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Ideals generated by minors of a symmetric matrix

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§0. Introduction

Let X be an n by n symmetric matrix with entries in a commutative Noetherian ring R with identity. R. Kutz investigated, in [11], ideals $I_p(X)$ generated by all the p by p minors of X. His main results states:

$$\text{depth } I_p(X) \leq \nu(p, n) := \frac{(n-p+1)(n-p+2)}{2}$$

and in case of equality the ideal $I_p(X)$ is perfect, i.e. depth $I_p(X) = pdR/I_p(X)$. Kutz used in his proof a technique applied for the first time by Hochster and Eagon in [10] to determinantal ideals associated with an arbitrary matrix.

In §2 of the present paper we extend some results of Kutz concerning the depth of $I_p(X)$ and prove that the height of $I_p(X)$ is also bounded by $\nu(p, n)$.

In §3 we construct a free complex, $\mathbf{L}(X)$, of length 3 which gives a free resolution of $I_{n-1}(X)$ when depth $I_{n-1}(X) = 3$.

All the proofs in §§2, 3 depend heavily on a lemma stated in §1 which contains in particular the structure theorem for non-singular quadratic forms over a local ring (see [12], Lemme 2).

In §4 we utilize the complex $\mathbf{L}(X)$ to describe the relation between the Poincaré series of local rings R and $R/I_{n-1}(X)$ when depth $I_{n-1}(X) = 3$.

§1. The fundamental lemma

(1.1) LEMMA (Micali-Villamayor). *Let R be a commutative ring with identity. Let $X = (x_{ij})$ be an n by n symmetric matrix with entries in R. Let $I_p(X)$ be the ideal of R generated by all the p by p minors of X, $1 \leq p \leq n$.*

I) *If x_{11} is invertible in R, then there exists an invertible matrix C such that*

1)

$$^{t}CXC = \left(\begin{array}{c|c} x_{11} & 0 \\ \hline 0 & X' \end{array}\right),$$

2) *the* $n-1$ *by* $n-1$ *matrix* X' *is symmetric and*

$$x'_{kj} = x_{kj} - \frac{x_{k1}\, x_{1j}}{x_{11}}, \qquad k, j = 2, \ldots, n,$$

3) $I_p(X) = I_{p-1}(X')$ for $p \geq 1$.

II) *If* $\det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} := a$ *is invertible in* R, *then there exists an invertible matrix* C *such that*

1)

$$ {}^t CXC = \left(\begin{array}{cc|c} x_{11} & x_{12} & \\ x_{21} & x_{22} & 0 \\ \hline \multicolumn{2}{c|}{0} & X'' \end{array}\right),$$

2) *the* $n-2$ *by* $n-2$ *matrix* X'' *is symmetric and* $x''_{kj} = x_{kj} - a_k x_{1j} - b_k x_{2j}$, *where*
$a_k = \dfrac{x_{1k}x_{22} - x_{2k}x_{12}}{a}$, $b_k = \dfrac{x_{2k}x_{11} - x_{1k}x_{21}}{a}$, $k, j = 3, \ldots, n$,

3) $I_p(X) = I_{p-2}(X'')$ *for* $p \geq 2$.

(1.2) *Remark.* We adopt the convention $I_0(X) = R$.

Proof. I) We define

$$C = \begin{pmatrix} 1 & -\frac{x_{12}}{x_{11}} & \cdots & -\frac{x_{1n}}{x_{11}} \\ & 1 & & O \\ & & \ddots & \\ & O & & 1 \end{pmatrix}$$

II) We define

$$C = \left(\begin{array}{cc|ccc} 1 & 0 & -a_3 & \cdots & -a_n \\ 0 & 1 & -b_3 & \cdots & -b_n \\ \hline & & 1 & & O \\ \multicolumn{2}{c|}{O} & & \ddots & \\ & & O & & 1 \end{array}\right)$$

(1.3) COROLLARY. *Let R be a local ring. If $I_1(X) = R$, then the hypothesis of either I) or II) of Lemma* (1.1) *holds* (*possibly after rearrangement of some rows and the same columns of X*).

Proof. From $I_1(X) = R$ it follows that some entry of X is invertible. If that entry lies on the main diagonal the hypothesis of *I*) holds. If this is not the case one may assume that x_{12} is invertible and all the entries on the main diagonal belong to the maximal ideal of R. Then $a = x_{11}x_{22} - x_{12}^2$ is invertible.

