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The homology of some groups of diffeomorphisms

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§1. Introduction

In this paper I will develop some methods, originated by Segal, for dealing with groups of homeomorphisms and diffeomorphisms. Their main application here is to proving the topological case of the Mather–Thurston theorem which relates groups of homeomorphisms to Haefliger's classifying space for foliations. However, §5 will discuss the C^r-case. Its conclusions are used in [8] to prove the C^r-version of this theorem, for $1 \leq r \leq \infty$.

The results can be stated as follows. Let X be a connected, topological n-manifold, possibly with boundary and with trivial ends. (Thus X is contained in a compact manifold $\bar{X}$, where $\bar{X} - X$ is a possibly empty union of components of the boundary of $\bar{X}$.) Let $\mathcal{H}om\, X$ be the homeomorphism group of X with the compact-open topology, and let $\text{Hom}\, X$ be the same group with the discrete topology. There is a natural map $B\,\text{Hom}\, X \to B\,\mathcal{H}om\, X$, and we denote its homotopy fibre by $\bar{B}\,\mathcal{H}om\, X$. Then the following is true:

ASSERTION. This homotopy fibre $\bar{B}\,\mathcal{H}om\, X$ has trivial integer homology.

A completely equivalent statement is that the mapping $B\,\text{Hom}\, X \to B\,\mathcal{H}om\, X$ is a homology equivalence, i.e. induces an isomorphism on homology with all local coefficients coming from $B\,\mathcal{H}om\, X$ (see [5]). In some cases the homology groups of $B\,\mathcal{H}om\, X$ are known (for instance the stability theorems of [6] V §5 give some information about $H_i(B\,\mathcal{H}om\, \mathbf{R}^n; \mathbf{Z})$ for $i \leq n+2$), and so this amounts to a computation of $H_*(B\,\text{Hom}\, X)$.

This assertion will be proved in §2 for all closed manifolds and for compact n-manifolds with boundary where $n \geq 2$. The case of non-compact manifolds and the case of the unit interval are not covered by this proof but do follow if we also make use of Segal [21]. (See Remark 2.16.) When X is closed this theorem is due to Thurston [23], but no proofs have yet been published except in the case $n = 1$

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(see [12] and [23]). However, in the C^r-case, $r \geq 1$, Mather [13] has worked out the details of the proofs of Thurston's results. See also [22].

More generally, if A is a closed subset of X there is a corresponding result for $\bar{B}\,\mathcal{H}om\,(X, \text{rel}\,A)$, where $\mathcal{H}om\,(X, \text{rel}\,A)$ is the group of all homeomorphisms of X which are the identity near A, with the direct-limit topology. (See §2.) In the case when X is compact and $\partial X \subseteq A$, this result is due to Mather [12] for $n = 1$ and to Thurston [23] for $n > 1$. Its main consequence is the following:

THEOREM 1.1. (Mather–Thurston) *$\bar{B}\Gamma_0^n$ is weakly contractible, i.e. has the weak homotopy type of a point.*

Here $\bar{B}\Gamma_0^n$ is Haefliger's classifying space for topological foliations of codimension n with trivialized normal bundles. Again, no published proof of this has appeared except when $n = 1$, see [12].

This paper is organized as follows. In §2, the above theorems are proved by means of an inductive technique very like the immersion-theoretic method described in [6] V. The arguments are based on two key results (Propositions 2.1 and 2.2). These are due to Segal, and I am very grateful to him for explaining them to me. The first one, which is proved in §4, says that for certain nice pairs of manifolds (X, Y), the inclusion $\bar{B}\,\mathcal{H}om_0\,(X, Y) \hookrightarrow \bar{B}\,\mathcal{H}om\,X$ is a weak equivalence, where $\mathcal{H}om_0\,(X, Y)$ is the identity component of the submonoid of $\mathcal{H}om\,X$ consisting of all homeomorphisms which embed Y into itself. The advantage of considering $\mathcal{H}om_0\,(X, Y)$ instead of $\mathcal{H}om\,X$ is that there is a restriction map from $\mathcal{H}om_0\,(X, Y)$ to $\mathcal{E}mb\,Y$, the monoid of self-embeddings of Y, and it is on the properties of this restriction map that the whole argument is based. Indeed, the second result, which is proved in §3, says that this restriction map gives rise to a homology fibration sequence. (See Proposition 3.8.) In §5, proofs are given of the extensions of these basic results to monoids of C^r-embeddings which are used in [8] to relate these monoids to the classifying space for foliations. Readers who are interested only in §5 should start there, referring back to §3 and §4 as necessary. The Appendix is an erratum to [8]. (The "thin" realization of $B\Gamma$ was used there instead of a "thick" realization.) Also, errors in the proof of the homology fibration theorem [9] Proposition 2 are corrected in Lemma 3.1 below.

I wish to thank D. B. A. Epstein very warmly for his detailed and constructive criticism. As well as pointing out the errors mentioned above, he suggested many improvements to the present paper, including the use of the category $\mathcal{HG}$ [24].

§2. The acyclicity of $\bar{B}\,\mathcal{H}om\,(X, \text{rel}\,A)$

We begin by establishing some notation which will be used throughout §§2–4. As already mentioned, X denotes a connected topological n-manifold, possibly with boundary, which has trivial ends. Y will denote an n-dimensional submanifold of X which is *cleanly embedded* in X (see [6] I §2). This means that the pair (X, Y) is locally homeomorphic to the pair $(\mathbf{R}^+\times\mathbf{R}^{n-1}, \mathbf{R}^+\times\mathbf{R}^+\times\mathbf{R}^{n-2})$ where $\mathbf{R}^+$ is the interval $[0,\infty)$. Thus Y is a closed subset of X. Also its frontier Fr Y is bicollared. We will call such a pair (X, Y) of manifolds a *nice manifold pair.* (Essentially the same definition was given in [8], modulo compactness assumptions.) Finally, A will denote a closed subset of X such that the set $(\operatorname{Fr} Y)-A$ is relatively compact in X.

