Theorem 16.1 of [CE]). Thus $H^1(\bar{G}_1 \times \bar{T}_1, \mathbf{C})$ is one dimensional and a generator must transgress to a non-zero class in $H^2(\bar{G}/\bar{G}_1 \times \bar{T}_1, \mathbf{C})$. (Otherwise $H^1(\bar{G}, \mathbf{C})$ would not be 0.) Conversely, any two dimensional class in $H^*(\bar{G}/\bar{G}_1 \times \bar{T}_1, \mathbf{C})$ must be in the image of the transgression in $\bar{G}_1 \times \bar{T}_1 \to \bar{G} \twoheadrightarrow \bar{G}/\bar{G}_1 \times \bar{T}_1$, since $H^2(\bar{G}, \mathbf{C}) = 0$. Thus there can be only one such class which shows that $H^2(\bar{G}/\bar{G}_1 \times \bar{T}_1, \mathbf{C})$ is one dimensional.

To see that a power of x is a non-zero top dimensional class note that $\bar{G}/\bar{G}_1 \times \bar{T}_1$ is Kaehler and x must be a multiple of the fundamental two form.

EXAMPLE 1(a). The Dynkin diagram for $\mathfrak{sl}(n+k)$ is just:

$$\circ\!-\!\!-\!\!-\circ\!-\!\!-\!\!-\!\!-\cdots\!-\!\!-\!\!-\!\!-\circ \quad (n+k-1 \quad \text{vertices}).$$

If we remove one of these vertices we get a parabolic of the form

$$\{\|a_{ij}\| \in \mathscr{sl}(n+k) \mid a_{ij}=0 \text{ for } i>k, j\le k\}$$

Here

$$\mathscr{G}_1 \oplus \mathscr{T}_1 \approx s(\mathscr{gl}(k) \oplus \mathscr{gl}(n)) = \{\|a_{ij}\| \in \mathscr{sl}(n+k) \mid a_{ij}=0 \text{ for } i>k, j\le k \text{ or } i\le k, j<k\}.$$

The codimension of the associated foliation is nk.

$$\left[\begin{array}{c|c} \mathscr{G}_1 \oplus \mathscr{T}_1 & \mathscr{N} \\ \hline \mathscr{N}^- & \mathscr{G}_1 \oplus \mathscr{T}_1 \end{array}\right]$$

Set $K=1$.

In doing computations, it is simpler to find maps on Weil polynomials if we replace $\mathscr{sl}(n+k)$ with $\mathscr{gl}(n+k)$ and $s(\mathscr{gl}(n) \oplus \mathscr{gl}(k))$ with $\mathscr{gl}(n) \oplus \mathscr{gl}(k)$. One easily verifies that doing the computation in $\mathscr{gl}(n+k)$ and then restricting to $\mathscr{sl}(n+k)$ is equivalent to doing the computation in $\mathscr{sl}(n+k)$. In fact, this corresponds to working with the foliation on $\Gamma\backslash SL(n+k, \mathbf{R}) \times \mathbf{R} = \Gamma\backslash GL(n+k, \mathbf{R})$ whose leaves are of the form $L \times \mathbf{R}$ where L is a leaf in $\Gamma\backslash SL(n+k, \mathbf{R})$.

So let $\bar{A} = S_{U(n)\times U(k)}/I \otimes H^*(U(n) \times U(k))$. The ring $S_{U(n)\times U(k)}/I$ is the cohomology ring of k planes in $n+k$-space. Let $d_i \in H^*(U(n))$ (resp. $e_i \in H^*(U(k))$) represent the primitive generators transgressing to the i^{th} Chern classes $\bar{d}_i$ in $S_{U(n)}$ (resp. $\bar{e}_i$ in $S_{U(k)}$). Then $S_{U(n)\times U(k)} = \mathbf{C}[\bar{d}_1, \ldots, \bar{d}_n, \bar{e}_1, \ldots, \bar{e}_k]$. To evaluate the map on characteristic classes $\bar{\rho}_1$, induced by inclusion $\rho_1 : U(n) \times U(k) \to U(n+k)$ recall the Whitney sum formula

$$\left(\sum_0^{n+k} \bar{g}_i\right) \to \left(\sum_0^n \bar{d}_i\right)\left(\sum_0^k \bar{e}_i\right) \tag{5.2}$$

where $\bar{g}_i$ is the i^{th} Chern class in $S_{U(n+k)}$ and $\bar{d}_0 = \bar{e}_0 = \bar{g}_0 = 1$. Thus I is generated by the relations $1 = (\sum_0^n \bar{d}_i)(\sum_0^k \bar{e}_i)$.

