On the decomposition of a class of plane quasiconformal mappings.

Comment. Math. Helvetici **53** (1978) 15–27 Birkhäuser Verlag, Basel

On the decomposition of a class of plane quasiconformal mappings

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Dedicated to Albert Pfluger on the occasion of his seventieth birthday

1. Introduction

Let Q_I denote the class of quasiconformal mappings $w=f(z)$ of the unit disk $U=\{|z|<1\}$ onto $U=\{|w|<1\}$ with the property that the boundary values of f are those of the identity:

$$f(e^{i\theta})\equiv e^{i\theta}, \qquad 0\le\theta<2\pi.$$

If $f\in Q_I$, then the complex dilatation μ of f,

$$\mu(z)=f_{\bar z}/f_z$$

will be said to belong to class $\mathscr{F}$. The maximal dilatation of f is

$$K[f]=\frac{1+k[f]}{1-k[f]}, \qquad k[f]=\operatorname*{ess\,sup}_{z\in U}|\mu(z)|=\|\mu\|_\infty.$$

To avoid triviality we assume $k[f]>0$.

We will be concerned with certain questions related to the possibility of *decomposing* a given $f\in Q_I$ into factors

$$f=f_2\circ f_1, \qquad f_i\in Q_I, \qquad K[f_i]<K[f], \qquad i=1,2. \tag{1.1}$$

In particular, we will see that a decomposition satisfying (1.1) always exists. This should be contrasted with the known fact [4, pp. 215–216] that from the

[1] Work done with support from National Science Foundation grant MCS 75-06656A01.

assumptions that $\mu \in \mathscr{F}$ and $0<t<1$ it does *not* follow that $t\mu \in \mathscr{F}$. Thus, the attempt to construct f_i, $i=1,2$, as[2]

$$f_i = f^{t_i\mu}, \qquad 0<t_i<1, \qquad i=1,2,$$

does *not* work. In fact, as shown by Gehring [2], a decomposition of f of type (1.1), where $K[f_i]=K^{1/2}$, $i=1,2$, does not necessarily exist.

In Theorem 1 the role played by f_1 will be a mapping

$$f_1(z)=h(z,t), \qquad z\in U, \qquad 0\le t\le t_0(k), \qquad k=k[f], \tag{1.2}$$

close to the identity, and depending on the positive parameter t. In this case f_2 assumes the role of a *variation* $\tilde{f}$ of the mapping f,

$$\tilde{f}(t,z)=f\circ h^{-1}, \qquad 0\le t\le t_0(k), \qquad \tilde{f}(0,z)=f(z). \tag{1.3}$$

The construction of $h(z,t)$ for Theorem 1 requires the application of the Hahn-Banach theorem. On the other hand the process is sufficiently constructive so that the dilatations of all mappings involved are capable of being estimated explicitly in terms of k. An alternative construction of the f_i, avoiding the Hahn-Banach theorem, is found in Section 4. The resulting Theorem 3 has the advantage of leading to a very simple estimate for $K[f_i]$, but the disadvantage that a bound on $K[f]$ is assumed.

As a corollary of Theorems 2 and 3 a potentially quantitative version of a result of Earle and Eells [1] on the decomposition of $f\in Q_I$ into $(1+\varepsilon)$-quasiconformal mappings f_i,

$$f=f_n\circ f_{n-1}\circ\cdots\circ f_2\circ f_1, \qquad f_i\in Q_I, \tag{1.4}$$

is obtained (Section 3).

A decomposition of type (1.4) give rise to an *interpolating chain* $\mathscr{C}=\{F_i\}$ for f *within* Q_I,

$$F_0(z)=z, \qquad F_n(z)=f(z), \qquad F_i=f_i\circ f_{i-1}\circ\cdots\circ f_1, \qquad i=1,2,\ldots,n,$$

which connects f to the identity. If

$$K[F_i\circ F_{i-1}^{-1}]<1+\varepsilon, \qquad i=1,2,\ldots,n,$$

[2] $f^{\varkappa}$ denotes the quasiconformal mapping of U onto U with complex dilatation $\varkappa(z)$, normalized so that $f^{\varkappa}(1)=1$, $f^{\varkappa}(i)=i$, $f^{\varkappa}(-1)=-1$.

we shall say that $\mathscr{C}$ has *link size* ε. We define

$$R(\mathscr{C}) = \max_i K[F_i].$$

In Section 5 we shall see that $R(\mathscr{C})$ can be bounded in terms of $K[f]$ alone.