We define $\mathcal{H}om\,(X, Y, \operatorname{rel} A)$ to be the subspace of $\mathcal{H}om\,(X, \operatorname{rel} A)$ consisting of all homeomorphisms h which embed Y in itself. Since h need not take Y onto Y, the elements of $\mathcal{H}om\,(X, Y, \operatorname{rel} A)$ form a monoid instead of a group. $\mathcal{H}om\,(X, Y, \operatorname{rel} A)$ is given the direct limit topology. Thus it is the limit $\varinjlim \mathcal{H}om\,(X, Y, \operatorname{rel} U)$, where U runs over the set of all open neighbourhoods of A in X, and where $\mathcal{H}om\,(X, Y, \operatorname{rel} U)=\{h\in\mathcal{H}om\,X: h(Y)\subseteq Y,\ h=\mathrm{id} \text{ on } U\}$ is given the compact-open topology. Since $\mathcal{H}om\,(X, Y, \operatorname{rel} U)$ is first countable, it is a k-space (i.e. a compactly generated Hausdorff space). Hence the direct limit $\mathcal{H}om\,(X, Y, \operatorname{rel} A)$ is also a k-space. We will work throughout this paper in Vogt's category $\mathcal{HG}$ (see [24] §5) which contains all k-spaces, and is better behaved than the category of k-spaces. In particular, any product $U\times V$ is to be understood as a product in $\mathcal{HG}$ and so has a topology which may be finer than the usual product topology. (More information about $\mathcal{HG}$ is given in §3 below.)

Now let $\mathcal{H}om_0\,(X, Y, \operatorname{rel} A)$ be the identity component of $\mathcal{H}om\,(X, Y, \operatorname{rel} A)$, and write $\operatorname{Hom}(X, Y, \operatorname{rel} A)$ and $\operatorname{Hom}_0(X, Y, \operatorname{rel} A)$ for the corresponding discrete monoids. Further, let $\operatorname{Emb}^X(Y, \operatorname{rel} A)$ (resp. $\operatorname{Emb}_0^X(Y, \operatorname{rel} A)$) denote the quotient $\operatorname{Hom}(X, Y, \operatorname{rel} A)/\sim$ (resp. $\operatorname{Hom}_0(X, Y, \operatorname{rel} A)/\sim$), where $h\sim h'$ if and only if $h=h'$ near Y. Thus $\operatorname{Emb}_0^X(Y, \operatorname{rel} A)$ is a discrete monoid whose elements can be thought of as germs at Y of embeddings of neighbourhoods of Y, which extend to homeomorphisms of X, are the identity near A, and moreover are homotopic to the identity through such embeddings. Note that such an embedding must take $\partial X\cap\partial Y$ into itself. Since $(\operatorname{Fr} Y)-A$ is relatively compact, it follows from the isotopy extension theorem of Edwards and Kirby that $\operatorname{Emb}_0^X(Y, \operatorname{rel} A)=\operatorname{Emb}_0^X(Y, \operatorname{rel} Y\cap A)$. (See Cor. 1.2 of [3], and the remark on p. 79 after its proof.) This need not be true for $\operatorname{Emb}^X(Y, \operatorname{rel} A)$. Also, $\operatorname{Emb}_0^X(Y, \operatorname{rel} A)=\operatorname{Emb}_0^Z(Y, \operatorname{rel} Z\cap A)$ whenever Z is a neighbourhood of Y in X.

Since $\operatorname{Emb}_0^X(Y, \operatorname{rel} A)$ does not have a natural (non-discrete, Hausdorff) topology, we define the associated topological monoid by a slightly different construction as follows. Let $\mathcal{E}mb\,(Y, \operatorname{rel} Y\cap A)$ be the space of all self-embeddings of Y which are the identity on some neighbourhood of $Y\cap A$, with the direct limit topology as before, and write $\mathcal{E}mb_0\,(Y, \operatorname{rel} Y\cap A)$ for its identity component. The monoid $\mathcal{E}mb_0^X(Y, \operatorname{rel} A)$ is defined to be the image of $\mathcal{H}om_0\,(X, Y, \operatorname{rel} A)$ in $\mathcal{E}mb_0\,(Y, \operatorname{rel} Y\cap A)$ under the restriction map, with the subspace topology. Since $(\mathrm{Fr}\, Y)-A$ is relatively compact, it follows from the results of [3] that the subspace and quotient topologies on $\mathcal{E}mb_0^X(Y, \mathrm{rel}\, A)$ coincide. Thus this monoid is a k-space. Notice also that $\mathcal{E}mb_0^X(Y, \mathrm{rel}\, A)$ equals $\mathcal{H}om_0(X, \mathrm{rel}\, A)$ if $Y=X$, and, by the isotopy extension theorem, equals $\mathcal{E}mb_0(Y, \mathrm{rel}\, A)$ if Y is a compact submanifold in the interior of X.