THEOREM 5.3. *Suppose $k<n$. Then the classes $c_1^{kn} h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell}$ for all $k<i_1< \cdots <i_\ell \le n$ are non-zero and linearly independent in $H^*(\Gamma\backslash SL(n+k, \mathbf{R}), \mathbf{R})$.*

Notation. In Theorem 5.3 and throughout this chapter, "for all $k<i_1< \cdots < i_\ell \le n$" includes the class involving no indices, i.e. $c_1^{kn} h_1 \cdots h_k$ in the above case.

Proof. The representation $\sigma_1 = \sigma_{|G_1\times T_1}: GL(n)\times GL(k) \to GL(kn)$ is the representation of $GL(n)\times GL(k)$ on $\mathbf{R}^n \otimes \mathbf{R}^k$ where the action of $GL(n)$ on $\mathbf{R}^n$ is the usual one and the action of $GL(k)$ on $\mathbf{R}^k$ is given by $A \mapsto (A^t)^{-1}$. It is not difficult to show that

$$\bar{\sigma}_1(c_i) = k\bar{d}_i + (-1)^i n\bar{e}_i + \text{mixed terms} \tag{5.4}$$

where a mixed term is a product of two or more of $\bar{e}_1, \bar{e}_2, \ldots, \bar{d}_1, \bar{d}_2, \ldots$. (For ease in notation, assume that $\bar{d}_i = d_i = 0$ for $i > n$, $\bar{e}_i = e_i = 0$ for $i > k$.)

In particular $\bar{\sigma}_1(c_1) = k\bar{d}_1 - n\bar{e}_1$ and in $S_{U(n)\times U(k)}/I$ the relation $\bar{e}_1 + \bar{d}_1 = 0$ implies $\bar{\sigma}_1(c_1) \equiv -(n+k)\bar{e}_1 \mod I$. Thus the map $\lambda: W_n \to \bar{A}$ of Theorem 4.5 sends c_1^{kn} to $[-(n+k)\bar{e}_1]^{kn}$ which, by Proposition 5.1, is a non-zero top dimensional class in $S_{U(n)\times U(k)}/I$, hereafter referred to as V.

Now $H^*(\bar{A}, \mathbf{C}) \approx H^*(U(n+k), \mathbf{C})$ is an exterior algebra on generators $g_1, \ldots, g_{n+k}$ where the dimension of g_i is $2i-1$. We show that $\lambda^*(c_1^{nk} h_1 \cdots h_k)$ is non-zero by taking its product with $g_1 \wedge \cdots \wedge g_n$ and obtaining a non-zero multiple of the top dimensional class $V \otimes e_1 \wedge \cdots \wedge e_k \wedge d_1 \wedge \cdots \wedge d_n \in \bar{A}$. Because $\lambda(c_1^{nk}) = V$ is top dimensional in $S_{U(n)\times U(k)}/I$, we can ignore all terms involving $\bar{e}_1, \ldots, \bar{e}_k, \bar{d}_1, \ldots, \bar{d}_n$ when multiplying V by a $\lambda(h_i)$. With this in mind we set

$$\lambda(h_i) = 1 \otimes (kd_i + (-1)^i ne_i) + \text{other terms}$$

and the other terms will drop out when multiplied by V (see equation 5.4). Thus

$$\lambda(c_1^{nk} h_1 \cdots h_k) = V \otimes \prod_1^k \left(kd_i + (-1)^i ne_i\right). \tag{5.5}$$

We now give cochain representatives for the generators $g_1, \ldots, g_n \in H^*(\bar{A}, \mathbf{C})$. Using Theorem 2.2 (ii) and (iii), and the fact that $K = 1$, we obtain from (5.2)

$$g_i = 1 \otimes d_i + 1 \otimes e_i + \text{mixed terms}$$

where, again, the mixed terms will vanish when multiplied by V. Thus $\lambda^*(c_1^{nk} h_1 \cdots h_k) g_1 \cdots g_n$ has a cochain representative in $\bar{A}$,

$$V \otimes \prod_1^k (kd_i + (-1)^i ne_i) \prod_1^n (d_i + e_i). \tag{5.6}$$

Since $k \neq n$, this is a non-zero multiple of $V \otimes e_1 \wedge \cdots \wedge e_k \wedge d_1 \wedge \cdots \wedge d_n$ so that $\lambda^*(c_1^{nk} h_1 \cdots h_k)$ must be non-zero.

The proof that $\lambda^*(c_1^{nk}h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell})$ is non-zero for $k < i_1 < \cdots < i_\ell \leq n$ is the same except that we multiply by

$$\prod_{\substack{i \neq i_1, \ldots, i_\ell \\ i \leq n}} g_i.$$

Furthermore, suppose a linear combination

$$\sum_{k < i_1 < \cdots < i_\ell \leq n} \alpha_{i_1 \cdots i_\ell} \lambda^*(c_1^{kn} h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell}) = 0. \tag{5.7}$$

Then, for fixed $i_1, \ldots, i_\ell$ multiply equation (5.7) by

$$\prod_{\substack{i \neq i_1, \ldots, i_\ell \\ i \leq n}} g_i$$

to show that $\alpha_{i_1 \cdots i_\ell} = 0$ so that these classes are linearly independent in $H^*(\bar{A}, \mathbf{C})$, hence by Theorem 4.5 in $H^*(\Gamma \backslash SL(n+k, \mathbf{R}), \mathbf{R})$. Q.E.D.

