Spectrum of a compact flat manifold.

Comment. Math. Helvetici **53** (1978) 613–621 Birkhäuser Verlag, Basel

Spectrum of a compact flat manifold

TOSHIKAZU SUNADA

1. Introduction

From the differential geometric view point, the celebrated Jacobi identity for a Dirichlet series associated with a lattice L in $\mathbf{R}^n$ is written as

$$\sum_{i=0}^{\infty} \exp(-\lambda_i t) = \mathrm{Vol}(L\backslash\mathbf{R}^n)(4\pi t)^{-n/2} \sum_{x\in L} \exp(-l(c_x)^2/4t) \qquad (t>0),$$

in which the series $0=\lambda_0<\lambda_1\leqq\lambda_2\leqq\cdots\uparrow\infty$ is the spectrum of the Laplacian $-\sum_{i=1}^{n}(\partial/\partial x_i)^2$ on a flat torus $L\backslash\mathbf{R}^n$, and $l(c_x)$ stands for the length of a closed geodesic c_x in $L\backslash\mathbf{R}^n$ belonging to the homotopy class x. Here we identify L with the fundamental group $\pi_1(L\backslash\mathbf{R}^n)$.

In this paper, we extend the Jacobi identity to the case of compact flat manifolds with the non trivial holonomy groups and apply to the isospectral problem of such manifolds.

Before giving the statement of our results, we recall some properties of closed geodesics in a flat manifold. Let M be an n-dimensional compact Riemannian manifold with flat curvature. Then the universal covering of M is isometric to the Euclidean space $\mathbf{R}^n$ and the covering transformation group C is isomorphic to a discrete uniform torsion free subgroup of the group of rigid motions acting on $\mathbf{R}^n$. We denote by $[\gamma]$ the conjugacy class containing γ in C and by $[C]$ the set of all conjugacy classes of C, which is naturally identified with the set of all free homotopy classes of mappings of the circle $S^1=\mathbf{R}/\mathbf{Z}$ into M. If we denote by $\mathrm{Geo}(M)$ the set of all closed geodesics $c\colon S^1\to M$ and by $M_{[\gamma]}$ the set of ones belonging to the homotopy class $[\gamma]$, then $M_{[\gamma]}$ is not empty, and we have of course

$$\mathrm{Geo}(M) = \bigcup_{[\gamma]\in[C]} M_{[\gamma]} \quad \text{(disjoint)}.$$

As was shown in [8], each $M_{[\gamma]}$ is a finite dimensional compact connected manifold whose fundamental group is isomorphic to the centralizer C_γ of γ in C, and any element in $M_{[\gamma]}$ has a common length, to be denoted $l(\gamma)$. Furthermore, the evaluation mapping $M_{[\gamma]} \to M$ defined by $c \mapsto c(0)$ is an immersion and induces a flat metric on $M_{[\gamma]}$.

Given an element γ of C, let $x \mapsto Ax + a$ be the corresponding motion, and we put

$$\alpha(\gamma) = |\det (A - I \mid \mathrm{Im}(A - I))|^{-1}$$

in which $A - I \mid \mathrm{Im}\,(A - I)$ stands for the restriction of $A - I$ to the image of $A - I$. This is well-defined because the restriction gives rise to a non-singular transformation of the subspace $\mathrm{Im}\,(A - I)$. We easily observe that $\alpha(\gamma)$ depends only upon the conjugacy class $[\gamma]$.

Using these terminologies one of our results is stated as follows:

THEOREM 1. *Let* $0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \uparrow \infty$ *be the spectrum of the Laplacian* Δ *on* M. *Then*

$$\sum_{i=0}^{\infty} \exp(-\lambda_i t) = \sum_{[\gamma] \in [C]} \alpha(\gamma) \operatorname{Vol}(M_{[\gamma]})(4\pi t)^{-\dim M_{[\gamma]}/2} \exp(-l(\gamma)^2/4t),$$

By a theorem of Bieberbach, any compact flat manifold M admits a normal Riemannian covering by a flat torus T_M. The spectrum of M is therefore a subsequence of that of T_M. But, as an application of the above identity, we can prove that the spectrum of M determines that of T_M. On the other hand, it was proved by M. Kneser that the number of isometry classes of flat tori with a given spectrum is finite (unpublished). Combining these facts, we can deduce the following.