The homotopy fibre of the natural map from $B\,\mathrm{Emb}_0^X(Y, \mathrm{rel}\, A)$ to $B\,\mathcal{E}mb_0^X(Y, \mathrm{rel}\, A)$ will be called $\bar{B}\,\mathcal{E}mb_0^X(Y, \mathrm{rel}\, A)$. Since $\mathcal{H}om(X, \mathrm{rel}\, A)$ is a group, it follows from Lemma 3.5 below that $\bar{B}\,\mathcal{H}om_0(X, \mathrm{rel}\, A) \simeq \bar{B}\,\mathcal{H}om(X, \mathrm{rel}\, A)$. (Because it is not known whether the group $\mathcal{H}om\, X$ has the homotopy type of a CW complex, the symbol $\simeq$ will be used to denote a *weak* equivalence.)

We are now ready to state the two propositions which form the basis for our argument.

PROPOSITION 2.1. *If* (X, Y) *is a nice pair and* A *is a closed subset of* X *such that* $(\mathrm{Fr}\, Y)-A$ *is relatively compact, then the inclusion*

$$\bar{B}\,\mathcal{H}om_0(X, Y, \mathrm{rel}\, A) \hookrightarrow \bar{B}\,\mathcal{H}om(X, \mathrm{rel}\, A)$$

is a weak equivalence.

PROPOSITION 2.2. *If* X, Y, *and* A *are as in Proposition* 2.1, *then the sequence*

$$\bar{B}\,\mathcal{H}om_0(X, \mathrm{rel}\, Y\cup A) \to \bar{B}\,\mathcal{H}om_0(X, Y, \mathrm{rel}\, A) \xrightarrow{\bar{\rho}} \bar{B}\,\mathcal{E}mb_0^X(Y, \mathrm{rel}\, A)$$

is an integer homology fibration sequence. In other words, the inclusion of $\bar{B}\,\mathcal{H}om_0(X, \mathrm{rel}\, Y\cup A)$ *into the homotopy fibre of* $\bar{\rho}$ *induces an isomorphism on (untwisted) integer homology. Here* $\bar{\rho}$ *is the map obtained by restriction of homeomorphisms to (a neighbourhood of)* Y.

These two propositions are due to Segal, and will be proved in §3 and §4. The following are immediate corollaries. Recall that a space is said to be acyclic if it has trivial integer homology.

COROLLARY 2.3. *If* $\bar{B}\,\mathcal{H}om_0(X, \mathrm{rel}\, Y\cup A)$ *is acyclic, then* $\bar{\rho}$ *is a homology equivalence, i.e. it induces an isomorphism on homology for all local coefficients.*

This follows easily by the arguments of [5].

COROLLARY 2.4. *Let X, Y, and A be as in Proposition* 2.1. *Then, if* $\bar{B}\,\mathcal{H}om\,(X, \text{rel } Z)$ *is acyclic for* $Z = A$, Y *and* $Y \cup A$, *it is acyclic for* $Z = Y \cap A$ *also.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \bar{B}\,\mathcal{H}om_0\,(X, \text{rel } Y \cup A) & \longrightarrow & \bar{B}\,\mathcal{H}om_0\,(X, Y, \text{rel } A) & \xrightarrow{\sigma} & \bar{B}\,\mathcal{E}mb_0^X\,(Y, \text{rel } A) \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\| \Big\downarrow{\scriptstyle\gamma} \\ \bar{B}\,\mathcal{H}om_0\,(X, \text{rel } Y) & \longrightarrow & \bar{B}\,\mathcal{H}om_0\,(X, Y, \text{rel } Y \cap A) & \xrightarrow{\tau} & \bar{B}\,\mathcal{E}mb_0^X\,(Y, \text{rel } Y \cap A), \end{array}$$

where the vertical maps α, β, γ are inclusions. It was remarked above that γ is an equality. Also, since for any Z

$$\bar{B}\,\mathcal{H}om_0\,(X, \text{rel } Z) \simeq \bar{B}\,\mathcal{H}om\,(X, \text{rel } Z),$$

our hypotheses imply that both $\bar{B}\,\mathcal{H}om_0\,(X, \text{rel } Y \cup A)$ and $\bar{B}\,\mathcal{H}om_0\,(X, \text{rel } Y)$ are acyclic. Therefore, by Corollary 2.3, both σ and τ are homology equivalences. But, by Proposition 2.1, $\bar{B}\,\mathcal{H}om_0\,(X, Y, \text{rel } A) \simeq \bar{B}\,\mathcal{H}om\,(X, \text{rel } A)$ and so is acyclic. Therefore, $\bar{B}\,\mathcal{E}mb_0^X\,(Y, \text{rel } A) = \bar{B}\,\mathcal{E}mb_0^X\,(Y, \text{rel } Y \cap A)$ is acyclic. It follows that $\bar{B}\,\mathcal{H}om\,(X, \text{rel } Y \cap A)$, which is weakly equivalent to $\bar{B}\,\mathcal{H}om_0\,(X, Y \text{ rel } Y \cap A)$ by Proposition 2.1, is acyclic as well. □

Our first aim is to prove:

THEOREM 2.5. *If X is as above and if $X - A$ is relatively compact, then* $\bar{B}\,\mathcal{H}om\,(X, \text{rel } A)$ *is acyclic.*

Of course, this implies that if X is any paracompact manifold then $\bar{B}\,\mathcal{H}om_c\,X$ is acyclic, where $\mathcal{H}om_c\,X$ is the group of all compactly supported homeomorphisms of X, with the direct limit topology.