Remark. An examination of the proof shows that the stipulation $k \neq n$ is not necessary when $k = n = 1$. This is Roussarie's example on $\Gamma \backslash SL(2, \mathbf{R})$. (see [GV]).

We now examine the remaining classes in $H^*(W_{kn}, \mathbf{R})$. We start by restricting our attention to the case where $k = 1$.

THEOREM 5.9. (*see* [KT2]). *On the foliation of* $\Gamma \backslash SL(n+1, \mathbf{R})$ *whose leaf is determined by the parabolic* $\mathcal{P} = \{\|a_{ij}\| \mid a_{ij} = 0$ *for* $i > 1$, $j = 1\}$ *the image of the characteristic map*

$$\Phi : H^*(W_n, \mathbf{R}) \to H^*(\Gamma \backslash SL(n+1), \mathbf{R}), \mathbf{R})$$

has the following properties:

(*i*) *Let* $q(c_1, \ldots, c_n)$ *be a monomial of deg.* $2i$. *Then the classes* $\Phi(q(c_1, \ldots, c_n) h_{n-i+1} h_{i_1} \cdots h_{i_\ell})$ *for all* $n - i + 1 < i_1 < \cdots < i_\ell \leq n$ *are linearly independent in* $H^*(\Gamma \backslash SL(n+1), \mathbf{R}), \mathbf{R})$. *These classes are multiples of the classes*

$$\Phi(c_1^n h_1 h_{i_1} \cdots h_{i_\ell}) \in H^*(\Gamma \backslash SL(n+1, \mathbf{R}), \mathbf{R}).$$

If $p(c_1, \ldots, c_n)$ *and* $q(c_1, \ldots, c_n)$ *are monomials of degree* $2i$ *then*

$\Phi(p(c_1, \ldots, c_n)h_{n-i+1}h_{i_1} \cdots h_{i_\ell})$ and $\Phi(q(c_1, \ldots, c_n)h_{n-i+1}h_{i_1} \cdots h_{i_\ell})$ are multiples of each other.

(ii) The image of the rigid classes under Φ is 0.

This theorem yields (in all codimensions) non-vanishing (in $H^*(F\Gamma_n^r, \mathbf{R})$) of all the classes which form the basis for $H^*(W_n)$ in Lemma 1.2 and which are deformable (see [HI1]). To obtain linear independence of these classes as we vary the monomial $q(c_1, \ldots, c_n)$, we must examine the other foliations we have analyzed, and compare them with this one. If we find that in other examples $q(c_1, \ldots, c_n)$ is a different multiple of c_1^i from the one it is in the $k=1$ example, then new linear independence relations can be obtained. In particular we have the following result:

PROPOSITION 5.10. *For $k<n$, consider the $GL(kn, \mathbf{C})$ bundles associated to the bundles*

$$\begin{array}{l} U(kn)\times U(1) \to U(kn+1) \\ \qquad\qquad\qquad\quad \downarrow \\ \qquad\qquad\qquad U(kn+1)/U(kn)\times U(1) \end{array}$$

$$\begin{array}{l} U(k)\times U(n) \to U(k+n) \\ \qquad\qquad\qquad\quad \downarrow \\ \qquad\qquad\qquad U(k+n)/U(k)\times U(n) \end{array}$$

by the representations σ used in Theorem 4.1. Let $p(c_1, \ldots, c_{kn})$ and $q(c_1, \ldots, c_{kn})$ be any polynomials of dimension $2kn$ in their Chern classes. Then if the ratios of the Chern numbers associated to $p(c_1, \ldots, c_{kn})$ and $q(c_1, \ldots, c_{kn})$ are different on these two bundles, the set of classes

$$\{q(c_1, \ldots, c_{kn})h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell}, p(c_1, \ldots, c_{kn})h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell} \quad k<i_1 < \cdots < i_\ell \leq n\}$$

are linearly independent in $H^*(F\Gamma_{kn}^r, \mathbf{R})$, $r\geq 2$.

Proof. On the $\Gamma\backslash SL(nk+1, \mathbf{R})$ example, we know that for any linear combination of these classes which is 0, the terms involving $q(c_1, \ldots, c_{kn})h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell}$ and $p(c_1, \ldots, c_{kn})h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell}$ must vanish pairwise. The same is true for the example $\Gamma\backslash SL(n+k, \mathbf{R})$. Thus we obtain two equations of the form

$$0=\alpha q(c_1, \ldots, c_{kn})h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell} + \beta p(c_1, \ldots, c_{kn})h_1 \cdots h_k h_{i_1} \cdots h_{i_\ell}$$

and the ratios of α to β in these two examples are different. But this can't happen so the classes must be linearly independent.