(1.4) COROLLARY. *Let R be a local ring. If $I_{n-1}(X) = R$, then there exists an invertible matrix C over R such that*

$$
{}^tCXC = \begin{pmatrix} \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & & & & O & \\ & \ddots & & & & \\ & & \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & & & \\ O & & & * & & \\ & & & & \ddots & \\ & & & & & * \\ & & & & & & u \end{pmatrix}
$$

where the starred i by i minors on the main diagonal are invertible, $i = 1$ and/or 2, $u \in R$.

Proof. If $I_{n-1}(X) = R$, then at least one entry of X is invertible. Using Lemma (1.1) one can transform X into a matrix of the kind *I*1) or *II*1) of the lemma with an r by r invertible matrix in the upper left corner, $r = 1$ or 2. If $n - 1 = r$ we are done. If not, then the remaining $n - r$ by $n - r$ matrix in the lower right corner has also at least one invertible entry. We again apply Lemma (1.1). Proceeding in this way we get the required result.

§2. Height and Depth of $I_p(X)$

The next theorem can be deduced from Kutz's results and the general theory of generically perfect ideals of Eagon and Northcott, [6]. To make the proof of Theorem (2.3) as self-contained as possible we indicate here a short proof of Theorem (2.1) along the lines presented in [5] for arbitrary determinantal ideals.

(2.1) THEOREM. *Let R be a commutative Noetherian ring with identity and X an n by n symmetric matrix with entries in R. Every minimal prime ideal of the ideal $I_p(X)$ has height at most equal to $\nu(p, n)$.*

Proof. We only sketch the proof and refer to [5], pp. 202–203, where possible.

We use induction on n. If $n \leq 2$ or $p = 1$ the theorem follows from the generalized principal ideal theorem of Krull.

Suppose $n>2$, $p>1$ and let P be a minimal prime ideal of $I_p(X)$. One may assume that R is local with maximal ideal P and that $I_p(X)$ is P-primary. If $I_1(X)=R$, then by Lemma (1.1) and Corollary (1.3) $I_p(X)=I_{p-r}(\tilde{X})$ where $r=1$ *or* 2, and $\tilde{X}$ is an $n-r$ by $n-r$ symmetric matrix over R. By the induction hypothesis *ht* $P\leq \nu(p-r, n-r)=\nu(p, n)$.

If $I_1(X)\subset P$ we consider a matrix

$$X+\left(\begin{array}{c|c} Z & 0 \\ \hline 0 & 0 \end{array}\right),$$

where Z is an indeterminate over R and proceed as in [5].

For the proof of the next theorem we record the following easy lemma.

(2.2) LEMMA. *Let K be a subring of a commutative ring R with identity and let $x_1, \ldots, x_q$ be a sequence of elements in R which are algebraically independent over K. Assume that t is a non-zero divisor in R belonging to $K[x_1, \ldots, x_s]$, $s<q$, and write $K'=K[x_1, \ldots, x_s]_{\{t^k\}}$, $R'=R_{\{t^k\}}$ for the localizations of the corresponding rings at the powers of t; moreover let $a'_{s+1}, \ldots, a'_q$ be elements of K'.*

Then $K'\subset R'$ and the elements $x_{s+1}-a'_{s+1}, \ldots, x_q-a'_q$ are algebraically independent over K'.

(2.3) THEOREM. *Let R be a commutative Noetherian ring with identity, K a Noetherian subring of R with the same identity. Let $\{x_{ij}\}$, $1\leq i\leq j\leq n$, be a sequence of elements of R which are algebraically independent over K. Assume that R is flat as an algebra over $K[\{x_{ij}\}]$. If we put $x_{ji}=x_{ij}$ for $j<i$ and define $X=(x_{ij})$, then depth $I_p(X)=\nu(p, n)$.*

Proof. We use certain arguments of Eagon from [4]. If $p=1$, then $I_p(X)$ is generated by $\{x_{ij}\}$, $i\leq j$, and therefore depth $I_1(X)=\nu(1, n)$. In fact, the sequence $\{x_{ij}\}$, $i\leq j$, is R-regular since R is flat over $K[\{x_{ij}\}]$.