THEOREM II. *There are only finitely many isometry classes of flat manifolds with a given spectrum.*

For fundamental properties of flat manifolds, see J. A. Wolf [9] or S. Kobayashi and K. Nomizu [4].

2. Proof of Theorem I

We denote by $E(n)$ the group of rigid motions of $\mathbf{R}^n$.

Let $M = C\backslash\mathbf{R}^n$ be a compact flat manifold, C being a torsion free discrete subgroup of $E(n)$, and let D be a fundamental domain of the group C in $\mathbf{R}^n$. The fundamental solution of the heat equation $\partial/\partial t+\Delta$ on M is given by

$$K(t;\ x,\ y)=\sum_{\gamma\in C}k(t;\ x,\ \gamma y),$$

where $k(t;\ x,\ y)=(4\pi t)^{-n/2}\exp(-\|x-y\|^2/4t)$, and hence we obtain

$$\sum_{i=0}^{\infty}\exp(-\lambda_i t)=\int_M K(t;\ x,\ x)\,dx=\sum_{\gamma\in C}\int_D k(t;\ x,\ \gamma x)dx.$$

By a well-known process (see A. Selberg [7]), the right hand side is rewritten as

$$\sum_{[\gamma]\in[C]}\int_{C_\gamma\backslash\mathbf{R}^n}k(t;\ x,\gamma x)\,dx.$$

To transform the expression

$$\int_{C_\gamma\backslash\mathbf{R}^n}k(t;x,\gamma x)\,dx \tag{1}$$

still further, we shall make use of the following subset in $\mathbf{R}^n$:

$$V_\gamma=\{x\in\mathbf{R}^n;\|\gamma x-x\|\leqq\|\gamma y-y\|\quad\text{for any}\quad y\in\mathbf{R}^n\}.$$

Evidently, V_γ is invariant under the action of C_γ and coincides with the critical point set of the function $f\colon x\to\|\gamma x-x\|^2$. More explicitly, it can be described as the affine subspace in $\mathbf{R}^n$ which consists of all x in $\mathbf{R}^n$ satisfying the equation $\gamma^2x-2\gamma x+x=0$. Hence, if $\gamma:x\mapsto Ax+a$, then V_γ is a translation of the subspace $\operatorname{Ker}(A-I)$.

Now, if x is an element of V_γ, then setting $\tilde{c}_x(t)=t\gamma x+(1-t)x$, we obtain a straight line $\tilde{c}_x:\mathbf{R}\to\mathbf{R}^n$ with $\tilde{c}_x(0)=x$ and $\tilde{c}_x(1)=\gamma x$, which satisfies $\tilde{c}_x(t+1)=\gamma\tilde{c}_x(t)$ for any $t\in\mathbf{R}$. In fact,

$$\begin{aligned}\tilde{c}_x(t+1)-\gamma\tilde{c}_x(t)&=(t+1)\gamma x-tx-t\gamma^2x-(1-t)\gamma x\\&=-t(\gamma^2x-2\gamma x+x)\\&=0.\end{aligned}$$

Therefore, $\tilde{c}_x$ yields a closed geodesic $c_x:S^1\to M$ lying in $M_{[\gamma]}$.

LEMMA 1. *The mapping* $V_\gamma\to M_{[\gamma]}$ *given by* $x\mapsto c_x$ *induces an isometry* φ_γ *of* $C_\gamma\backslash V_\gamma$ *onto* $M_{[\gamma]}$.