We will prove Theorem 2.5 by an inductive procedure, analogous to the immersion-theoretic method mentioned in [6] essay V. Because it is a double induction over X and A, we have to modify the scheme given there. Our starting points are Lemmas 2.6 and 2.6′ below. The inductive step is always completed by an application of Corollary 2.4. We will often have occasion to use the fact that if (X, Y) is a nice pair for which $Y \subseteq \text{Int}\, X$, then the spaces $\bar{B}\,\mathcal{H}om\,(X, \text{rel } \overline{X - Y})$ and $\bar{B}\,\mathcal{H}om\,(Y, \text{rel } \partial Y)$ are identical. We will refer to this as "excision." For convenience, we will often denote $\mathcal{H}om\,(X, \text{rel } \partial X)$ by $\mathcal{H}om\,(X, \text{rel } \partial)$. Also, if

$\lambda > 0$, we will write λD^n for the closed disc of radius λ centered at the origin in $\mathbf{R}^n$.

LEMMA 2.6. *$\bar{B}\,\mathcal{H}om\,(D^n, \mathrm{rel}\,\partial)$ is acyclic.*

Proof. Mather proved in [11] that $B\,\mathrm{Hom}\,(D^n, \mathrm{rel}\,\partial)$ is acyclic. Since $\mathcal{H}om\,(D^n, \mathrm{rel}\,\partial)$ is contractible by the Alexander trick, the result follows immediately. □

LEMMA 2.7. *For all $n \geq 1$ and $k = 0, \ldots, n$, $\bar{B}\,\mathcal{H}om\,(S^k \times D^{n-k}, \mathrm{rel}\,\partial)$ is acyclic.*

Proof. By induction on k. For $k = 0$, this follows immediately from Lemma 2.6. So, suppose it has been proved for all $k' < k$, where $k \leq n$. Decompose S^k as $L \cup M$, where L is the lower disc $\{x \in S^k : x_{k+1} \leq \frac{1}{2}\}$ and M is the upper disc $\{x \in S^k : x_{k+1} \geq -\frac{1}{2}\}$, and let Y and A be the complements in $X = S^k \times D^{n-k}$ of certain thickenings of L and M, respectively. For instance, we may take $Y = S^k \times D^{n-k} - \mathrm{Int}\,(L \times \frac{1}{2}D^{n-k})$ and $A = S^k \times D^{n-k} - \mathrm{Int}\,(M \times \frac{1}{2}D^{n-k})$. Then (X, Y) is a nice manifold pair. It follows easily from Lemma 2.6 that $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\,Z)$ is acyclic when $Z = A$ or Y. Similarly, using the inductive hypothesis, one sees that $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\,Y \cup A)$ is acyclic. Therefore, $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\,Y \cap A)$ is acyclic by Corollary 2.4. But, by excision, $\mathcal{H}om\,(X, \mathrm{rel}\,Y \cap A) \cong \mathcal{H}om\,(S^k \times D^{n-k}, \mathrm{rel}\,\partial)$ and so the inductive step is complete. □

LEMMA 2.8. *Let (X, A) be a nice manifold pair with $\partial X \subseteq A$. Then, if (X, A) has a finite handle decomposition, $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\,A)$ is acyclic.*

Proof. The proof is by induction on the number, p, of handles in a handle decomposition

$$A = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_p = X$$

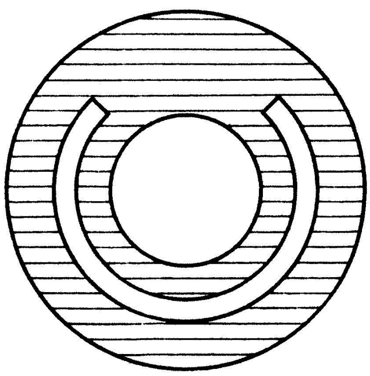

Figure 2.1. Here $Y \subseteq S^1 \times D^1$ is shaded.

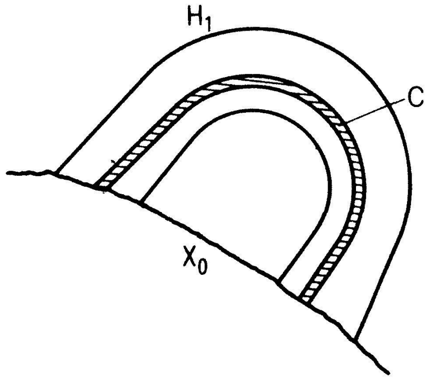

Figure 2.2

of (X, A). Here, each handle $H_i = \overline{X_i - X_{i-1}}$ is a cleanly embedded submanifold of X such that $(H_i, H_i \cap X_{i-1}) \cong (D^k, \partial D^k) \times D^{n-k}$ for some k. Suppose, inductively, that we have proved the lemma for all pairs (Z, B) which have a handle decomposition with $<p$ handles, and consider the pair (X, A) above. Let $(H_1, H_1 \cap X_0) \cong (D^k, \partial D^k) \times D^{n-k}$ and let C denote the core $D^k \times \frac{1}{2} D^{n-k} \subseteq D^k \times D^{n-k}$ of H_1.

Then, because $\partial X \subseteq A = X_0$, the pairs $(X, X_0 \cup C)$ and $(X, X_0 \cup H_1) = (X, X_1)$ are homeomorphic. Therefore, $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, X_0 \cup C)$ is acyclic by the inductive hypothesis. Also, if $Y = X \dot{-} \mathrm{Int}\, H_1$, it is easy to see that both $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, Y)$ and $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, Y \cup X_0 \cup C)$ are acyclic. (The latter space is homeomorphic to $\bar{B}\,\mathcal{H}om\,(D^{k+1} \times S^{n-k-1}, \mathrm{rel}\, \partial)$ by excision, and so is acyclic by Lemma 2.7.) It now follows from Corollary 2.4. that $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, X_0)$ is acyclic, which completes the inductive step. □

NOTE 2.9. The details of the proof of Theorem C in [8] may be filled in by the arguments of Lemma 2.7 and 2.8. Care should be taken to round off corners properly.