Now we argue by induction on n, assuming $n>1$, $p>1$. Let $u_1, \ldots, u_l$ be a maximal R-regular sequence contained in $I_p(X)$. By Theorem (2.1) we know that $l\leq\nu(p, n)$, hence in view of $p>1$ we have $l<(n^2+n)/2=\nu(1, n)$. Write $I=I_p(X)$, $J=(u_1, \ldots, u_l)$ for short. Since I consists of zero divisors on J, there exists a prime ideal P associated to J and containing I. Thus $l=$depth $J=$depth $I=$depth P. By $l<(n^2+n)/2$ we must have $x_{ij}\notin P$ for some i, j. We consider two cases:

I) $i=j$; one may assume without loss of generality that $i=1$. Write $t=x_{11}$, $K'=K[x_{11}, x_{12}, \ldots, x_{1n}]_{\{t^k\}}$.

II) $i\neq j$ and all elements on the main diagonal belong to P. As above one may assume $i=1$, $j=2$. Write, in this case, $t=x_{11}x_{22}-x_{12}^2$, $K'=K[x_{11},\ldots,x_{1n}, x_{22},\ldots,x_{2n}]_{\{t^k\}}$. Of course $t\notin P$.

In both cases write $R'=R_{\{t^k\}}$. For an ideal $\mathfrak{A}$ of R let $\mathfrak{A}'$ denote $\mathfrak{A}R'$. Thus we have $J'\subset I'\subset P'$, P' is proper and depth $J'=l$ since J' is generated by an R'-regular sequence $u_1,\ldots,u_l$. On the other hand, P' is an associated prime of J' because $t\notin P$. Therefore $l=$ depth $J'=$ depth $I'=$ depth P'. Observe that $I'=I_p(X)R'$ is the ideal in R' generated by all the p by p minors of X. Since t is invertible in R' we may apply Lemma (1.1) to R' and X. We conclude that $I'=I_{p-r}(\tilde{X})R'$, where $r=1$ or 2 depending on case *I*) or *II*), and $\tilde{X}=(x_{kj}-a'_{kj})$ is the $n-r$ by $n-r$ symmetric matrix with entries in R', $a'_{kj}\in K'$, $r<k, j\leq n$. By Lemma (2.2) we infer that the elements $\{x_{kj}-a'_{kj}\}$, $r<k\leq j\leq n$, are algebraically independent over K'. Moreover $K'[\{x_{kj}-a'_{kj}\}]$, $r<k\leq j\leq n$, is equal to $K[\{x_{ij}\}]_{\{t^k\}}$, $1\leq i\leq j\leq n$, and R' is flat over $K[\{x_{ij}\}]_{\{t^k\}}$, $1\leq i\leq j\leq n$. Hence by the induction hypothesis we finally get $l=$ depth $I'=$ depth $I_{p-r}(\tilde{X})R'=\nu(p-r, n-r)=\nu(p,n)$.

(2.4) COROLLARY. *Let K be a commutative Noetherian ring and $R=K[\{x_{ij}\}]$, $1\leq i\leq j\leq n$, a polynomial ring over K in $(n^2+n)/2$ indeterminates $\{x_{ij}\}$ · Put $X=(x_{ij})$. Then depth $I_p(X)=\nu(p,n)$.*

(2.5) *Remark.* Corollary (2.4) was proved by R. Kutz in [11, Proposition 6.2] under the additional assumption that K is an integral domain.

(2.6) COROLLARY. *Let R be a local algebra over a field K and let $\{x_{ij}\}$, $1\leq i\leq j\leq n$, be a regular sequence in R. Then depth $I_p(X)=\nu(p,n)$ for a symmetric matrix $X=(x_{ij})$.*

Proof. Since $\{x_{ij}\}$ are algebraically independent over K and R is flat over $K[\{x_{ij}\}]$([8, Proposition 1]), the corollary follows immediately from Theorem (2.3).

Using the method of the proof of Theorem (2.3) one can also prove

(2.7) THEOREM. *Let R be a commutative Noetherian ring with identity and K a Noetherian subring of R with the same identity. Let $\{x_{ij}\}$, $1\leq i\leq r$, $1\leq j\leq s$, be a sequence of elements of R which are algebraically independent over K and let X denote an r by s matrix (x_{ij}). Assume that R is flat as an algebra over $K[\{x_{ij}\}]$. Then depth $I_t(X)=(r-t+1)(s-t+1)$ where $I_t(X)$ is an ideal of R generated by all the t by t minors of X.*

(2.8) COROLLARY. *Let R be a local algebra over a field K and let $\{x_{ij}\}$, $1\leq i\leq r$, $1\leq j\leq s$, be a regular sequence in R. Then epth $I_t(X)=(r-t+1)(s-t+1)$ where $X=(x_{ij})$.*

§3. A Free Resolution of $I_{n-1}(X)$

Let R be a commutative ring with identity and $X=(x_{ij})$ a symmetric n by n matrix with entries in R. Write $Y=(y_{ij})$ for the matrix of cofactors of X, i.e.

