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Pseudoconcave homogeneous surfaces

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0. Introduction

The purpose of this note is to list all pseudoconcave, 2-dimensional, homogeneous complex manifolds (*pseudoconcave homogeneous surfaces*). One would expect such manifolds to have compactifications as almost homogeneous surfaces. This turns out to be the case, but one can not proceed directly, because there *are* pseudoconcave surfaces which are not compactifiable [6].

It was noted in [5] that the only pseudoconcave Lie groups are the compact ones. With one type of exception this is also the case for pseudoconcave homogeneous surfaces: *Other than the compact homogeneous surfaces, the only pseudoconcave homogeneous surfaces are the Hirzebruch surfaces with their exceptional divisors removed.*

1. Historical remarks

For our purposes a homogeneous complex manifold is the quotient space of a connected, *complex* Lie group G by a closed subgroup H . Homogeneous algebraic surfaces were of course studied by the Italians (e.g. see [2]). The simply-connected, compact (not necessarily algebraic) homogeneous surfaces were classified by Wang [24]. Later on Tits [23] classified all compact homogeneous surfaces. Using Kodaira's work and going through the cases of transcendence degree of the function field as well as the possibilities for an Albanese fibration, Potters [20] gave a complete list of almost homogeneous compact surfaces. *Almost homogeneous* means that a connected, complex Lie group of automorphisms has an open orbit. In the case of compact Kähler manifolds there is a very good classification by Borel and Remmert [9]: Every compact homogeneous

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Kähler manifold is the product of a homogeneous projective rational manifold with a complex torus.

The purpose of the above remarks is to point out that *quite* a lot is known in the *compact* case. The non-compact case appears to be more complicated. Nevertheless in the “Stein direction” there are a number of interesting results. For example, when G is either reductive or nilpotent, Matsushima ([16], [17]) gave very good descriptions of the Stein quotients G/H . More generally, if G is reductive and G/H is holomorphically separable, then Barth and Otte [7] point out that the homogeneous space G/H is a Zariski open subset of an affine algebraic variety.

It is easy to see (e.g. [10]) that given a homogeneous space G/H there is a complex Lie group J containing H so that the fibration $G/H \rightarrow G/J$ (the “separation” map) identifies *exactly* the points which the holomorphic functions identify (i.e. $p, q \in G/H, p \sim q \Leftrightarrow f(p) = f(q) \forall f \in \mathcal{O}(G/H)$). Hence $\mathcal{O}(G/J)$ separates points on G/J . If for example G is reductive, then the Barth–Otte theorem tells us that G/J is Zariski open in an affine algebraic variety. It would be reasonable to hope that $\mathcal{O}(G/J) \simeq \mathbf{C}$. This is in fact the case when G/J is Stein [10] or when G/H is itself a complex Lie group [18]. However, for example, one can construct homogeneous spaces which are simply \mathbf{C}^* -bundle spaces over $\mathbf{C}^2 \setminus \{0, 0\}$ such that the holomorphic functions live on the base [7]. Even so, it is still quite reasonable to study homogeneous spaces G/H with $\mathcal{O}(G/H) \simeq \mathbf{C}$ as well as the other extreme where $\mathcal{O}(G/H)$ separates points. In some special cases (e.g. when G is nilpotent) one can classify G/H when $\mathcal{O}(G/H) \simeq \mathbf{C}$ (see [11] and [8]). But even for solvable G we do not know what these manifolds are. Thus the stronger curvature assumption of *pseudoconcavity* seems warranted.

DEFINITION. A complex manifold X is called *pseudoconcave* if it contains a relatively compact open subset W with ∂W defined⁽³⁾ by a C^2 -function φ whose Levi form restricted to the complex tangent plane of ∂W at any $p \in \partial W$ has at least one negative eigenvalue.