Of course this proposition has analogues involving three or more Chern numbers evaluated on three or more different bundles associated to different foliations.

We will compute the ratios of the monomials $c_2 c_1^{2n-2}$ and c_1^{2n} for the codimension $2n$ foliations on $\Gamma \backslash SL(2n+1, \mathbf{R})$ and on $\Gamma \backslash SL(n+2, \mathbf{R})$, but, before proceeding further, we digress to analyze the ring structure of $S_{U(n)\times U(k)}/I \approx H^*(U(n+k)/U(n)\times U(k))$.

PROPOSITION 5.11. *The cohomology algebra $S_{U(n)\times U(k)}/I$ is isomorphic to an algebra which is generated as a vector space by n-tuples of integers $(\ell_1, \dots, \ell_n)$ satisfying $0 \le \ell_1 \le \ell_2 \le \cdots \le \ell_n \le k$. The cohomology class associated to $(\ell_1, \dots, \ell_n)$ has dimension $2(\ell_1 + \cdots + \ell_n)$. The class $\bar{e}_i \in S_{U(n)\times U(k)}/I$ is sent to $(0, \dots, 0, i)$ under this isomorphism and we have the following formula for cup products of classes with $\bar{e}_i$:*

$$(0, \dots, 0, i) \cup (\ell_1, \dots, \ell_n) = \sum (\sigma_1, \dots, \sigma_n) \tag{5.12}$$

where the summation is taken over n-tuples satisfying $\dim(\sigma_1, \dots, \sigma_n) = \dim(\ell_1, \dots, \ell_n) + 2i$, $\ell_j \le \sigma_j \le \ell_{j+1}$ *for* $j = 1, \dots n-1$, $\ell_n \le \sigma_n \le k$.

Proposition 5.11 is proven in [HP], Volume II, but it is stated there in terms of homology classes and intersection cycles instead of cohomology and cup products. Formula (5.12) is referred to there as Pieri's formula. The fact that $\bar{e}_i$ corresponds to the class $(0, \dots, 0, i)$ is shown in [C]. We caution that these two references use different notation conventions.

Now let $k = 2$ and write

$$\xi_{ij} = (0, \dots, 0, \underbrace{1, \dots, 1, \underbrace{2, \dots, 2}_{j \text{ entries}}}_{i \text{ entries}})$$

Then we can write .

$$\bar{e}_1^t = \sum_{i+j=t, i \ge j} \alpha_{i,j} \xi_{i,j}$$

where $\alpha_{i,j} \in \mathbf{Z}$.

LEMMA 5.13.

$$\alpha_{i,j}=\binom{i+j}{i}-\binom{i+j}{i+1}.$$

Proof: From formula (5.12)

$$\xi_{i,j}\cup\bar{e}_i=\xi_{i+1,j}+(1-\delta_{ij})\xi_{i,j+1}$$

where δ_{ij} is the Kronecker delta. From this we get

$$\alpha_{i,j}=\alpha_{i-1,j}+\alpha_{i,j-1}$$

$\left(\text{Note that } \alpha_{i-1,i}=\binom{2i-1}{i-1}-\binom{2i-1}{i}=0\right)$. The lemma is easily proven for arbitrary i and $j=0$ or $j=1$ $\left(\text{for } j=0, \text{ use the convention } \binom{i}{i+1}=0.\right)$ Proceeding inductively, assume the lemma proven for all $i<k$ and $j\le i$ and all $i=k$ and $j<\ell\le k$. Then

$$\alpha_{k,\ell}=\alpha_{k-1,\ell}+\alpha_{k,\ell-1},$$

so

$$\begin{aligned}\alpha_{k,\ell}&=\binom{k+\ell-1}{k-1}-\binom{k+\ell-1}{k}+\binom{k+\ell-1}{k}-\binom{k+\ell-1}{k+1}\\&=\binom{k+\ell-1}{k-1}-\binom{k+\ell-1}{k+1}=\frac{(k+\ell)!(k-\ell+1)!}{\ell!(k+1)!}\\&=\binom{k+\ell}{k}-\binom{k+\ell}{k+1}\quad \text{Q.E.D.}\end{aligned}$$

From Proposition 5.11 it is not difficult to deduce that in $S_{U(2)\times U(n)}/I$

$$\bar{e}_2\bar{e}_1^{2n-2}=\alpha_{n-1,n-1}\xi_{n,n}\quad\text{and}\quad \bar{e}_1^{2n}=\alpha_{n,n}\xi_{n,n}.$$