$y_{ij} = (-1)^{i+j} X^i_j$ where X^i_j stands for the minor of X obtained by deleting the i-th column and the j-th row of X. The matrix Y is also symmetric. We are fixing the matrix X (and hence Y) throughout this section.

Let $M_n(R)$ be the free R-module of all n by n matrices over R and $A_n(R)$ the free submodule of $M_n(R)$ consisting of all alternating matrices. Furthermore, let $tr: M_n(R) \to R$ denote the trace map.

We have a free complex of length 3 associated with X:

$$\mathbf{L}(X): 0 \longrightarrow A_n(R) \xrightarrow{d_3} \mathrm{Ker}\,(M_n(R) \xrightarrow{tr} R) \xrightarrow{d_2} M_n(R)/A_n(R) \xrightarrow{d_1} R,$$

where the corresponding differentials are defined as follows:

$$d_1(M \bmod A_n(R)) = \mathrm{tr}\,(YM),$$
$$d_2(N) = XN \bmod A_n(R),$$
$$d_3(A) = AX.$$

d_1 and d_3 are well defined because the trace of the product of a symmetric and an alternating matrices is 0. Observe that $H_0(\mathbf{L}(X)) = R/I_{n-1}(X)$.

Now we can state the main result of this section.

(3.1) THEOREM. *Let R be a commutative Noetherian ring with identity. Let $X = (x_{ij})$ be an n by n symmetric matrix with entries in R. If depth $I_{n-1}(X) = 3$ (the largest possible), then the complex $\mathbf{L}(X)$ is acyclic and gives a free resolution of $R/I_{n-1}(X)$.*

The proof of (3.1) requires several preliminary lemmata.

(3.2) LEMMA. *Let $\varphi: R \to R'$ be a ring homomorphism, $X = (x_{ij})$ a symmetric matrix over R, and $X' = (\varphi(x_{ij}))$. Then the complexes $\mathbf{L}(X) \otimes_R R'$ and $\mathbf{L}(X')$ are isomorphic over R'.*

(3.3) LEMMA. *The complexes $\mathbf{L}({}^tCXC)$ and $\mathbf{L}(X)$ are isomorphic for an arbitrary invertible n by n matrix C.*

Proof. Let F be a free R-module of rank n and let F^* be the dual module of F. A map $f: F^* \to F$ is said to be *symmetric* if, with respect to some (and therefore every) basis and dual basis of F and F^*, the matrix of f is symmetric.

We are going to prove the lemma by assigning to a symmetric map $f: F^* \to F$

a free complex $\mathbf{L}(f)$ of length 3 and showing that $\mathbf{L}(X)$ and $\mathbf{L}({}^tCXC)$ are both isomorphic with $\mathbf{L}(f)$. The passage from $\mathbf{L}(f)$ to $\mathbf{L}(X)$ corresponds to fixing a basis of F and taking the dual basis of F^*, and further passage to $\mathbf{L}({}^tCXC)$ corresponds to a change of bases.

An invariant basis-free description of our complex can be given as follows:

$$\mathbf{L}(f): 0 \longrightarrow \bigwedge^2(F^*) \xrightarrow{\partial_3} \operatorname{Ker}(F^* \otimes F \xrightarrow{ev} R) \xrightarrow{\partial_2} S_2(F) \xrightarrow{\partial_1} R,$$

where ev stands for the evaluation map, $S_2(F)$ is the second symmetric power of F, and $\bigwedge^2(F^*)$ the second exterior power of F^*.