Pseudoconcave manifolds are close relatives of compact manifolds (e.g. $\mathcal{O}(X) \simeq \mathbf{C}$, and certain cohomology groups are finite dimensional). The reader can find the basic properties derived in [1]. Pseudoconcave quotient spaces arise quite naturally. For example, if the isotropy subgroup of a point p in a compact complex space X acts transitively on $X \setminus \{p\}$, then $X \setminus \{p\}$ is a (strongly) pseudoconcave homogeneous space. Andreotti and Grauert [3] observed that the Siegel upper half plane modulo the modular group is pseudoconcave. More generally,

³ “Defined” means that W corresponds to $\{\varphi < 0\}$, $\partial W = \{\varphi = 0\}$ and $d\varphi \neq 0$ on W .

any bounded symmetric domain modulo an arithmetic subgroup is pseudoconcave [20]. Applying finiteness and transcendence degree theorems ([1], [4]), these results have numerous interesting ramifications.

2. Statement of the result and a quick sketch of the proof

Our main result is the following:

THEOREM. *Let X be a pseudoconcave, 2-dimensional, complex manifold which admits a transitive complex Lie group of holomorphic automorphisms (i.e. $X = G/H$, where H is a closed subgroup of G). Then X is one of the following:*

- (1) *A compact 2-dimensional torus;*
- (2) *The product of \mathbf{P}_1 with an elliptic curve;*
- (3) *A homogeneous Hopf surface;*
- (4) *A homogeneous rational manifold (i.e. $\mathbf{P}_1 \times \mathbf{P}_1$ or \mathbf{P}_2);*
- (5) *A Hirzebruch surface Σ_n $n = 1, 2, \dots$, with its exceptional divisor⁽⁴⁾ removed.*

The reader should note that the only non-compact possibilities occur in (5). Cases (1), (2) and (4) are well-known. The homogeneous Hopf surfaces are formed by dividing $\mathbf{C}^2 \setminus \{0, 0\}$ by a discrete finitely generated abelian group of linear transformations Γ . The group Γ contains one copy of \mathbf{Z} which is generated by a dilation $(z_1, z_2) \rightarrow (\alpha z_1, \alpha z_2)$, where $|\alpha| > 1$. The torsion part is generated by diagonal matrices whose entries are roots of unity. These facts are routinely derivable from the explicit remarks of Kodaira ([14], p. 694 ff).

Our method of proof is to use the adjoint action of G on a Grassmann manifold. This identifies $G/N_G(H^0)$ with the orbit of G on the point corresponding to the Lie algebra \mathfrak{h} of H sitting as a subspace of the Lie algebra \mathfrak{g} of G . This combined with the Albanese fibration was the main tool of Borel and Remmert [9]! Letting $N := N_G(H^0)$, this fibration $G/H \rightarrow G/N$ allows us to study G/H in three separate cases depending on the dimension of G/N .

If G/H is 0-dimensional, then X is isomorphic to a complex Lie group S modulo a properly discontinuous subgroup Γ . Since in this case S must be 2-dimensional, it is quite easy to write down all possible such quotients, and, using

⁴Every Σ_n contains a unique exceptional rational curve T_n with $T_n \cdot T_n = -n$. We note that $\Sigma_1 \setminus T_1$ is the same as $\mathbf{P}_2 \setminus \{\text{point}\}$, so we redefine Σ_1 to be \mathbf{P}_2 .

the remark of [5], to show that the only possibility for a pseudoconcave S/Γ is a complex torus.

If G/N is 1-dimensional, then the fiber $N/H^0/H/H^0$ is either \mathbf{C} , \mathbf{C}^* , or an elliptic curve. The base G/N must be \mathbf{P}_1 , because N contains the radical of G . If in this case G/H is a torus bundle over \mathbf{P}_1 , then it is either a product or a Hopf surface. The \mathbf{C}^* -bundle is eliminated by the pseudoconcavity, and a \mathbf{C} -bundle compactifies to a Hirzebruch surface by adding an exceptional⁽⁵⁾ rational curve at infinity.

If G/N is 2-dimensional, then G/H is a covering space over the pseudoconcave G/N . By using Andreotti's function field theorem, we compactify G/N to an algebraic variety V in the Grassmann manifold. We then extend the action of G to the minimal desingularization \tilde{V} of V . For pseudoconcavity reasons it is again easy to show that \tilde{V} is a Hirzebruch surface. In particular we see that G/N is simply connected! So $G/H = G/N$ and the classification is finished.