Before proceeding we must list a number of known results and an immediate corollary of one of them for reference later. In what follows

$$\|\varphi\| = \iint_U |\varphi(z)|\, dx\, dy,$$

and $\mathscr{B}$ denotes the Banach space of functions $\varphi(z)$ holomorphic in U, $\|\varphi\| < \infty$. $\mathscr{N}$ will be the class of all complex valued measurable functions $\nu(z)$, $z \in U$, such that

$$\|\nu\|_\infty < \infty, \qquad \iint_U \nu(z)\varphi(z)\, dx\, dy = 0 \quad \text{for all} \quad \varphi \in \mathscr{B}.$$

For a given quasiconformal mapping g of U onto U, Q_g denotes the class of quasiconformal mappings of U onto U which agree with g on ∂U. Each class Q_g contains at least one *extremal* member, G, in the sense that $K[G]$ is minimal.

We recall the following ([4], [6]):

THEOREM A. *If $\mu \in \mathscr{F}$ then, for any function $\varphi \in \mathscr{B}$,*

$$\left| \iint_U \frac{\mu(z)\varphi(z)}{1-|\mu(z)|^2}\, dx\, dy \right| \le \iint_U \frac{|\mu(z)|^2}{1-|\mu(z)|^2} |\varphi(z)|\, dx\, dy. \tag{1.5}$$

THEOREM B. *Suppose g is a quasiconformal mapping of U onto U with complex dilatation $\varkappa(z)$. If G is an extremal mapping in Q_g, $K[G] = (1+k^*)/(1-k^*)$, then*

$$\frac{k^*}{1-k^*} \le I[\varkappa] + \Delta[\varkappa], \tag{1.6}$$

where

$$I[\varkappa]=\sup_{\substack{\varphi\in\mathscr{B}\\ \|\varphi\|\le 1}}\left|\iint_U \frac{\varkappa(z)\varphi(z)}{1-|\varkappa(z)|^2}\,dx\,dy\right|, \tag{1.7}$$

and

$$\Delta[\varkappa]=\sup_{\substack{\varphi\in\mathscr{B}\\ \|\varphi\|\le 1}}\iint_U \frac{|\varkappa(z)|^2}{1-|\varkappa(z)|^2}|\varphi(z)|\,dx\,dy. \tag{1.8}$$

If $\varkappa(z)=t\nu(z)$, $0\le t<1/\|\nu\|_\infty$, then

$$\iint_U \frac{\varkappa\varphi}{1-|\varkappa|^2}\,dx\,dy=t\iint_U \nu\varphi\,dx\,dy+\iint_U \frac{\varkappa|\varkappa|^2}{1-|\varkappa|^2}\,\varphi\,dx\,dy.$$

If $\nu\in\mathscr{N}$, then the first integral on the right hand side vanishes, and we obtain

$$I[\varkappa]\le\frac{t^3\|\nu\|_\infty^3}{1-t^2\|\nu\|_\infty^2},\qquad \Delta[\varkappa]\le\frac{t^2\|\nu\|_\infty^2}{1-t^2\|\nu\|_\infty^2}.$$

As a corollary of Theorem B, we therefore have the following.

THEOREM C. *Suppose $\nu\in\mathscr{N}$, $0\le t<1/\|\nu\|_\infty$, and suppose $g=f^{t\nu}$. If G is an extremal mapping in Q_g, $K[G]=(1+k^*)/(1-k^*)$, then*

$$\frac{k^*}{1-k^*}\le\frac{t^2\|\nu\|_\infty^2}{1-t\|\nu\|_\infty}. \tag{1.9}$$

2. Variation of f in the class Q_I.

Following the notation of Section 1, we will prove the following.