To determine the differentials of $\mathbf{L}(f)$ we define a map $g: F \to F^*$ by requiring commutativity of the following diagram:

$$\begin{array}{ccc} \bigwedge^{n-1}(F^*) & \xrightarrow{f} & \bigwedge^{n-1}(F) \\ \wr\| & & \wr\| \\ F & \xrightarrow{g} & F^* \end{array}$$

where the vertical maps are the canonical isomorphisms. Then the composition $F \otimes F \xrightarrow{1 \otimes g} F \otimes F^* \xrightarrow{ev} R$ induces ∂_1 on $S_2(F)$ and the map $F^* \otimes F \xrightarrow{f \otimes 1} F \otimes F \xrightarrow{\eta} S_2(F)$ induces ∂_2, where η is the canonical epimorphism. Finally, ∂_3 is induced by $\bigwedge^2(F^*) \xrightarrow{\gamma} F^* \otimes F^* \xrightarrow{1 \otimes f} F^* \otimes F$, where $\gamma(u \wedge w) = u \otimes w - w \otimes u$.

The next lemma needs some information about the $n-2$ by $n-2$ minors of X. To fix the notation let X_{ij}^{kl} be the minor of X obtained by leaving out the i-th and j-th rows, and the k-th and l-th columns of X, $i \neq j$, $k \neq l$. Observe that $X_{ij}^{kl} = X_{kl}^{ij}$ because X is symmetric.

We define two functions:

$$\sigma(i, j) = \begin{cases} 1 & \text{if } i < j \\ 0 & \text{if } i = j, \quad i, j \in N; \\ -1 & \text{if } i > j \end{cases}$$

$$T(i, j, k, l) = (-1)^{i+j+k+l} \sigma(i, j) \sigma(k, l) X_{ij}^{kl}, \quad i, j, k, l, \in N$$

By the Laplace expansion we get the following formulas:

$$(\#)\ \sum_{l=1}^{n} x_{sl}T(i, j, k, l) = \begin{cases} 0 & \text{if } s \neq i, s \neq j, \\ -y_{ik} & \text{if } s = i, i \neq j, \\ y_{ik} & \text{if } s = j, i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

We write $\{E_{ij}\}$ for the standard basis of $M_n(R)$; if $F_{ij} = E_{ij} - E_{ji}$, then $\{F_{ij}\}$, $i < j$, is a basis of $A_n(R)$.

(3.4) LEMMA. *For* $i < j$

$$YF_{ij} = \Big(\sum_{p<q} (-1)^{i+j+p+q} X_{ij}^{pq} F_{pq}\Big) X.$$

Proof. Write $\alpha = i + j + p + q$ for short. Using $(\#)$ we have

$$\sum_{p<q} (-1)^{\alpha} X_{ij}^{pq} F_{pq} X =$$

$$\sum_{p<q} (-1)^{\alpha} X_{ij}^{pq} \Big(\sum_{s} x_{qs} E_{ps} - \sum_{s} x_{ps} E_{qs}\Big) =$$

$$\sum_{p,s} \Big(\sum_{q>p} (-1)^{\alpha} x_{qs} X_{ij}^{pq} + \sum_{q<p} (-1)^{\alpha-1} x_{qs} X_{ij}^{pq}\Big) E_{ps} =$$

$$\sum_{p,s} \Big(\sum_{q} x_{sq} T(i, j, p, q)\Big) E_{ps} =$$

$$\sum_{p} y_{ip} E_{pj} - \sum_{p} y_{jp} E_{pi} = Y(E_{ij} - E_{ji}) = YF_{ij}.$$

(3.5) COROLLARY. *For an arbitrary alternating matrix B there exists an alternating matrix A such that* $YB = AX$.

(3.6) LEMMA. *If X is a symmetric invertible matrix, then* $\mathbf{L}(X)$ *is exact.*

Proof. $\operatorname{Ker} d_1 = \operatorname{Im} d_2$. Let $M \bmod A_n(R) \in \operatorname{Ker} d_1$; this means that $\operatorname{tr}(YM) = 0$ and therefore $YM \in \operatorname{Ker}(M_n(R) \xrightarrow{\operatorname{tr}} R)$. Hence

$$M \bmod A_n(R) = d_2[(\det X)^{-1} YM] \in \operatorname{Im} d_2.$$

Ker $d_2 = \mathrm{Im}\, d_3$. Let $N \in \mathrm{Ker}\, d_2$, i.e. $B := XN$ is alternating. Multiplying by Y and using Corollary (3.5) we get $N = [(\det X)^{-1}A]X \in \mathrm{Im}\, d_3$, where A is an alternating matrix.
Ker $d_3 = 0$ is obvious.