We can sharpen Lemma 2.8, showing that $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, A)$ is acyclic for any pair where X is compact and $\partial X \subseteq A$, by the usual trick of using a handlebody cover of (X, A). We first remark:

LEMMA 2.10. *If $A_1 \supseteq A_2 \supseteq \cdots$ is a sequence of compact subsets of X with intersection A such that $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, A_i)$ is acyclic for each i, then $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, A)$ is acyclic.*

Proof. This is immediate. □

LEMMA 2.11. *Let X be a compact manifold, and $A \subseteq X$ a closed subset containing ∂X. Then $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, A)$ is acyclic.*

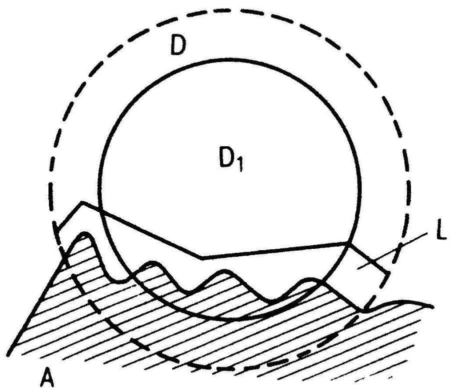

Figure 2.3

Proof. Since $\partial X \subseteq A$, it follows from Lemma 2.10 that we may assume that A contains a collar neighbourhood of ∂X. Then $\overline{X-A}$ has a finite covering by the interiors $U_1, \ldots, U_l$ of closed discs $D_1, \ldots, D_l$ which are cleanly embedded in $\operatorname{Int} X$. Let us suppose that the lemma has been proved for all pairs (Y, B) as above, where $\overline{Y-B}$ has a covering by $<k$ such discs, and let $\overline{X-A}$ be covered by $U_1, \ldots, U_k$, where $k \geq 1$. Since D_1 is cleanly embedded in $\operatorname{Int} X$, it is contained in the interior of a larger closed disc, D say. Choose a sequence $L_1 \supseteq L_2 \supseteq \cdots$ of PL-submanifolds of D which have intersection $A \cap D$ and are such that the pairs $(X, \overline{X-D} \cup L_i)$ all satisfy the conditions of Lemma 2.8. Then each $\bar{B}\,\mathcal{H}om\,(X, \operatorname{rel} \overline{X-D} \cup L_i)$ is acyclic. Moreover, by the inductive hypothesis $\bar{B}\,\mathcal{H}om\,(X, \operatorname{rel} Z)$ is acyclic when Z is $D_1 \cup A$ or $D_1 \cup A \cup (\overline{X-D} \cup L_i)$. Therefore, by Corollary 2.4, it is acyclic when $Z = (D_1 \cup A) \cap (\overline{X-D} \cup L_i) = A \cup (D_1 \cap L_i)$. Hence, by Lemma 2.10, $\bar{B}\,\mathcal{H}om\,(X, \operatorname{rel} A)$ is acyclic, which completes the inductive step. □

In order to deal with the case when $\partial X \not\subseteq A$, it is convenient first to consider pairs of the form $(X \times I, X \times \{0\} \cup A \times I)$, where A is a closed subset of X containing ∂X, since this will enable us to handle the boundary region. As analogues of Lemmas 2.6 and 2.8, we have:

LEMMA 2.6′. *If* $n \geq 1$, $\bar{B}\,\mathcal{H}om\,(D^n \times I, \operatorname{rel}\,(D^n \times \{0\} \cup \partial D^n \times I))$ *is acyclic.*

Proof. It is not hard to prove that $B\,\mathrm{Hom}\,(D^n \times I, \operatorname{rel}\,(D^n \times \{0\} \cup \partial D^n \times I))$ is acyclic, using Mather's techniques in [11]. Details will be left to the reader. Since $\mathcal{H}om\,(D^n \times I, \operatorname{rel}\,(D^n \times \{0\} \cup \partial D^n \times I))$ is contractible by the Alexander trick, the result follows. □

LEMMA 2.8′. *If the pair* (X, A) *is as in Lemma* 2.8 *then*

$$\bar{B}\,\mathcal{H}om\,(X \times I, \operatorname{rel}\,(X \times \{0\} \cup A \times I))$$

is acyclic.

Proof. In the case that X has dimension 0, so that $X-A$ is a collection of points, this result is proved by Segal in [21] §2. When $\dim X \geqslant 1$, it may be proved by modifying the proofs of Lemmas 2.7 and 2.8 in the obvious way, using Lemma 2.6′ in place of Lemma 2.6. □

With these preliminaries, we may now prove Theorem 2.5.

Proof of Theorem 2.5. We have to show that $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, A)$ is acyclic when $X-A$ is relatively compact. Recall that X has the form $\bar{X}-B$ where $\bar{X}$ is compact and B is a union of components of $\partial\bar{X}$. Since $X-A$ is relatively compact, A must contain a neighbourhood of B in X. Therefore $\mathcal{H}om\,(X, \mathrm{rel}\, A) \cong \mathcal{H}om\,(\bar{X}, \mathrm{rel}\, A \cup B)$, and it suffices to consider the case when X is compact.