From Lemma 5.13, it follows that

$$\bar{e}_2\bar{e}_1^{2n-2}=\frac{n+1}{4n-2}\,\bar{e}_1^{2n}=\frac{\alpha_{n-1,n-1}}{\alpha_{n,n}}\,\bar{e}_1^{2n}. \tag{5.14}$$

We now return to the computation of the monomials $c_2c_1^{kn-2}$ and c_1^{kn}. Recall that, if $\bar{d}_i$ is the i^{th} symmetric polynomial in $b_1, \ldots, b_n$, and $\bar{e}_i$ is the i^{th} symmetric polynomial in $a_1, \ldots, a_k$, then $\bar{\sigma}_1(c_i)$ is the i^{th} symmetric polynomial in $\{b_i - a_j \mid 1 \leq i \leq n,\ 1 \leq j \leq k\}$. Computation then yields that

$$\bar{\sigma}_1(c_1) = k\bar{d}_1 - n\bar{e}_1$$

$$\bar{\sigma}_1(c_2) = n\bar{e}_2 + k\bar{d}_2 + \binom{n}{2}\bar{e}_1^2 + \binom{k}{2}\bar{d}_1^2 - (kn-1)\bar{e}_1\bar{d}_1.$$

Since $\bar{d}_1 + \bar{e}_1 \equiv 0 \bmod I$ and $\bar{d}_2 + \bar{e}_2 + \bar{e}_1\bar{d}_1 \equiv 0 \bmod I$ we get

$$\bar{\sigma}_1(c_1) \equiv -(n+k)\bar{e}_1 \bmod I$$

$$\bar{\sigma}_1(c_2) \equiv (n-k)\bar{e}_2 + \left[\binom{n+k}{2} + k - 1\right]\bar{e}_1^2 \bmod I.$$

We want to show that the ratios of $\bar{\sigma}_1(c_1^{2n})$ and $\bar{\sigma}_1(c_2c_1^{2n-2})$ are different in $S_{U(n)\times U(2)}/I$ and $S_{U(2n)\times U(1)}/I$. In $S_{U(2n)\times U(1)}/I$ we have

$$\bar{\sigma}_1(c_1^{2n}) = (2n+1)^{2n}\bar{e}_1^{2n}$$

$$\bar{\sigma}_1(c_2c_1^{2n-2}) = (2n+1)^{2n-2}\binom{2n+1}{2}\bar{e}_1^{2n}$$

(since in this case $\bar{e}_2 = 0$). In $S_{U(n)\times U(2)}/I$ we have (using 5.14)

$$\bar{\sigma}_1(c_1^{2n}) = (n+2)^{2n}\bar{e}_1^{2n}$$

$$\bar{\sigma}_1(c_2c_1^{2n-2}) = (n+2)^{2n-2}\left(\left[\binom{n+2}{2} + 1\right] + (n-2)\frac{\alpha_{n-1,n-1}}{\alpha_{n,n}}\right)\bar{e}_1^{2n}.$$

It is tedious but not difficult to show that the ratios $\bar{\sigma}_1(c_2c_1^{2n-2})/\bar{\sigma}_1(c_1^{2n})$ are equal in these two cases if and only if $n = 1$. As a corollary to Proposition 5.10 we get

COROLLARY 5.15. *For $r \geq 2$, $n > 2$ in $H^*(F\Gamma_{2n}^r, \mathbf{R})$ the set of classes*

$$\left\{ \begin{matrix} c_1^{2n}h_1h_2h_{i_1}\cdots h_{i_\ell} \\ c_2c_1^{2n-2}h_1h_2h_{i_1}\cdots h_{i_\ell} \end{matrix} \;\middle|\; 2 < i_1 < \cdots < i_\ell \leq n \right\}$$

is linearly independent.

EXAMPLE 1(b). We now do the same calculations done in Example 1(a), but we consider the foliations on $\Gamma\backslash SL(n+k,\mathbf{R})/SO(n)\times SO(k)$. This will give linear independence relations for classes in $H^*(WO_n,\mathbf{R})$.

Note that there is a commutative diagram:

$$\begin{array}{ccc} H^*(B\Gamma_n^r,\mathbf{R}) & \leftarrow & H^*(WO_n,\mathbf{R}) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \eta^*} \\ H^*(F\Gamma_n^r,\mathbf{R}) & \leftarrow & H^*(W_n,\mathbf{R}) \end{array}$$

where i^* is induced by inclusion $F\Gamma_n^r\subset B\Gamma_n^r$ and $\eta: WO_n\to W_n$ is also the natural inclusion. The horizontal maps are the characteristic maps for the classifying spaces. Thus linear independence in $H^*(F\Gamma_n^r,\mathbf{R})$ of classes in the image of η^* proves linear independence of these classes in $H^*(B\Gamma_n^r,\mathbf{R})$. For this reason, using Example 1(a) when $k=1$, we automatically obtain