In the course of the proof of Theorem (3.1) we will need the following corollary from the "*Lemme d'acyclicité*" of Peskine-Szpiro (see [3], Corollary 4.2).

(3.7) LEMMA. *Let R be a Noetherian ring, and let*

$$\mathbf{L}: 0 \to L_3 \to L_2 \to L_1 \to L_0$$

be a complex of finitely generated free R-modules. If for every prime ideal $P \subset R$ with depth $P < 3$ the localized complex $\mathbf{L}_P$ is exact, then $\mathbf{L}$ is exact.

Proof of Theorem (3.1). By Lemma (3.7) it is enough to prove that $\mathbf{L}(X)_P$ is exact for every prime P with depth $P < 3$. Since depth $I_{n-1}(X) = 3$ we infer that $I_{n-1}(X) \not\subset P$ for such a P, and hence $I_{n-1}(X_P) = R_P$ where X_P is a matrix X considered over R_P. Since $\mathbf{L}(X)_P \simeq \mathbf{L}(X_P)$ by Lemma (3.2) it suffices to prove the theorem for R local and X with $I_{n-1}(X) = R$.

Under these assumptions and by Corollary (1.4) there exists an invertible matrix C such that ${}^tCXC = X'X''$ where $x'_{ij} = 0$ for $i < n$, $j = n$, $i = n$, $j < n$, $x'_{nn} = 1$, $x''_{ii} = 1$ for $i < n$, $x''_{nn} = u$, $x''_{ij} = 0$ for $i \neq j$, and the matrix X' is invertible. Observe that X' and X'' commute with each other. By Lemma (3.3) it is enough to prove that $\mathbf{L}(X'X'')$ is exact. By direct computation one proves that $\mathbf{L}(X'')$ is exact.

Write d_p, d'_p, d''_p for the differentials of $\mathbf{L}(X'X'')$, $\mathbf{L}(X')$, $\mathbf{L}(X'')$, respectively, and Y', Y'' for the matrices of cofactors of X', X'', respectively. Note that $Y = Y'Y''$.
Ker $d_1 = \mathrm{Im}\, d_2$
For any matrix $Q = (q_{ij})$ we have an equality

$$Q = X''\tilde{Q} + (1-u)\sum_{j<n} q_{nj}E_{nj}, \qquad (\#\#)$$

where $\tilde{q}_{ij} = q_{ij}$ for $(i, j) \neq (n, n)$, and $\tilde{q}_{nn} = -\sum_{i=1}^{n-1} q_{ii}$; hence tr $\tilde{Q} = 0$.

Suppose that M mod $A_n(R) \in \mathrm{Ker}\, d_1$; one can assume that M is triangular with zeros under the main diagonal. Since $y'_{nj} = 0$ for $j < n$ we get, by applying (##) to $Y'M$, an equality $Y'M = X''W$ with tr $W = 0$. Multiplying both sides by X' and using the invertibility of $\det X'$ we finally get:

$$M \bmod A_n(R) = X'X''[(\det X')^{-1}W] \bmod A_n(R) \in \mathrm{Im}\, d_2.$$

Ker $d_2 = \mathrm{Im}\, d_3$

We can assume that the entry u in the lower right corner of X'' belongs to the maximal ideal of R because otherwise $X'X''$ is invertible and we are done by Lemma (3.6).

Let $Q \in \operatorname{Ker} d_2$, i.e. $\operatorname{tr} Q = 0$ and $B := X'X''Q$ is alternating. Multiplying by Y' gives $X''Q = (\det X')^{-1} Y'B$ and $\operatorname{tr}(X''Q) = 0$. This together with $\operatorname{tr} Q = 0$ implies that $q_{nn} = 0$, because $1 - u$ is invertible. Note that $X''QY' = (\det X')^{-1} Y'BY'$ is alternating and a simple calculation shows that $\operatorname{tr}(QY') = 0$. This means that $QY' \in \operatorname{Ker}(M_n(R) \xrightarrow{\operatorname{tr}} R)$. Since $X''(QY')$ is alternating we get $QY' = DX''$, for some alternating D, from the exactness of $\mathbf{L}(X'')$. Therefore $Q = [(\det X')^{-1}D]X'X'' \in \operatorname{Im} d_3$.
$\operatorname{Ker} d_3 = 0$ is obvious.