3. Details of the proof

We follow the outline given above, introducing preparatory material as we need it. The basic tool is the action of G on the Grassmann ([23], [9]): Let $\text{ad}: G \rightarrow \text{Hom}(\mathfrak{g}, \mathfrak{g})$ be the adjoint representation. For a closed subgroup H we consider its Lie algebra \mathfrak{h} to be a point in the Grassmann manifold $G_{k,n}$ of k -planes in n -space, where $k := \dim_{\mathbf{C}} H$ and $n := \dim_{\mathbf{C}} G$. For $g \in G$ it is clear that $\text{ad}(g)$ acts on $G_{k,n}$. We consider the map $g \rightarrow \text{ad}(g)(\mathfrak{h})$ which sends G to its orbit on \mathfrak{h} . Let $N := N_G(H^0)$ be the normalizer of the connected component H^0 in G , then N is also a closed subgroup of G with G/N being canonically identified with this orbit. We have the fibration

$$G/H \xrightarrow{\alpha} G/N \rightarrow G_{k,n},$$

where the fiber $N/H^0/H/H^0$ of α is a complex Lie group modulo a discrete subgroup.

Case 1. Suppose $\dim_{\mathbf{C}} G/N = 0$. Then X can be realized as a simply-connected, complex Lie group S modulo a discrete subgroup Γ . If S is abelian,

⁵ We use the word exceptional for a curve which can be blown down, but the quotient may be singular. Differing slightly from [13], we identify Σ_1 with \mathbf{P}_2 .

then S/Γ is again an abelian Lie group which, since it is pseudoconcave, must be a torus [5]. If S is non-abelian, then it has \mathbf{C}^2 as its underlying manifold with group structure given by

$$(a_1, b_1)(a_2, b_2) = (a_1 + a_2, e^{a_1}b_2 + b_1).$$

PROPOSITION. *Let S be a 2-dimensional, complex Lie group and let Γ be a discrete subgroup of S . Then S/Γ can be realized as an abelian Lie group modulo a finite subgroup.*

Proof. This is clear if S itself is abelian. Thus we may assume that S is \mathbf{C}^2 with the above non-abelian structure. Let $T := \{(2\pi in, b) \in \Gamma \mid n \in \mathbf{Z}\}$. Then T is a normal subgroup of Γ which acts on the left as a group of translations. An elementary calculation shows that the derived group $\Gamma^{(1)}$ is contained in $T_2 := \{(0, b) \in \Gamma\}$ which acts discontinuously as a group of translations of the second variable. If there exists a non-trivial $h = (0, c) \in T_2$, then for $g := (a, b) \in \Gamma$ we see that

$$g^{-n}hg^n = (0, e^{-na}c). \tag{*}$$

It is also of use to note that

$$g^n = (na, b(1 - e^{na})(1 - e^a)^{-1}). \tag{**}$$

If Γ is abelian, then we use a simple affine transformation (see [22]) to facilitate a description of S/Γ : Let $g := (a, b) \in \Gamma$ and suppose that $e^a \neq 1$. Thus we may make the change of variables $(z_1, z_2) \rightarrow (z_1, z_2 - b(1 - e^a)^{-1})$ so that in this new system (using the old letters) $g(z_1, z_2) = (z_1 + a, e^a z_2)$. If h is an arbitrary element of Γ , then $h(z_1, z_2) = (z_1 + c, e^c z_2 + t(h))$, where $h = (c, d)$ in the old system and $t(h) = d + b(1 - e^c)(1 - e^a)^{-1}$. The statement that g and h commute amounts to $(e^a - 1)t(h) = 0$. Thus $t(h) = 0$ for all $h \in \Gamma$. Hence the general element $h \in \Gamma$ is defined by $h(z_1, z_2) = (z_1 + c, e^c z_2)$. The restriction of Γ to the z_1 -axis is a faithful representation. Thus Γ is free abelian having rank at most 2.