Proof. Take a typical element $c \in M_{[\gamma]}$. Then there exist a lifting $\tilde{c}: \mathbf{R} \to \mathbf{R}^n$ of c satisfying $\tilde{c}(t+1) = \gamma\tilde{c}(t)$ for any t. Since $\tilde{c}$ is a straight line, it is given by the equation $\tilde{c}(t) = t\gamma\tilde{c}(0) + (1-t)\tilde{c}(0)$ and therefore $\gamma^2\tilde{c}(0) = \tilde{c}(2) = 2\gamma\tilde{c}(0) - \tilde{c}(0)$, which implies that $\tilde{c}(0)$ lies in V_γ and $c_{\tilde{c}(0)} = c$. Hence the mapping $V_\gamma \to M_{[\gamma]}$ is surjective.

Next suppose that $c_x = c_y$. Then there exists some μ of C with $\tilde{c}_x(t) = \mu\tilde{c}_y(t)$ $(t \in \mathbf{R})$. Since $\gamma\mu y = \tilde{c}_x(1) = \mu\gamma y$, it follows that μ lies in C_γ and $x = \mu y$. The converse is obviously true, so the mapping $\varphi_r : C_\gamma \backslash V_\gamma \to M_{[\gamma]}$ is bijective.

Finally, notice the following commutative diagram

$$\begin{array}{ccc} C_\gamma \backslash V_\gamma & \xrightarrow{\ i\ } & \\ \varphi_\gamma \downarrow & & C\backslash\mathbf{R}^n = M, \\ M_{[\gamma]} & \longrightarrow & \end{array}$$

i being the isometric immersion induced from the injection: $V_\gamma \subset \mathbf{R}^n$. It follows immediately that φ_γ is isometric.

From now on we shall identify $C_\gamma \backslash V_\gamma$ with $M_{[\gamma]}$ via φ_γ.

Let $\tilde{\omega}$ be the orthogonal projection of $\mathbf{R}^n$ onto the affine sub-space V_γ. Since $\tilde{\omega}$ is C_γ-equivariant, it gives rise to a Riemannian submersion $\tilde{\omega} : C_\gamma \backslash \mathbf{R}^n \to C_\gamma \backslash V_\gamma$ with fibers isometric to affine sub-spaces in $\mathbf{R}^n$, and thus the term (1) becomes

$$\int_{C_\gamma \backslash V_\gamma} dy \int_{\tilde{\omega}^{-1}(y)} k(t;\, x,\, \gamma x)\, dx$$

LEMMA 2. *For any* $y \in C_\gamma \backslash V_\gamma$,

$$\int_{\omega^{-1}(y)} k(t;\, x,\, \gamma x)\, dx = \alpha(\gamma)(4\pi t)^{-\dim V_\gamma/2} \exp(-l(\gamma)^2/4t).$$

In particular, its value does not depend upon y.

Proof. Let $y_0 \in V_\gamma$ be an element lying over y, and we put

$$W_\gamma = \mathrm{Im}\,(A - I) + y_0.$$

Since $\mathrm{Im}(A-I)$ is the orthogonal complement of $\mathrm{Ker}\,(A-I)$, it follows easily that W_γ is identified with the fiber of $\tilde{\omega}$ over y_0. Hence, we have

$$\int_{\tilde{\omega}^{-1}(y)} k(t;\, x,\, \gamma x)\, dx = (4\pi t)^{-n/2} \int_{W_\gamma} \exp(-\|(A-I)x + a\|^2/4t)\, dx.$$