THEOREM 5.16. (*see* [KT1] *and* [KT2].) *Let $q(c_1,\ldots,c_n)$ be a monomial of degree $2i$. Then for fixed $n-i+1$ odd and all sequences of odd numbers $i_1,\ldots,i_\ell$ with $n-i+1<i_1<\cdots<i_\ell\le n$ the classes $q(c_1,\ldots,c_n)h_{n-i+1}h_{i_1}\cdots h_{i_\ell}$ are linearly independent in $H^*(B\Gamma_n^r,\mathbf{R})$.*

In particular for dimension $q(c_1,\ldots,c_n)=2n$ these classes are of the form $q(c_1,\ldots,c_n)h_1h_{i_1}\cdots h_{i_\ell}$.

However, when $k>1$ and dimension $q(c_1,\ldots,c_{kn})=2kn$ the non-vanishing classes are all of the form

$$q(c_1,\ldots,c_{kn})h_1h_2\cdots h_kh_{i_1}\cdots h_{i_\ell}.$$

These classes are all divisible by h_2 and so they are not in the image of η^*. Thus we get no information about $H^*(WO_{kn},\mathbf{R})$ from these examples. We reduce by $SO(n)\times SO(k)$ to rectify this situation.

THEOREM 5.17. *Suppose $k\le n$ and consider the characteristic map $\Phi: H^*(WO_{kn},\mathbf{R})\to H^*(\Gamma\backslash SL(n+k,\mathbf{R})/SO(n)\times SO(k),\mathbf{R})$ induced by the foliation determined by the parabolic Lie algebra*

$$\mathcal{P}=\{\|a_{ij}\|\in \mathfrak{sl}(n+k)\mid a_{ij}=0,\ i>k,\ j\le k\}.$$

Then the classes $\Phi(c_1^{kn}(\prod_{i\,\mathrm{odd}\le k}h_i)h_{i_1}\cdots h_{i_\ell}$ where $k<i_1<\cdots<i_\ell\le n$ and $i_1,\ldots,i_\ell$ are odd, are linearly independent.

Remark. Note that $k=n$ is now allowed. This is because the cancellation in

Equation 5.6 which occurs when $k=n$ happens only when dealing with h_i for i even.

Proof. $\bar{A}=S_{U(k)\times U(n)}/I\otimes S_{SO(k)\times SO(n)}\otimes H^*(U(k)\times U(n))$. Let $\bar{V}_i$ (resp. $\bar{U}_i$) denote the i^{th} Pontrjagin class in $SO(k)$ (resp. $SO(n)$). $H^*(\bar{A},\mathbf{C})\approx H^*(U(n+k)/SO(n)\times SO(k),\mathbf{C})\approx E\otimes P$. Here E is an exterior algebra on the suspensions of the odd Chern classes in $U(n+k)$, and in $\bar{A}$ generators are of the form

$$g_{2i-1}=\sum_{\substack{s+t=2i-1\\ s\text{ odd}}}\bar{e}_t\otimes 1\otimes d_s+\sum_{\substack{s+t=2i-1\\ t\text{ odd}}}\bar{d}_s\otimes 1\otimes e_t$$

where $2i-1\leq n+k$. P is the algebra $\mathbf{C}[\bar{U}_i,\bar{V}_j,\bar{X}_k,\bar{X}_n]/J$ where $\bar{X}_k$ (resp. $\bar{X}_n$) is the Euler class in $S_{SO(k)}$ (resp. $S_{SO(n)}$) if k (resp. n) is even, and $\bar{X}_k=O$ (resp. $\bar{X}_n=O$) otherwise. J is generated by the relations $(\sum\bar{U}_i)(\sum\bar{V}_i)=1$ and $\bar{X}_k^2=\bar{V}_{k/2}$ (for k even), $\bar{X}_n^2=\bar{U}_{n/2}$ (for n even).

The proof now proceeds along lines similar to the proof of Theorem 5.3. We multiply the class $\lambda^*(c_1^{kn}(\prod_{i\text{ odd}\leq k}h_i)h_{i_1}\cdots h_{i_\ell})$ by the class $\prod_{n\geq i\text{ odd}\neq i_1,\ldots,i_\ell}g_i$ to obtain a non-zero multiple of the top dimensional class $V\otimes(\prod_{i\text{ odd}\leq k}e_i)(\prod_{j\text{ odd}\leq n}d_j)$ in $\bar{A}$. (When n or k is even, we must also multiply by the Euler classes $\bar{X}_n$ or $\bar{X}_k$ to get a top dimenisonal class.) At any rate since $\lambda^*(c_1^{nk}(\prod_{k\geq i\text{ odd}}h_i)h_{i_1}\cdots h_{i_\ell})$ cups with a non-zero class to give a non-zero multiple of the top dimensional class, it must also be non-zero. We leave the rest of the details to the reader.