Let $q: \mathbf{C}^2 \rightarrow S/\Gamma$ be the quotient map. If Γ has rank 2, then $q\{z_2 = 0\}$ is an elliptic curve E . But $h(E)$ is also an elliptic curve for all $h \in S$. If $h(z_1, z_2) = (z_1 + c, e^c z_2 + t(h))$, then $h\{z_2 = 0\} = \{(z_1, t(h)) \mid z_1 \in \mathbf{C}\}$. Now $q(z_1, t(h)) = q(z'_1, t(h))$ iff there is an $f \in \Gamma$ such that $f(z_1, t(h)) = (z'_1, t(h))$ (i.e. $(z_1 + c, e^c t(h)) = (z'_1, t(h))$). In particular, for $h(E) = qh\{z_2 = 0\}$ to be a torus it would be necessary for Γ to contain a rank 2 subgroup with $e^c = 1$ for all h in that group. This is of course impossible. Even if $\text{rank } \Gamma = 1$, the same argument shows that Γ would have to contain a rank 1 subgroup $\Gamma^\#$ whose elements act on \mathbf{C}^2 by

$(z_1, z_2) \rightarrow (z_1 + 2\pi in, z_2)$, $n \in \mathbb{Z}$, and such that Γ/Γ^* is finite. Thus S/Γ is $\mathbb{C}^* \times \mathbb{C}$ modulo a finite abelian group.

In summary we have shown above that if Γ is abelian then either S/Γ is $\mathbb{C}^* \times \mathbb{C}$ modulo a finite group or $e^a = 1$ for all $g = (a, b) \in \Gamma$. In the latter case $\Gamma = T$, Γ acts as a lattice on \mathbb{C}^2 , and S/Γ is obviously realizable as a complex Lie group,

It remains to consider the non-abelian case. In this situation we know that T_2 contains a non-trivial element $(0, c)$. Thus if there exists some $g = (a, b) \in \Gamma$ with a not a root of unity, then (*) implies that $g^n h g^{-n}$ (or $g^{-n} h g^n$) has a cluster point. Since Γ is a discrete group, this is impossible and consequently for every $g = (a, b) \in \Gamma$ we know that $a = 2\pi iq$ for some $q \in \mathbb{Q}$. In particular this says that Γ/T is a torsion group. Considering two separate cases, we show that Γ/T is finite.

First, assume that $\text{rank } T_2 = 1$. So there exists $c \in \mathbb{C}$ such that $T_2 = \{(0, nc) \mid n \in \mathbb{Z}\}$. Take $h = (0, c) \in T_2$ and $g = (2\pi iq, b) \in \Gamma$. Then

$$ghg^{-1}h^{-1} = (0, c(e^{2\pi iq} - 1)) \in T_2.$$

This is only possible if q is an integer or an integer plus $\frac{1}{2}$. Therefore Γ/T is certainly finite in this case.

If $\text{rank } T_2 = 2$, then we note that given a class in Γ/T we can pick a representative $g = (2\pi iq, b)$ where $|b| < M$ and M is some a priori bound determined by Γ . Furthermore if $Nq = m \in \mathbb{Z}$, then $g^N = (2\pi im, 0) \in T$. Hence we may additionally pick a representative $h = (2\pi ir, b)$ with $|r| < |m|$. If Γ/T were infinite then we could therefore find infinitely many different elements of Γ in a compact region of \mathbb{C}^2 . This is contrary to the fact that Γ is discrete. Hence Γ/T is finite and S/Γ is the quotient by a finite group of automorphisms of the space \mathbb{C}^2/T which is obviously realizable as a complex Lie group.

COROLLARY. *Let $X = S/\Gamma$ be a homogeneous space where S is a 2-dimensional complex Lie group and Γ is a discrete subgroup. If X is pseudoconcave, then it is a compact torus.*

Proof. By the above proposition, X has an abelian Lie group G as a finite cover. Thus G is also pseudoconcave. But [5] implies that G is a torus, and therefore X is likewise.