THEOREM 1. *There exist functions $\delta(k)>0$, $t_0(k)>0$, and $C(k)$, defined for $0<k<1$, with the following properties. If $f\in Q_I$, $\|\mu\|_\infty=k=(K-1)/(K+1)$, $0\le t\le t_0(k)$, then there exists a mapping $h(z,t)\in Q_I$ such that*

$$K[h]\le 1+C(k)t \tag{2.1}$$

and

$$K[f \circ h^{-1}] \leq K - \delta(k)t. \tag{2.2}$$

Proof. The expression

$$L_\mu[\varphi] = \iint_U \frac{\mu}{1-|\mu|^2} \varphi \, dx \, dy$$

defines a bounded linear functional over $\mathcal{B}$, By the Hahn-Banach and Riesz representation theorems[3] there exists a complex valued measurable function $\tau(z)$, $z \in U$, such that

$$\|\tau\|_\infty = \sup_{z \in U} |\tau(z)| = \|L_\mu\|,$$

and

$$\iint_U \frac{\mu}{1-|\mu|^2} \varphi \, dx \, dy = \iint_U \tau\varphi \, dx \, dy, \quad \text{for all} \quad \varphi \in \mathcal{B}.$$

Hence,

$$\nu(z) = \frac{\mu(z)}{1-|\mu(z)|^2} - \tau(z) \in \mathcal{N}, \tag{2.3}$$

while, by Theorem A,

$$\|\tau\|_\infty = \|L_\mu\| \leq \frac{k^2}{1-k^2}. \tag{2.4}$$

Let

$$g_1 = f^{t\nu}, \qquad g_2 = f \circ g_1^{-1}, \qquad \left(0 \leq t < \frac{1}{\|\nu\|_\infty}\right). \tag{2.5}$$

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[3] Applications of the Hahn–Banach theorem in closely related situations can be found in [3] and [5].

The complex dilatation of g_2 is

$$\mu_{g_2}(\zeta)=\frac{\mu(z)-t\nu(z)}{1-t\overline{\nu(z)}\mu(z)}\frac{g_{1z}}{\overline{g_{1z}}}, \qquad (\zeta=g_1(z)). \tag{2.6}$$

We will show that there exist $\delta'(k)>0$, and $t_0'(k)>0$, such that

$$|\mu_{g_2}(\zeta)|\leq k-\delta'(k)t, \qquad 0\leq t\leq t_0'(k), \qquad z\in U, \qquad (\zeta=g_1(z)). \tag{2.7}$$

Let $\alpha=\alpha(k)$, $0<\alpha(k)<1$, be the solution of the equation

$$\frac{\alpha}{1-\alpha^2}=\frac{1}{2}\left(\frac{k^2}{1-k^2}+\frac{k}{1-k^2}\right)=\frac{k}{2(1-k)}. \tag{2.8}$$

Let

$$S_1=\{z\in U:|\mu(z)|\leq\alpha\},$$
$$S_2=\{z\in U:\alpha<|\mu(z)|\leq k\}.$$

Since $|\mu(z)|\leq\alpha<k$ for $z\in S_1$, it is obvious from (2.6) that there exist $\delta_1(k)>0$, $t_1(k)>0$, such that

$$|\mu_{g_2}(\zeta)|\leq k-\delta_1 t, \qquad 0\leq t\leq t_1, \qquad z\in S_1. \tag{2.9}$$

By (2.6),

$$|\mu_{g_2}(\zeta)|^2=\frac{|\mu|^2-2t\,\mathrm{Re}\,(\nu\bar\mu)+t^2|\nu|^2}{1-2t\,\mathrm{Re}\,(\nu\bar\mu)+t^2|\nu|^2|\mu|^2}.$$

Therefore, for $z\in S_2$, we have the development

$$|\mu_{g_2}(\zeta)|=|\mu(z)|-t\frac{1-|\mu(z)|^2}{|\mu(z)|}\mathrm{Re}\,[\nu(z)\overline{\mu(z)}]+0(t^2), \tag{2.10}$$

where the $0(t^2)$ term is uniform both with respect to z and to k, providing k is bounded away from 1. By (2.3),

$$\mathrm{Re}\,(\nu\bar\mu)=\mathrm{Re}\left[\frac{|\mu|^2}{1-|\mu|^2}-\tau\bar\mu\right]\geq\frac{|\mu|^2}{1-|\mu|^2}-|\tau|\,|\mu|=|\mu|\left[\frac{|\mu|}{1-|\mu|^2}-|\tau|\right].$$