Suppose additionally that the pair $(\partial X, A \cap \partial X)$ satisfies the conditions of Lemma 2.8. By Lemma 2.10 we may also suppose that A is a product near ∂X, that is, that there is a collar neighbourhood $\partial X \times I \hookrightarrow X$ of $\partial X \equiv \partial X \times \{0\}$ such that $A \cap (\partial X \times I) = (A \cap \partial X) \times I$. Set $Y = X - \partial X \times [0, \frac{1}{2})$. Then $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, Y \cup A)$ is acyclic by Lemma 2.8′. Also, by Lemma 2.11, $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, Z)$ is acyclic when $Z = A \cup \partial X$ and $A \cup \partial X \cup Y$. Therefore, since our hypotheses ensure that $Y \cup A$ is a nice submanifold of X, it follows from Corollary 2.4 that $\bar{B}\,\mathcal{H}om\,(X, \mathrm{rel}\, A)$ is acyclic.

The proof for general A is completed by an inductive argument as in Lemma 2.11. Thus one argues by induction over the number of discs in ∂X which cover the set $\overline{\partial X - A}$. Further details are left to the reader. □

COROLLARY 2.12. *$\bar{B}\,\mathcal{H}om\, D^n$ is acyclic.*

COROLLARY 2.13. *The restriction map $B\,\mathrm{Hom}_0\, D^n \to B\,\mathrm{Hom}_0\, S^{n-1}$ induces an isomorphism on integer homology.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} B\,\mathrm{Hom}_0\, D^n & \xrightarrow{\alpha} & B\,\mathrm{Hom}_0\, S^{n-1} \\ \downarrow & & \downarrow \\ B\,\mathcal{H}om_0\, D^n & \xrightarrow{\beta} & B\,\mathcal{H}om_0\, S^{n-1}. \end{array}$$

The vertical maps are homology equivalences by Theorem 2.5. Also, because the map $\mathcal{H}om_0\, D^n \to \mathcal{H}om_0\, S^{n-1}$ splits and has contractible fibre, β is a homotopy equivalence. Thus α must induce an isomorphism on homology for all coefficient systems on $B\,\mathrm{Hom}_0\, S^{n-1}$ coming from $B\,\mathcal{H}om_0\, S^{n-1}$. These coefficient systems will always be trivial, since $B\,\mathcal{H}om_0\, S^{n-1}$ is simply connected. □

Note. The same argument as that given above shows that

$$B \operatorname{Hom} D^n \to B \operatorname{Hom} S^{n-1}$$

is a homology isomorphism for all local coefficients induced from $B\,\mathscr{H}om\, S^{n-1}$. Now $\operatorname{Hom}_0 S^{n-1}$ is perfect (in fact simple – see [3] p. 80) and therefore any abelian coefficient system on $B \operatorname{Hom} S^{n-1}$ is induced from $\pi_0(\mathscr{H}om\, S^{n-1})$. (The definition of abelian coefficient system is given in §3.) It follows that $B \operatorname{Hom} D^n \to B \operatorname{Hom} S^{n-1}$ is an abelian homology equivalence. We remark that $\pi_0(\mathscr{H}om\, S^{n-1}) = \mathbf{Z}_2$ if $n \neq 5$ and is unknown if $n=5$ (see [6]).

COROLLARY 2.14. *If (X, Y) is a nice pair in which Y is a compact submanifold of* $\operatorname{Int} X$, *then $\bar{B}\,\mathscr{E}mb_0^X\, Y$ is weakly contractible.*

Proof. By replacing X by a compact neighbourhood of Y in $\operatorname{Int} X$ we may suppose that X is compact. Since both $\bar{B}\,\mathscr{H}om_0\,(X, \operatorname{rel} \partial)$ and $\bar{B}\,\mathscr{H}om_0(X, \operatorname{rel} Y \cup \partial)$ are then acyclic by Theorem 2.5, it follows from Corollary 2.3 that $\bar{B}\,\mathscr{E}mb_0^X\, Y$ is acyclic. However, $\pi_1 \bar{B}\,\mathscr{E}mb_0^X\, Y$ is abelian (see Corollary 5.5) and so the homotopy groups of $\bar{B}\,\mathscr{E}mb_0^X\, Y$ all vanish. $\square$

COROLLARY 2.15. *$\bar{B}\,\mathscr{E}mb\,\mathbf{R}^n$ is weakly contractible.*

Proof. Since $\mathscr{E}mb\,\mathbf{R}^n \simeq \mathscr{H}om\,\mathbf{R}^n$ (see [7], Theorem 1), it follows from Lemma 3.5 that $\bar{B}\,\mathscr{E}mb\,\mathbf{R}^n \simeq \bar{B}\,\mathscr{E}mb_0\,\mathbf{R}^n$. Consider the maps

$$\mathscr{E}mb_0^{\mathbf{R}^n} D^n \xleftarrow{\ \rho\ } \mathscr{E}mb_0\,(\mathbf{R}^n, D^n) \xhookrightarrow{\ i\ } \mathscr{E}mb_0\,\mathbf{R}^n.$$

It is easy to check that the inclusion i is a weak equivalence. Further, by [3], the restriction ρ is a weak fibration whose fibre consists of all embeddings which are the identity on D^n. Since this fibre is contractible, ρ is a weak equivalence. Segal considers the corresponding diagram for B Emb in [21] (2.7), (2.8) and shows that

$$B \operatorname{Emb}_0^{\mathbf{R}^n} D^n \xleftarrow{\ \simeq\ } B \operatorname{Emb}_0\,(\mathbf{R}^n, D^n) \xhookrightarrow{\ \simeq\ } B \operatorname{Emb}_0 \mathbf{R}^n.$$

Thus $\bar{B}\,\mathscr{E}mb_0\,\mathbf{R}^n \simeq \bar{B}\,\mathscr{E}mb_0^{\mathbf{R}^n} D^n$, and the result follows from Corollary 2.14. $\square$