Case 2. Suppose $\dim_{\mathbb{C}} G/N = 1$. Then the fact that X has no non-constant holomorphic functions implies that $\mathcal{O}(G/N) \cong \mathbb{C}$ and consequently G/N is a compact Riemann surface. We now apply a theorem of Borel and Remmert ([9], p. 435): The base G/N is rational (i.e. it is \mathbf{P}_1) and N is connected. Consequently X is a bundle space over \mathbf{P}_1 whose fiber is either \mathbb{C} , \mathbb{C}^* or an elliptic curve. The \mathbb{C}^* case can be eliminated rather easily, because such bundles are classified by

their Chern numbers. If the Chern number is $n > 0$ then the bundle has a non-trivial section s , and if z is a fiber coordinate, then sz^{-n} is a well-defined holomorphic function on X . If $n < 0$, then the dual bundle has a section s and sz^n does the job.

The case of an elliptic curve as a fiber is handled classically [23], being a product in the Kähler case and a homogeneous Hopf surface otherwise. It only remains to consider \mathbf{C} -fiber bundles over \mathbf{P}_1 . Call such a bundle space E . We note that E has a natural compactification \bar{E} which comes from adding the section s at infinity. It is clear that the automorphisms of E extend to \bar{E} . Thus E is a compact, rational, almost homogeneous surface which is a \mathbf{P}_1 -bundle over \mathbf{P}_1 . Hence \bar{E} is a Hirzebruch surface, where we blow down the fixed set when its self intersection number is -1 . This finishes *Case 2*.

Case 3. Suppose $\dim_{\mathbf{C}} G/N = 2$. Let W be the relatively compact set in G/H which displays its pseudoconcavity, and let $\Omega := \alpha(W)$, where $\alpha : G/H \rightarrow G/N$ is the fibration map. Certainly $\partial\Omega \subset \alpha(\partial W)$. Thus Ω is pseudoconcave in the following more general sense: Given $p \in \partial\Omega$, there is an open neighborhood U of p in G/N so that every function holomorphic on $U \cap \Omega$ extends to a function holomorphic on U . One finds such a U by taking $q \in \partial W$ with $\alpha(q) = p$. Then one constructs a neighborhood \tilde{U} of q in G/H which is small enough so that $\alpha|_{\tilde{U}}$ is biholomorphic, and such that \tilde{U} has the extendibility property required of U . Finally one defines U as $\alpha(\tilde{U})$. For a normal complex space Y containing a relatively compact open set having this more general pseudoconcave property, Andreotti [1] proved the following:

The field $\mathfrak{R}(Y)$ of meromorphic functions on Y is an algebraic function field having transcendence degree $t(Y)$ at most $\dim Y$ over \mathbf{C} .

It follows that we have Andreotti's transcendence degree theorem for $Y := G/N$. But Y is contained in $G_{k,n}$ which is in turn contained in some \mathbf{P}_N . We now follow a standard proof of Chow's Theorem: Let V be the smallest compact algebraic variety which contains Y (i.e. V is the intersection of all such varieties which contain Y). It is clear that V is irreducible. Suppose $k := \dim_{\mathbf{C}} V > 2$. Then $\mathfrak{R}(V) = \mathbf{C}(f_1, \dots, f_k)[g]$, where f_1, \dots, f_k are algebraically independent rational functions on V and g is some algebraically dependent function of maximal degree. Since $t(Y) = 2$, there exists a polynomial $P \in \mathbf{C}[X_1, \dots, X_k]$ so that $P(f_1, \dots, f_k) \equiv 0$ on Y . But f_1, \dots, f_k are algebraically independent on V . So $V \cap \{P(f_1, \dots, f_k) = 0\}$ is an algebraic subvariety of dimension $k - 1$ which contains Y . This contradicts the minimality of V and implies that $\dim_{\mathbf{C}} V = 2$. Thus *there is an irreducible, 2-dimensional, compact, algebraic subvariety of $G_{k,n}$ which contains G/N as an open subset.*

Now G acts on the whole Grassmann manifold. So for $g \in G$ it follows that $g(V)$ is an algebraic subvariety of $G_{k,n}$. But $g(V) \cap V$ contains Y as an open subset. Consequently $g(V) = V$, and we have extended the action of G to all of V . We now need the following:

LEMMA. *Let g be a biholomorphic map $g: V \rightarrow V$ of a 2-dimensional analytic space. Let $\rho: \tilde{V} \rightarrow V$ be the minimal desingularization of V . Then there is a unique biholomorphic map $\tilde{g}: \tilde{V} \rightarrow \tilde{V}$ so that*

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{g}} & \tilde{V} \\ \rho \downarrow & & \downarrow \rho \\ V & \xrightarrow{g} & V \end{array}$$

is commutative.