On the other hand, $\|(A-I)(z+y_0)+a\|^2=\|(A-I)z\|^2+l(\gamma)^2$ for any $z\in\mathbf{R}^n$ because $2((A-I)z, (A-I)y_0+a)=d_{y_0}f(z)=0$, and $\|\gamma y_0-y_0\|=l(\gamma)$. Thus, after the substitution

$$w=\frac{1}{\sqrt{4\pi t}}(A-I)z$$

the right hand side becomes

$$(4\pi t)^{-n/2+\dim \operatorname{Im}(A-I)/2}\,|\det(A-I\,|\,\operatorname{Im}(A-I))|^{-1}\exp(-l(\gamma)^2/4t)\times$$

$$\times\int_{\operatorname{Im}(A-I)}\exp(-\pi\|w\|^2)\,dw.$$

Using here the well-known equality: $\int_{\mathbf{R}}\exp(-\pi x^2)\,dx=1$, we get

$$\int_{\tilde\omega^{-1}(y)} k(t;\,x,\,\gamma x)\,dx=\alpha(\gamma)(4\pi t)^{-\dim V_\gamma/2}\exp(-l(\gamma)^2/4t),$$

as asserted.

Theorem I is an immediate consquence of Lemma 2.

3. Proof of Theorem II

Let $M=C\backslash\mathbf{R}^n$ be a compact flat manifold. By a theorem of Bieberbach, C contains the unique lattice $L=C\cap\mathbf{R}^n$ that is normal and maximal abelian in C. Thus we have exact sequence of groups

$$0\to L\xrightarrow{i}C\xrightarrow{\pi}F\to 1,$$

where i denotes the natural inclusion and F is the image of C by the homomorphism $\pi:(A,a)\in O(n)\cdot\mathbf{R}^n=E(n)\mapsto A\in O(n)$ with kernel $\mathbf{R}^n$. F is clearly a finite subgroup of $O(n)$, and $F(L)=L$. Geometrically speaking, the group F can be regarded as the holonomy group of M. Further, setting $T_M=L\backslash\mathbf{R}^n$, we obtain a normal Riemannian covering by the torus: $T_M\to M$, in which the deck transformation group is isomorphic to F.

We shall first prove the following.

LEMMA 3. *Let* $M = C\backslash\mathbf{R}^n$ *be a compact flat manifold, and let* $0 \to L \to C \to F \to 1$ *be the exact sequence determined by* C. *Then an element* γ *of* C *lies in* L *if and only if* $\dim V_\gamma = n$.

Proof. If γ lies in L, then C_γ contains L, hence $\dim V_\gamma = \text{rank } L = n$. Conversely suppose that $\dim V_\gamma = n$. If $\gamma = (A, a)$, then $\dim V_\gamma = \textit{dim} \operatorname{Ker}(A - I)$, so $\dim V_\gamma = n$ implies $A = I$.

An easy argument shows that the spectrum of M determines the following Dirichlet series

$$D(t) = (4\pi t)^{-n/2} \sum \alpha(\gamma) \operatorname{Vol}(M_{[\gamma]}) \exp(-l(\gamma)^2/4t), \tag{2}$$

summed over $[\gamma]$ with $\dim M_{[\gamma]} = n$, which is a partial sum of the right hand side of the Jacobi indentity (Theorem I). But from the above lemma such elements γ lie in L because $\dim M_{[\gamma]} = \dim V_\gamma$, and in this case $[\gamma] = F(\gamma) \subset L$, $\alpha(\gamma) = 1$, $l(\gamma) = \|\gamma\|$. Further, if $0 \to L_\gamma \to C_\gamma \to F_\gamma \to 1$ is the exact sequence determined by C_γ, then clearly $L_\gamma = L$ and $F_\gamma = \{g \in F;\ g\gamma = \gamma\}$. Hence, we get

$$\operatorname{Vol}(M_{[\gamma]}) = \operatorname{Vol}(L\backslash\mathbf{R}^n)/{}^{\#}F_\gamma.$$

Substituting these values into the right hand side of (2), we obtain

$$D(t) = (4\pi t)^{-n/2} \operatorname{Vol}(L\backslash\mathbf{R}^n) \sum_{[\gamma] \in L/F} ({}^{\#}F_\gamma)^{-1} \exp(-\|\gamma\|^2/4t).$$