Therefore, by (2.4), (2.8), the coefficient of $-t$ in (2.10) is bounded below by

$$|\mu|\left[1-\frac{1-|\mu|^2}{|\mu|}|\tau|\right]\geq\alpha\left[1-\frac{1-\alpha^2}{\alpha}\frac{k^2}{1-k^2}\right]=\left(\frac{1-k}{1+k}\right)\alpha, \qquad z\in S_2.$$

Hence, there exist $\delta_2(k)>0$, $t_2(k)>0$, such that

$$|\mu_{g_2}(\zeta)|\leq k-\delta_2 t, \qquad 0\leq t\leq t_2, \qquad z\in S_2.$$

Taking $\delta'(k)=\min(\delta_1,\delta_2)$, $t_0'(k)=\min(t_1,t_2)$, therefore establishes (2.7).

Next, we correct for the fact that g_1 does not necessarily belong to Q_I. By Theorem C, only a relatively small correction is required. Namely, let G_1 be extremal for the class Q_{g_1}. In view of (2.3), (2.4),

$$\|\nu\|_\infty\leq\frac{k}{1-k}. \tag{2.11}$$

Thus, Theorem C provides an estimate of the type

$$K[G_1]\leq 1+C'(k)t^2, \qquad 0\leq t\leq\frac{1-k}{2k}. \tag{2.12}$$

The mapping

$$h=G_1^{-1}\circ g_1 \tag{2.13}$$

evidently belongs to Q_I, and in view of (2.5), (2.11), and (2.12), it has the desired property (2.1) when $\delta(k)$, $t_0(k)$, $C(k)$ are chosen appropriately. On the other hand,

$$\tilde{f}=f\circ h^{-1}=g_2\circ G_1.$$

Thus, by employing (2.7) and (2.12), after possibly modifying $\delta(k)$, $t_0(k)$, $C(k)$, we obtain (2.2).

3. Decomposition of f

In Theorem 1 we may choose t as some specific value, say $t=\min[t_0(k),(K-1)/(2C(k))]$. As is easily seen by following the computations of

Section 2, the functions $t_0(k)$, $\delta(k)$, and $C(k)$ occurring in assertions (2.1) and (2.2) can be chosen as *continuous* functions of k, $0 \leq k < 1$. Writing $h = f_1$, $f \circ h^{-1} = f_2$, we can therefore assert the following.

THEOREM 2. *There exists a function $\Phi(K)$, defined for $1 \leq K < \infty$, with the following properties:*

(i) *$\Phi(K)$ is continuous, $1 \leq K < \infty$, $\Phi(1) = 1$,* (3.1)

(ii) *$\Phi(K) < K$, $1 < K < \infty$,* (3.2)

(iii) *If $f \in Q_I$, $K[f] = K$, then there exist $f_1 \in Q_I$, $f_2 \in Q_I$, such that $f = f_2 \circ f_1$, $K[f_i] \leq \Phi(K)$, $i = 1, 2$.* (3.3)

Suppose now the decomposition process referred to in (3.3) is iterated. Stage j results in a decomposition of f into 2^j factors each having a maximal dilatation not more than $\Phi_j(K[f])$, where

$$\Phi_1(x) = \Phi(x), \qquad \Phi_{j+1}(x) = \Phi(\Phi_j(x)), \qquad j = 1, 2, \ldots, .$$

As a consequence of (3.2)

$$\Phi_{j+1}(x) \leq \Phi_j(x), \qquad j = 1, 2, \ldots,$$

and, with the help of (3.1), it therefore follows that

$$\lim_{j \to \infty} \Phi_j(x) = 1, \qquad 1 \leq x < \infty.$$

Thus we see that *an interpolating chain $\mathscr{C} = \{F_i\}$* connecting f to the identity within *Q_I, with link size ε, exists*[4] [1].

It would be straightforward to convert the above to an estimate of the value of $n = N(K, \varepsilon)$ required to achieve the decomposition (1.4).