Proof. The maps $\rho: \tilde{V} \rightarrow V$ and $g \circ \rho: \tilde{V} \rightarrow V$ are both minimal resolutions of the singularities of V . Thus the lemma follows directly from the uniqueness of a minimal resolution in dimension 2 (e.g. see [15]).

Hence every automorphism g of V is uniquely liftable to an automorphism \tilde{g} of the minimal desingularization \tilde{V} of V . Thus G/N can be compactified to a non-singular algebraic surface \tilde{V} where the group G acts and renders it almost homogeneous.

It is easy to see that an open orbit is the complement of a proper analytic subset (e.g. [21]). Thus $Y = G/N$ is Zariski open in \tilde{V} . Let $S := \tilde{V} \setminus Y$ be the fixed set. The only algebraic, almost homogeneous surfaces \tilde{V} such that $\mathcal{O}(\tilde{V} \setminus S) \simeq \mathbb{C}$ are rational surfaces. One can see this by looking at Potter's classification [21] where the only case which needs checking is that of the \mathbf{P}_1 -bundles over elliptic curves. These are so explicitly described that constructing non-constant holomorphic functions on the complement of the fixed set is a triviality. For example in Type II (see p. 252 of [21]), the function $f([w_1: w_2], z) := \exp(2\pi i w_1 (w_2 b)^{-1})$ is well-defined and holomorphic on the complement of S .

In summary we have shown that G/N compactifies to a Hirzebruch surface Σ_n by adding the fixed rational curve T_n which has self-intersection number $-n$ and (possibly) finitely many more points. The group G , assuming that it is acting effectively, is therefore contained in the stabilizer of the set of the finitely many points which were added.

If $S = \emptyset$, then $V = G/N$ is compact rational and is either $\mathbf{P}_1 \times \mathbf{P}_1$ or \mathbf{P}_2 . Every compact rational almost homogeneous surface can be blown down to some Σ_n . Thus the minimality of \tilde{V} implies that it is a Hirzebruch surface. The only

exceptional curve in Σ_n is T_n . Hence, when it is not empty, S consists of T_n plus isolated points. However, by a very general theorem of Oeljeklaus [19], if S contains an isolated point, then $\tilde{V} = \mathbf{P}_2$.

It is shown in [13] that $\Sigma_{2n} \setminus T_{2n}$ is homeomorphic to $S^2 \times (S^2 \setminus \{p\})$ and $\Sigma_{2n+1} \setminus T_{2n+1}$ is homeomorphic to $\mathbf{P}_2 \setminus \{p\}$. Thus G/N is simply-connected. Hence $G/H = G/N$ and our homogeneous space is in class (4) or (5) of the theorem.

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Added in proof.

The Proposition in Case 1 is false as stated; a corrected version appears in the thesis of J. Snow. The point is to eliminate the parallelizable case S/Γ in the classification. Since S is solvable, this follows immediately from a general theorem of Huckleberry and D. Snow (Three fibrations for pseudoconcave homogeneous manifolds, to appear). However as S is 2-dimensional, one may eliminate this case directly. Since $\mathcal{O}(S/\Gamma) = \mathbf{C}$, there is a proper normal subgroup L of S with closed orbits in S/Γ , yielding the homogeneous fibration $S/\Gamma \rightarrow S/L \cdot \Gamma = T$ [8, Satz 3.2]. Clearly T is a torus. One checks quite easily (see later in the paper) that there are no pseudoconcave homogeneous \mathbf{C} or \mathbf{C}^* bundles over an elliptic curve. Thus S/Γ must be compact and a fortiori S is abelian.