Now we put

$$L_c = \{x \in L;\ \|x\|^2 = c\}$$

for each $c \geqq 0$. It is clear that $F(L_c) = L_c$ and $L/F = \bigcup_c L_c/F$ (disjoint). The number of elements in L_c is calculated as

$${}^{\#}L_c = \sum_{[\gamma] \in L_c/F} (F/F_\gamma) = {}^{\#}F \sum_{[\gamma] \in L_c/F} ({}^{\#}F_\gamma)^{-1},$$

so we obtain

$$\begin{aligned} D(t) &= (4\pi t)^{-n/2} \operatorname{Vol}(L\backslash\mathbf{R}^n) \sum_c \sum_{[\gamma] \in L_c/F} ({}^{\#}F_\gamma)^{-1} \exp(-c/4t) \\ &= (4\pi t)^{-n/2} \operatorname{Vol}(L\backslash\mathbf{R}^n)({}^{\#}F)^{-1} \sum_c {}^{\#}L_c \exp(-c/4t) \\ &= (4\pi t)^{-n/2} \operatorname{Vol}(L\backslash\mathbf{R}^n)({}^{\#}F)^{-1} \sum_{\gamma \in L} \exp(-\|\gamma\|^2/4t). \end{aligned}$$

If we use the Jacobi identity for the flat torus $T_M = L\backslash\mathbf{R}^n$, this last expression is equal to

$$({}^{\#}F)^{-1} \sum_{x \in L^*} \exp(-4\pi^2 \|x\|^2 t),$$

in which L^* denotes the dual lattice of L consisting of vectors x with $(x, L) \subset \mathbf{Z}$. Since $4\pi^2 \|x\|^2 (x \in L^*)$ is the spectrum of T_M, we have consequently the following.

PROPOSITION. *The spectrum of M determines that of T_M and the order of the holonomy group of M.*

Now we consider the correspondence $M \mapsto T_M$. It gives rise to a mapping, with finite fibers, of the set of isometry classes of compact flat manifolds into that of flat tori. To prove this, let Aut (L) be the subgroup of $O(n)$ consisting of $A \in O(n)$ with $A(L) = L$, which is obviously finite. Hence there are only finitely many subgroups of Aut (L). Let $M' = C'\backslash\mathbf{R}^n$ be any flat manifold such that the corresponding torus $T_{M'}$ is isometric to $T_M = L\backslash\mathbf{R}^n$. Then there exists an element k of $O(n)$ with $k^{-1}C'k \cap \mathbf{R}^n = L$, and therefore an exact sequence $0 \to L \xrightarrow{i} k^{-1}C'k \xrightarrow{\pi} F \to 1$ with $F \subset \mathrm{Aut}(L)$. On the other hand, it is a standard fact that for each subgroup $F \subset \mathrm{Aut}(L)$ the cohomology group $H^2(F, L)$ is finite, whence there are finitely many torsion free discrete subgroups $C_1, C_2, \ldots, C_h$ of $E(n)$ such that $C_1 \cap \mathbf{R}^n = L$, $\pi(C_i) = F$ $(i = 1, 2, \ldots, h)$, and any subgroup of $E(n)$ with same properties is equivalent to certain C_i as group extension of L by F. Especially, $k^{-1}C'k$ is equivalent to some C_i. Then, there exists an element a of $\mathbf{R}^n$ with $(I, a)C_i(I, a)^{-1} = k^{-1}C'k$, from which it follows that $M' = C'\backslash\mathbf{R}^n$ is isometric to $C_i\backslash\mathbf{R}^n$. This completes the proof.

In order to prove Theorem II, it is therefore enough to show that there are only finite number of isometry classes of flat tori with a given spectrum. But this is just a Kneser's result.