4. An alternative decomposition algorithm

We will now indicate a more symmetric and somewhat more elementary procedure for arriving at the factors f_1, f_2 of Theorem 2. However, the success of

[4] The desirability of approaching the Earle–Eells results in this fashion occurred to the author as a sequel to an oral communication from Professor A. Marden whom he would also like to thank for helpful remarks.

the procedure will be guaranteed only if $K[f]$ is sufficiently small. From Section 5 it appears, on the other hand, that under certain circumstances it may be useful to apply the decomposition process of Theorem 2 *in conjunction* with that of Theorem 3.

THEOREM 3. *If* $K[f] < M = (3+\sqrt{5})/2 = 2.61803398\ldots$, *then the assertions of Theorem* 2 *hold with*

$$\Phi(x) = x^{3/2} - x + 1, \qquad 1 \le x < M. \tag{4.1}$$

Proof. Let $g_1(z)$ be a quasiconformal mapping of U onto U with complex dilatation

$$\mu_1(z) = t\mu(z), \qquad t = \frac{K+1}{(K^{1/2}+1)^2}.$$

This makes

$$K[g_1] = \frac{1+tk}{1-tk} = K^{1/2}. \tag{4.2}$$

Let $g_2 = f \circ g_1^{-1}$. The complex dilatation μ_2 of g_2 is

$$\mu_2 \circ g_1(z) = \frac{\mu(z) - \mu_1(z)}{1 - \overline{\mu(z)}\mu_1(z)} \frac{g_{1z}}{\overline{g_{1z}}} = \frac{(1-t)\mu(z)}{1 - t\,|\mu(z)|^2} \frac{g_{1z}}{\overline{g_{1z}}}.$$

Hence,

$$\|\mu_2\|_\infty = \frac{(1-t)k}{1-tk^2}, \qquad \frac{1+\|\mu_2\|_\infty}{1-\|\mu_2\|_\infty} = K^{1/2}; \tag{4.3}$$

that is,

$$f = g_2 \circ g_1, \qquad K[g_i] = K^{1/2}, \qquad i = 1, 2,. \tag{4.4}$$

As in Section 2, we correct for the fact that g_1 need not belong to Q_I by introducing an extremal mapping G_1 from the class Q_{g_1}. Let

$$K[G_1] = K_1^* = \frac{1+k_1^*}{1-k_1^*}. \tag{4.5}$$

The mappings f_1, f_2 are then defined by means of

$$f_1 = G_1^{-1} \circ g_1, \qquad f_2 = g_2 \circ G_1. \tag{4.6}$$

By (4.4) and (4.5),

$$K[f_i] \le K^{1/2} K_1^*, \qquad i = 1, 2. \tag{4.7}$$

We now proceed to apply Theorems A and B, and (4.2), to estimate K_1^* from above. Since $\mu_1(z) = t\mu(z)$,

$$\iint_U \frac{\mu_1 \varphi}{1 - |\mu_1|^2}\, dx\, dy = t \iint_U \frac{\mu\varphi}{1 - |\mu|^2}\, dx\, dy - t(1 - t^2) \iint_U \frac{\mu |\mu|^2 |\varphi|}{(1 - |\mu|^2)(1 - t^2 |\mu|^2)}\, dx\, dy.$$

Using Theorem A to estimate the first integral on the right side, we deduce that

$$\left| \iint_U \frac{\mu_1 \varphi}{1 - |\mu_1|^2}\, dx\, dy \right| \le t \iint_U \frac{|\mu|^2 |\varphi|}{1 - |\mu|^2}\, dx\, dy + t(1 - t^2) \iint_U \frac{|\mu|^3 |\varphi|}{(1 - |\mu|^2)(1 - t^2 |\mu|^2)}\, dx\, dy.$$

Therefore,

$$I[\mu_1] \le \frac{tk^2}{1 - k^2} + \frac{t(1 - t^2)k^3}{(1 - k^2)(1 - t^2 k^2)} = \frac{(1 - t^2 k)tk^2}{(1 - k)(1 - t^2 k^2)}.$$

Evidently,

$$\Delta[\mu_1] \le \frac{t^2 k^2}{1 - t^2 k^2}.$$

Hence, by Theorem B,

$$\frac{k_1^*}{1 - k_1^*} \le \frac{(1 + t)tk^2}{(1 - k)(1 + tk)},$$

and, therefore,

$$K_1^* \le 1 + \frac{2(1 + t)tk^2}{(1 - k)(1 + tk)} \le 1 + \frac{4tk^2}{(1 - k)(1 + tk)}. \tag{4.8}$$