An analogue to Proposition 5.10 and the computations done in 1(a) yield:

COROLLARY 5.19. *In $H^*(B\Gamma_{2n}^r,\mathbf{R})$, $r\geq 2$ the set of classes*

$$\left\{\begin{matrix} c_1^{2n}h_1h_{i_1}\cdots h_{i_\ell}\\ c_2c_1^{2n-2}h_1h_{i_1}\cdots h_{i_\ell}\end{matrix}\;\middle|\;\begin{matrix}1<i_1<\cdots<i_\ell\leq n\\ i_1,\ldots,i_\ell\text{ odd}\end{matrix}\right\}$$

is linearly independent.

In Examples 2 through 4 the techniques for calculations are the same as in Examples 1(a) and (b). We state the results but omit the details.

EXAMPLE 2. The Dynkin diagram for $Sp(n,\mathbf{C})$ is $\circ\!-\!\!-\!\!-\circ\!-\!\!-\!\!-\cdots-\!\!-\!\!-\circ\Leftarrow\circ$ with n vertices. By removing the vertex with a double bond at the end, we obtain a parabolic with $\mathcal{G}_1^{\mathbf{C}} \oplus \mathcal{T}_1^{\mathbf{C}} \approx \mathfrak{gl}(n, \mathbf{C})$. In

$$\mathfrak{sp}(n, \mathbf{R}) = \left\{ \left[\begin{array}{c|c} -A^t & S_2 \\ \hline S_1 & A \end{array} \right] \right\}$$

where $A \in \mathfrak{gl}(n, \mathbf{R})$ and S_1, S_2 are symmetric $n \times n$ matrices,

$$\mathcal{P} = \{X \in \mathfrak{sp}(n, \mathbf{R}) \mid S_1 = 0\}$$
$$\mathcal{G}_1 \oplus \mathcal{T}_1 = \{X \in \mathfrak{sp}(n, \mathbf{R}) \mid S_1 = S_2 = 0\}$$

and

$$\mathcal{N} = \{X \in \mathfrak{sp}(n, \mathbf{R}) \mid S_1 = A = 0\}.$$

The representation σ is the usual representation of $GL(n, \mathbf{C})$ on symmetric 2-tensors on $\mathbf{C}^n$.

THEOREM 5.20. *In $H^*(\Gamma \backslash Sp(n, \mathbf{R}), \mathbf{R})$ the classes*

$$\Phi(c_1^{n(n+1)/2}(\textstyle\prod_{n \geq i \text{ odd}} h_i) h_{2i_1} \cdots h_{2i_\ell} \text{ for } 2 \leq 2i_1 < \cdots < 2i_\ell \leq n$$

are linearly independent for the codimension $n(n+1)/2$ foliation determined by $\mathcal{P}$.

To obtain linear independence relations as the monomial $q(c_1, \ldots, c_{n(n+1)/2})$ varies, one can use an analogue to Propositions 5.10. For example,

PROPOSITION 5.21. (i) *In $H^*(B\Gamma^r_{n(n+1)/2}, \mathbf{R})$, $r \geq 2$, $n \geq 3$, the classes $c_1^{n(n+1)/2} \prod_{n \geq i \text{ odd}} h_i$ and $c_2^2 c_1^{[n(n+1)/2]-4} \prod_{n \geq i \text{ odd}} h_i$ are linearly independent.*

(ii) *In $H^*(F\Gamma^r_{n(n+1)/2}, \mathbf{R})$, $r \geq 2$, $n \geq 3$, the set of classes*

$$\left\{ \begin{array}{l|c} c_1^{n(n+1)/2} \left(\displaystyle\prod_{n \geq i \text{ odd}} h_i \right) h_{2i_1} \cdots h_{2i_\ell} & \\ & 1 \leq i_1 < \cdots < i_\ell \leq n/2 \\ c_2^2 c_1^{[n(n+1)/2]-4} \left(\displaystyle\prod_{n \geq i \text{ odd}} h_i \right) h_{2i_1} \cdots h_{2i_\ell} & \end{array} \right\}$$

are linearly independent.

Proof. One compares the ratios of Chern numbers $c_2^2 c_1^{[n(n+1)/2]-4}$ and $c_1^{n(n+1)/2}$ for the codimension $n(n+1)/2$ foliations on $\Gamma \backslash Sp(n, \mathbf{R})$ and on $\Gamma \backslash SL(n(n+1)/2 + 1, \mathbf{R})$. These ratios are always different and the proof proceeds as in Proposition 5.10.