(3.8) *Remark.* The proof simplifies considerably when 2 is invertible in R. In this case one can transform X as in Corollary (1.4) to a diagonal matrix and the proof of the exactness of $\mathbf{L}(X)$ for a diagonal matrix is straightforward.

(3.9) COROLLARY (Kutz). *If depth $I_{n-1}(X) = 3$, then $I_{n-1}(X)$ is perfect.*
From Corollary (2.4) we infer

(3.10) COROLLARY. *$I_{n-1}(X)$ is a generically perfect ideal (see* [6], §8, *for the definition).*
Corollaries (2.7) and (3.9) give together

(3.11) COROLLARY. *Let R be a local algebra over a field K and $X = (x_{ij})$ a symmetric matrix over R such that $\{x_{ij}\}$, $1 \leq i \leq j \leq n$, form an R-sequence. Then*

a) *$I_{n-1}(X)$ is perfect and $\mathbf{L}(X)$ is the minimal free resolution of $R/I_{n-1}(X)$.*

b) *if R is regular, $R/I_{n-1}(X)$ is a Cohen-Macaulay ring of type $(n^2 - n)/2$.*

The results of Eagon, Northcott and Hochster (see in particular [9, Theorem 3]) lead to the following corollary.

(3.12) COROLLARY. *Let X be a symmetric matrix over a Noetherian ring R. The complex $\mathbf{L}(X)$ is depth-sensitive, i.e. for any finitely generated R-module E such that $I_{n-1}(X)E \neq E$ we have*

$$\text{depth } (I_{n-1}(X), E) + q = 3,$$

where q is the index of the largest non-vanishing homology module of the complex $\mathbf{L}(X) \otimes_R E$.

(3.13) *Remark.* When the first version of this paper had been written I received from J. Herzog a preprint of S. Goto and S. Tachibana, [7]. The authors constructed a complex of length 3 identical with $\mathbf{L}(X)$ when 2 is invertible in R, and proved in this case (by different methods) Theorem (3.1).

§4. An Application to the Poincaré Series

We recall that if R is a local ring with residue field K, the Poincaré series $\mathcal{P}_R$ of R is the power series

$$\sum_{p=0}^{\infty} (\dim_K \operatorname{Tor}_p^R(K, K))t^p.$$

(4.1) THEOREM*. *Let R be a local ring, Z an n by n symmetric matrix with entries in the maximal ideal* $\mathfrak{m}$ *of R, $n>1$, and $S=R/I_{n-1}(Z)$. Assume that depth $I_{n-1}(Z)=3$. If $n>2$, then $I_{n-1}(Z)$ is a Golod ideal (see* [1], *Definition* 3.6), *and*

$$\mathcal{P}_R/\mathcal{P}_S=(1+t)^r/(1-t^2)^{3-r} \text{ if } n=2, \quad \text{where } r=\dim_K(I_1(Z)+\mathfrak{m}^2)/\mathfrak{m}^2,$$

$$\mathcal{P}_R/\mathcal{P}_S=1-\left(\frac{n^2+n}{2}\right)t^2-(n^2-1)t^3-\left(\frac{n^2-n}{2}\right)t^4 \quad \text{if} \quad n>2.$$

Proof. If $n=2$, then the ideal $I_{n-1}(Z)$ is a complete intersection and the corresponding formula is well known.

Let $n>2$; since $I_{n-1}(Z)$ generically perfect (Corollary (3.10)) we can use Theorem 6.2 of [1] which states that $I_{n-1}(Z)$ is a Golod ideal in R if and only if $I_{n-1}(X)$ is a Golod ideal in the power series ring $K[[x_{ij}]]$, $1\leq i\leq j\leq n$, where $X=(x_{ij})$. Observe that depth $I_{n-1}(X)=3$ by Corollary (2.6). (It is Theorem 6.2 of [1] which needs the hypothesis that the entries of Z belong to the maximal ideal of R.) Write $R'=K[[x_{ij}]]$, $1\leq i\leq j\leq n$, $S'=R'/I_{n-1}(X)$ for short. To prove that $I_{n-1}(X)$ is Golod it suffices (by Theorems 3.5 and 6.2 of [1]) to show that the algebra $\operatorname{Tor}^{R'}(S', K)$ has trivial Massey products. Since $\mathbf{L}(X)$ is a free resolution of S' over R' we know that $\operatorname{Tor}^{R'}(S', K)=\mathbf{L}(X)\otimes_{R'}K$.