REMARK 2.16. Segal proves in [21] that if X is any compact C^r-manifold, where $0 \le r \le \infty$, then $B \operatorname{Diff}^r(X \times [0,\infty), \operatorname{rel} X \times \{0\})$ is acyclic. Since

$$\mathscr{D}iff^r(X \times [0,\infty), \operatorname{rel} X \times \{0\})$$

is contractible, $\bar{B}\,\mathscr{D}iff^r(X \times [0,\infty), \operatorname{rel} X \times \{0\})$ is acyclic too. (Here $\mathscr{D}iff^r X$ denotes the group of C^r-diffeomorphisms of X, with the compactopen C^r-topology.) Therefore, if X is any manifold with trivial ends, one can prove that $\bar{B}\,\mathcal{H}om\,X$ is acyclic by using Segal's result instead of Lemma 2.8$'$. More generally, Segal's argument can be adapted to show that if A is any closed subset of a compact manifold X then $B\ \mathrm{Diff}^r\,(X\times[0,\infty),\mathrm{rel}\,X\times\{0\}\cup A\times[0,\infty))$ is acyclic. Hence, $\bar{B}\,\mathcal{H}om\,(X,\mathrm{rel}\,A)$ is acyclic for any pair (X,A), where X is a manifold with trivial ends and A is a closed subset of X which is a product near ∞.

We finish this section by proving Theorem 1.1.

Proof of Theorem 1.1. We will show that $\bar{B}\Gamma_0^n\simeq\bar{B}\,\mathcal{E}mb\,\mathbf{R}^n$ and then apply Corollary 2.15.

Haefliger defines $\bar{B}\Gamma_0^n$ in [4] II 4 to be the homotopy fibre of the 'differential' $\nu:B\Gamma_0^n\to B\,\mathcal{T}op_n$, where Γ_0^n is the topological groupoid of germs of homeomorphisms of $\mathbf{R}^n$ with the sheaf topology and $\mathcal{T}op_n$ is the quasi-topological group of germs at 0 of homeomorphisms of $\mathbf{R}^n$ which fix 0. Thus $B\,\mathcal{T}op_n$ classifies (numerable) microbundles. (Recall that we are working in Vogt's category $\mathcal{H}\mathcal{G}$. Since this is isomorphic to the category of quasi-topological spaces, we will consider both $\mathcal{T}op_n$ and $B\,\mathcal{T}op_n$ to belong to it. Also, we use thick realizations here: see the beginning of §3.)

Consider the following commutative diagram:

$$\begin{array}{ccccc}
B\,\mathrm{Emb}\,\mathbf{R}^n & \xleftarrow{\alpha_1} & \mathrm{Emb}\mathbf{R}^n\,\backslash\!\backslash\,\mathbf{R}^n & \xrightarrow{\beta_1} & B\Gamma_0^n \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\nu} \\
B\,\mathcal{E}mb\,\mathbf{R}^n & \xleftarrow{\alpha_2} & \mathcal{E}mb\,\mathbf{R}^n\,\backslash\!\backslash\,\mathbf{R}^n & \xrightarrow{\beta_2} & B\,\mathcal{T}op_n.
\end{array}$$

Here, the spaces $\mathrm{Emb}\,\mathbf{R}^n\,\backslash\!\backslash\,\mathbf{R}^n$ and $\mathcal{E}mb\,\mathbf{R}^n\,\backslash\!\backslash\,\mathbf{R}^n$ are formed, as explained in §3 below, from the natural actions of the monoids of embeddings on $\mathbf{R}^n$. The left-hand vertical maps are the obvious bijections. The maps α_1,α_2 are obtained by collapsing $\mathbf{R}^n$ to a point, and are weak equivalences by Lemma 3.1. The map β_1 is induced by the functor $\mathcal{C}(\mathrm{Emb}\,\mathbf{R}^n\,\backslash\!\backslash\,\mathbf{R}^n)\to\Gamma_0^n$ (notation as in §3) which is the identity on objects and which takes the morphism $m:x\to mx$ to the germ of m at x. Similarly, β_2 is induced by the functor which takes the morphism $m:x\to mx$ to the germ of $\mu_{mx}^{-1}m\mu_x$ at 0, where μ_x is translation by x. Segal shows in [21] §1 that β_1 is a weak equivalence. The map β_2 is also a weak equivalence. This is perhaps most easily seen by considering the commutative diagram

$$\begin{tikzcd}
\mathcal{E}mb\,\mathbf{R}^n\,\backslash\!\backslash\,\mathbf{R}^n \arrow[r, "\beta_2"] \arrow[d, "\simeq"', "\alpha_2"] & B\,\mathcal{T}op_n \\
B\,\mathcal{E}mb\,\mathbf{R}^n & B\,\mathcal{H}om\,(\mathbf{R}^n,0) \arrow[l, hook', "\alpha_5"] \arrow[ul, "\alpha_4"] \arrow[u, "\alpha_3"']
\end{tikzcd}$$

where $\mathscr{H}om(\mathbf{R}^n, 0)$ is the group of homeomorphisms of $\mathbf{R}^n$ which fix 0, with the compact-open topology. The maps α_4, α_5 are the obvious inclusions, and α_3 is induced by the homomorphism which takes an element of $\mathscr{H}om(\mathbf{R}^n, 0)$ to its germ at 0. Since $\mathscr{E}mb\,\mathbf{R}^n \simeq \mathscr{H}om(\mathbf{R}^n, 0)$ by [7], the map α_5 is an equivalence. Kister also shows that $B\,\mathscr{H}om(\mathbf{R}^n, 0)$ classifies microbundles over finite complexes. Hence α_3 is a weak equivalence. Thus β_2 is a weak equivalence. It follows that $\bar{B}\,\mathscr{E}mb\,\mathbf{R}^n \simeq \bar{B}\Gamma_0^n$ as required. □