For the sake of completeness, we present here the proof which is based on the so-called Mahler's criterion (see A. Borel [2]). For this we denote by H^n the set of positive definite symmetric matrices of degree n, that is naturally identified with the coset space $O(n)\backslash GL(n, \mathbf{R})$. The group $GL(n, \mathbf{Z})$ of all integer matrices of determinants ± 1 acts on H^n as $A \mapsto {}^t gAg$. Then the isometry classes of n-dimensional flat tori are parametrized by the quotient $H^n/GL(n, \mathbf{Z})$ via the correspondence $g\mathbf{Z}^n\backslash\mathbf{R}^n \leftrightarrow {}^t gg$. Further, if $A = {}^t gg$, then the Jacobi identity for the torus $g\mathbf{Z}^n\backslash\mathbf{R}^n$ can be rewritten as

$$\sum_{i=0}^{\infty} \exp(-\lambda_i t) = |\det A|^{1/2}(4\pi t)^{-n/2} \sum_{z\in\mathbf{Z}^n} \exp(-{}^t zAz/4t).$$

Therefore the spectrum determines the value $|\det A|$, the set $\{{}^t zAz;\ z\in\mathbf{Z}^n\}$, and especially the value $\min_{z\in\mathbf{Z}^n-0} {}^t zAz$.

Given a A_0 of H^n, we consider the set

$$X=\left\{A\in H^n;\ \sum_{z\in\mathbf{Z}^n} \exp(-{}^t zAzt)=\sum_{z\in\mathbf{Z}^n} \exp(-{}^t zA_0zt) \quad \text{for any positive } t\right\}.$$

which is clearly a $GL(n,\mathbf{Z})$-invariant real analytic subset of H^n, and the image of X by the canonical projection $p: H^n \to H^n/GL(n,\mathbf{Z})$ coincides with the set of isometry classes of flat tori with the same spectrum as $g_0\mathbf{Z}^n\backslash\mathbf{R}^n$, g_0 being an element of $GL(n, \mathbf{R})$ with ${}^t g_0 g_0 = A_0$. Since the functions $A \mapsto \det A$ and $A \mapsto (\min_{z\in\mathbf{Z}^n-0} {}^t zAz)^{-1}$ are constant on X, it follows from Mahlar's criterion that the image $p(X)$ is compact in $H^n/GL(n, \mathbf{Z})$.

On the other hand, X is discrete in H^n. To prove this, let X_α be any connected component of X. Then any two points of X_α are joined by a piecewise analytic curve in X_α. Let $\varphi: s \to \varphi(s)$ be any analytic curve in X_α. Then as mentioned above, the subset $\{{}^t z\varphi(s)z;\ z\in\mathbf{Z}^n\}$ of $\mathbf{R}$ does not depend upon s, and hence for each $z\in\mathbf{Z}^n$ $s\mapsto {}^t z\varphi(s)z$ is a constant function. From this φ is a constant curve and thus X consists of only one point. The analyticity of X implies discreteness of X as desired.

Finally, noting that $GL(n, \mathbf{Z})$ acts discontinuously on H^n, we see that $p(X)$ is a finite subset of $H^n/GL(n, \mathbf{Z})$.

REFERENCES

[1] M. Berger, P. Gauduchon and E. Mazet, *Le Spectre d'une variété riemannienne*, Lecture Notes in Mathematics, 194. Springer Verlag 1971.

[2] A. Borel, *Introduction aux groupes arithmétiques*, Hermann, Paris, 1969.

[3] Y. Colin de Verdiere, *Spectre du laplacien et longueurs des géodésiques périodiques II*, Composito Mathematica, *27* (1973) 159–184.

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. II, Interscience Publishers, New York, 1969.

[5] H. B. Lawson and S. T. Yau, *Compact manifolds of non positive curvature*, J. Diff. Geom., 7(1972), 211–228.

[6] H. P. McKean, *Selberg's trace formula as applied to a compact Rieman surface*, Comm. Pure Appl. Math., *25* (1972), 225–246.
[7] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces with applications to Dirichlet series*, J. Indian Math. Soc., *20* (1956), 47–87.
[8] T. Sunada, *Closed geodesics in a locally symmetric space*, Tôhoku Math. J., *30* (1978), 59–68.
[9] J. A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, New York, 1967.

Received November 21, 1977

Department of mathematics
Nagoya University
Chikusa-ku, Nagoya