Substituting $K=(1+k)/(1-k)$, and the value of t as specified in (4.2) into the outer inequality in (4.8), we obtain

$$K_1^* \leq 1 + K^{-1/2}(K-1)(K^{1/2}-1). \tag{4.9}$$

Therefore, by (4.7),

$$K[f_i] \leq K^{3/2} - K + 1, \qquad i = 1, 2.$$

This completes the proof of Theorem 3. The upper bound M on $K[f]$ is the solution of the equation

$$M^{3/2} - M + 1 = M, \qquad (M > 1).$$

5. Bounds for $R(\mathscr{C})$

The principal result of the present section will be to establish that for a given $f \in Q_I$ an interpolating chain $\mathscr{C}$ may be constructed for which $R(\mathscr{C})$ is bounded in terms of $K[f]$.

LEMMA 5.1. *Suppose $f \in Q_I$, $K[f] = K \leq \frac{5}{4}$. Then, for any $\varepsilon > 0$, there exists an interpolating chain $\mathscr{C}$ for f with link size ε such that*

$$R(\mathscr{C}) \leq e^{2(K-1)}.$$

Proof. Let

$$x_0 = 1 + a_0, \qquad a_0 = K - 1. \tag{5.1}$$

Let $\Phi(x)$ be defined by (4.1), and let

$$x_{j+1} = \Phi(x_j), \qquad j = 1, 2, \ldots, . \tag{5.2}$$

In line with the remarks of Section 3 it will suffice to establish that the recursion formula (5.2) implies that

$$(x_j)^{(2^j)} \leq e^{2(K-1)}, \qquad j = 1, 2, \ldots . \tag{5.3}$$

We first note that

$$\Phi(x)=x^{3/2}-x+1\le 1+\tfrac{1}{2}(x-1)+\tfrac{1}{2}(x-1)^2, \qquad x\ge 1. \tag{5.4}$$

Let $\{a_j\}$ be any sequence of real numbers such that

$$a_{j+1}-a_j\ge 2^{-j}a_j^2, \qquad j=0,1,2,\dots. \tag{5.5}$$

Suppose

$$1\le x_m\le 1+2^{-m}a_m \tag{5.6}$$

for some nonnegative integer m. It then follows, by (5.2), (5.4), (5.5), that

$$1\le x_{m+1}\le 1+2^{-m-1}a_{m+1}.$$

Thus, by induction, (5.6) holds for $m=0,1,2,\dots$. For any number $b>0$, the sequence

$$a_j=\frac{b}{1+2^{-j+1}b}, \qquad j=0,1,2,\dots, \tag{5.7}$$

will satisfy (5.5). Therefore, if

$$\frac{b}{1+2b}=a_0=K-1, \quad \text{i.e.} \quad b=\frac{K-1}{3-2K}, \tag{5.8}$$

we conclude that

$$x_j\le 1+2^{-j}a_j\le 1+2^{-j}b, \qquad j=0,1,2,\dots,$$

and, therefore,

$$(x_j)^{(2^j)}\le(1+2^{-j}b)^{(2^j)}\le e^b\le e^{2(K-1)}, \qquad j=1,2,\dots,$$

as was to be shown.

THEOREM 4. *There exists a function $\Psi(K)$, defined for $1\le K<\infty$, with the following property: If $f\in Q_I$, then, for any $\varepsilon>0$, there exists an interpolating chain $\mathscr{C}$*

for f with link size ε such that

$$R(\mathscr{C}) \leq \Psi(K).$$

Proof. Given $f \in Q_I$ we apply the process of Section 3 to find

$$u_l \in Q_I, \qquad K[u_l] \leq \tfrac{5}{4}, \qquad l = 1, 2, \ldots, \qquad N = N(K, 0.25),$$

such that

$$f = u_N \circ u_{N-1} \circ \cdots \circ u_2 \circ u_1.$$

By applying Lemma 5.1 to each factor u_l we arrive at an interpolating chain $\mathscr{C}$ for f with

$$R(\mathscr{C}) \leq e^{2(K-1)N(K,0.25)}.$$

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Received May 16, 1977