EXAMPLE 3. Consider $SO(2n, \mathbf{C})$ as all non-singular complex $n \times n$ matrices which preserve the bilinear form

$$\left[\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right]$$

Then

$$\mathfrak{so}(2n, \mathbf{C}) = \left\{ \left| \begin{array}{c|c} -A^t & B_2 \\ \hline B_1 & A \end{array} \right| \;\middle|\; A \in \mathfrak{gl}(n, \mathbf{C}) \text{ and } B_i \text{ are skew symmetric} \right\}.$$

The Dynkin diagram for $SO(2n, \mathbf{C})$ is $\circ\!-\!\!-\!\circ\!-\!\!-\cdots-\!\!-\!\circ\!<^{\circ}_{\circ}$ .By removing a vertex at the fork, we get the parabolic $P^{\mathbf{C}}$ with Lie algebra

$$\begin{aligned} \mathcal{P}^{\mathbf{C}} &= \{X \in \mathfrak{so}(2n, \mathbf{C}) \mid B_1 = 0\} \\ \mathcal{G}_1^{\mathbf{C}} \oplus \mathcal{T}_1^{\mathbf{C}} &= \{X \in \mathfrak{so}(2n, \mathbf{C}) \mid B_1 = B_2 = 0\} \approx \mathfrak{gl}(n, \mathbf{C}) \\ \mathcal{N}^{-\mathbf{C}} &= \{X \in \mathfrak{so}(2n, \mathbf{C}) \mid A = B_2 = 0\}. \end{aligned}$$

For a real form we use

$$\mathfrak{so}(n, n) = \left\{ \left[\begin{array}{c|c} -A^t & B_2 \\ \hline B_1 & A \end{array} \right] \;\middle|\; A \in \mathfrak{gl}(n, \mathbf{R}) \text{ and } B_i \text{ are real skew symmetric} \right\}.$$

THEOREM 5.22. *Suppose n is not a power of 2. Then the set of classes*

$$\left\{ \begin{array}{l} \Phi\left(c_1^{n(n-1)/2}\left(\prod_{n-1 \geq i \text{ odd}} h_i\right) h_{2i_1} \cdots h_{2i_\ell}\right) \\ \\ \Phi\left(c_1^{n(n-1)/2}\left(\prod_{n-1 \geq i \text{ odd}} h_i\right) h_n h_{2i_1} \cdots h_{2i_\ell}\right) \end{array} \;\middle|\; 1 \leq i_1 < \cdots < i_\ell \leq \frac{n-1}{2} \right\}$$

are linearly independent in $H^*(\Gamma \backslash SO(n, n), \mathbf{R})$ for the codimension $n(n-1)/2$ foliation determined by $\mathcal{P}$.

EXAMPLE 4. Let $SO(2n+1, \mathbf{C})$ be all non-singular complex matrices preserving the form

$$\left[\begin{array}{cc|c} \begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array} & & \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \\ \hline 0 \cdots 0 & & 1 \end{array}\right]$$

Then

$$\mathfrak{so}(2n+1,\mathbf{C}) = \left\{ \left[\begin{array}{c|c|c} -A^t & B_2 & a \\ \hline B_1 & A & b \\ \hline -b & -a & 0 \end{array}\right] \middle| \begin{array}{l} A \in \mathfrak{gl}(n,\mathbf{C}) \\ B_i \text{ are skew symmetric} \\ \text{and } a,\, b \in \mathbf{C}^n \end{array} \right\}.$$

The Dynkin diagram for $SO(2n+1,\mathbb{C})$, is $\circ - \circ - \ldots - \circ \Rightarrow \circ$. By removing the end vertex with the double bond, we get

$$\begin{aligned} \mathscr{P}^{\mathbf{C}} &= \{X \in \mathfrak{so}(2n+1,\mathbf{C}) \mid B_1 = 0,\, b = 0\} \\ \mathscr{G}_1^{\mathbf{C}} \oplus \mathscr{T}_1^{\mathbf{C}} &= \{X \in \mathfrak{so}(2n+1,\mathbf{C}) \mid B_1 = B_2 = 0,\, a = b = 0\} \\ \mathscr{N}^{-\mathbf{C}} &= \{X \in \mathfrak{so}(2n+1,\mathbf{C}) \mid A = B_2 = 0,\, a = 0\}. \end{aligned}$$

For a real form we use $SO(n+1, n)$,

$$\mathfrak{so}(n+1,n) = \left\{ \left[\begin{array}{c|c|c} -A^t & B_2 & a \\ \hline B_1 & A & b \\ \hline -b & -a & 0 \end{array}\right] \middle| \text{all entries are real} \right\}$$

THEOREM 5.23. *Suppose $n+1$ is not a power of 2. Then the classes $\Phi(c_1^{n(n+1)/2}(\prod_{n \ge i \text{ odd}} h_i) h_{2i_\ell} \cdots h_{2i_\ell})$ where $1 \le i_1 < \cdots < i_\ell \le n/2$ are linearly independent in $H^*(\Gamma \backslash SO(n, n+1), \mathbf{R})$.*

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