We are going to prove that $\mathbf{L}(X)$ can be endowed with the structure of a differential graded commutative algebra over R' in such a way that the induced multiplication on $\mathbf{L}(X)\otimes_{R'}K$ is trivial. This result implies that $\operatorname{Tor}^{R'}(S', K)$ has trivial Massey products, hence $I_{n-1}(X)$ is a Golod ideal and consequently, applying once again Theorems 3.5 and 6.2 of [1], we get the required formula for $\mathcal{P}_R/\mathcal{P}_S$.

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* I am grateful to L. Avramov who drew my attention to an erroneous formulation of Thm. (4.1) in an earlier version of the paper.

Write $\mathbf{L}=\mathbf{L}(X)$ for short. Let $S_2(\mathbf{L})$ denote the second symmetric power of the complex $\mathbf{L}$. A commutative multiplication on $\mathbf{L}$ defines a differential graded homomorphism $S_2(\mathbf{L})\to\mathbf{L}$, which is the identity on the canonical image of $\mathbf{L}$ in $S_2(\mathbf{L})$, and vice versa (by Proposition (1.1) of [2]). Moreover, this multiplication is associative because $\mathbf{L}$ is of length 3. So we only must define a map of complexes:

$$\begin{array}{ccccccccc}
S_2(\mathbf{L}): \cdots \longrightarrow & L_1\otimes L_2+L_3 & \xrightarrow{\delta_3} & L_1\otimes L_1+L_2 & \xrightarrow{\delta_2} & L_1 & \xrightarrow{\delta_1} & L_0 \\
\downarrow{\scriptstyle\varphi} & {\scriptstyle\varphi_2}\searrow \ \ \ // & & {\scriptstyle\varphi_1}\searrow \ \ \ // & & \| & & \| \\
\mathbf{L} \ : \quad 0 \longrightarrow & L_3 & \xrightarrow{\quad d_3 \quad} & L_2 & \xrightarrow{d_2} & L_1 & \xrightarrow{d_1} & L_0
\end{array}$$

To define φ_1 we fix a basis $U_{ij}=E_{ij} \bmod A_n(R)$, $1\leqslant i\leqslant j\leqslant n$, of L_1, and a basis $W_{pq}=E_{pq}$, $p\neq q$, $W_p=E_{pp}-E_{nn}$, $p=1,\ldots,n-1$, of L_2.

We put

$$\varphi_1(U_{ij}\otimes U_{kl})=\sum_{\alpha\neq 1} T(k,j,i,\alpha)W_{\alpha 1}+\sum_{\alpha\neq j} T(l,i,k,\alpha)W_{\alpha j}+$$

$$\begin{cases} T(k,j,i,l)(W_l-W_j) & \text{if } j\neq n,\, l\neq n,\\ T(k,j,i,l)W_l & \text{if } j=n,\, l\neq n,\\ -T(k,j,i,l)W_j & \text{if } j\neq n,\, l=n,\\ 0 & \text{if } j=l=n.\end{cases}$$

Let $\mathfrak{m}'$ denote the maximal ideal of R'. Note that $\varphi_1(L_1\otimes L_1)\subset\mathfrak{m}'L_2$ because $n>2$. It follows from this definition of φ_1 that $\varphi_1\delta_3(L_1\otimes L_2)\subset\mathfrak{m}'^2L_2$. Since $\mathbf{L}$ is exact there exists presisely one map φ_2 making the above diagram commutative. We show that $\varphi_2(L_1\otimes L_2)\subset\mathfrak{m}'L_3$. But this is equivalent to the implication $d_3(b)\in\mathfrak{m}'^2L_2\Rightarrow b\in\mathfrak{m}'L_3$, $b\in L_3$. The last statement follows simply from the definition of d_3 and linear independence of $\{x_{ij} \bmod \mathfrak{m}'^2\}$, $i\leqslant j$, over K.

(4.2) *Remark.* If not all entries of Z belong to the maximal ideal of R and depth $I_{n-1}(Z)=3$, then by Lemma (1.1) $I_{n-1}(Z)=I_p(Z')$ for some symmetric matrix Z' with all the entries in the maximal ideal and for some p. Therefore, Theorem (4.1) applies also for such matrices.

(4.3) *Remark.* Theorem (4.1) has also been proved independently by J. Herzog and M. Steurich.

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