§3. Homology fibrations formed from monoids

The aim of this section is to prove Proposition 2.2. Before beginning the proof we will recall some facts about classifying spaces of monoids. As in §2 we work in Vogt's category $\mathscr{H}\mathscr{G}$ [24]. This is a full subcategory of the category $\mathscr{T}$ of topological spaces which is convenient to use because it contains the spaces of interest to us, and has a categorical product which commutes with quotient maps. In fact, there is a retraction $k: \mathscr{T} \to \mathscr{H}\mathscr{G}$ which preserves weak homotopy type and is defined as follows: for each space X in $\mathscr{T}$ the space $k(X)$ has the same underlying set as X and has the finest topology such that any continuous map from a compact Hausdorff space S to X is continuous when considered as a map $S \to k(X)$. Thus $\mathscr{H}\mathscr{G}$ contains all k-spaces, and is closed under taking quotients. In particular, if $\mathscr{C}$ is a topological category, such as the groupoid Γ, whose spaces of objects and of morphisms are in $\mathscr{H}\mathscr{G}$, then any realization of $\mathscr{C}$ also belongs to $\mathscr{H}\mathscr{G}$.

A *topological monoid* $\mathscr{M}$ is a space (in $\mathscr{H}\mathscr{G}$) with a strictly associative multiplication $\mathscr{M} \times \mathscr{M} \to \mathscr{M}$ and identity element e. We will assume throughout this paper that $\mathscr{M}$ is *good* (see [19] Appendix A). This means that the inclusion $e \hookrightarrow \mathscr{M}$ is a closed cofibration. For this, it suffices that $\mathscr{M}$ be normal and locally contractible, and so all the monoids mentioned in §2 are good.

Suppose now that $\mathscr{M}$ acts on the left on a space Z. We denote by $\mathscr{C}(\mathscr{M} \backslash\backslash Z)$ the (topological) category whose space of objects is Z and whose space of morphisms is $\mathscr{M} \times Z$, where the pair (m, z) represents the morphism $m: z \to mz$. The realization of this category will be called $\mathscr{M} \backslash\backslash Z$. For reasons which will soon be apparent, we use the ordinary or "thin" realization here, except when it is explicitly mentioned to the contrary. When $\mathscr{M}$ is good, the simplicial space associated to $\mathscr{C}(\mathscr{M} \backslash\backslash Z)$ is also good and all its realizations have the same weak homotopy type (see [19] Appendix A). Therefore, as far as homotopy theoretic statements go, the choice of realization for $\mathscr{C}(\mathscr{M} \backslash\backslash Z)$ is immaterial. (However, one must always use a thickened realization for $B\Gamma$. For the thin realization of $B\Gamma$ may well have a different homotopy type from the thickened ones, and it is the latter which classify foliations. For example, Haefliger uses the join realization in [4], which is a thickened realization: see [21] §4.)

We will often be in the situation when two monoids $\mathcal{M}$ and $\mathcal{N}$ act on Z, one on the left and one on the right. We may form $\mathcal{M}\backslash\!\backslash Z$ as above, and $Z/\!/\mathcal{N}$ similarly. If the actions of $\mathcal{M}$ and $\mathcal{N}$ commute, then $\mathcal{M}$ and $\mathcal{N}$ also act on $Z/\!/\mathcal{N}$ and $\mathcal{M}\backslash\!\backslash Z$, respectively, so that we may form $\mathcal{M}\backslash\!\backslash(Z/\!/\mathcal{N})$ and $(\mathcal{M}\backslash\!\backslash Z)/\!/\mathcal{N}$. It is not difficult to check that these spaces are homeomorphic. (The proof uses the fact that products and quotients commute in the category $\mathcal{HG}$. See [24] 3.8.) The two spaces $*/\!/\mathcal{M}$ and $\mathcal{M}\backslash\!\backslash *$, where $*$ denotes a one point space, are also homeomorphic, see [17] p. 83, and are called $B\mathcal{M}$.

Now, recall from [9] that a coefficient system $\mathcal{A}$ of abelian groups on Y is called *abelian* if it is the pullback over $Y\to K(G,1)$ of a coefficient system on $K(G,1)$, where G is an abelian group. Further, a map $f:X\to Y$ is called a *homology* (resp. *abelian homology, integer homology*) *equivalence* if it induces an isomorphism $H_*(X, f^*\mathcal{A})\to H_*(Y,\mathcal{A})$ for all coefficient systems $\mathcal{A}$ on Y (resp. for all abelian systems, for $\mathcal{A}=\mathbf{Z}$). Finally a sequence of maps $F\xrightarrow{\alpha}E\xrightarrow{\beta}B$ (where B is connected) is called a *homotopy* (resp. an *abelian homology*) *fibration sequence* if there is an associated map of F into the homotopy fibre of β which is a weak homotopy (resp. an abelian homology) equivalence. We will usually be concerned with sequences of pointed spaces in which $F\subset\beta^{-1}(*)$. In this situation there is a natural inclusion of F into the homotopy fibre of β at $*$. For example, if $\mathcal{K}$ is the kernel of the monoid homomorphism $\mathcal{M}\to\mathcal{N}$, then the image of $B\mathcal{K}$ in $B\mathcal{N}$ is the single point $B\{e\}$. (Notice that if we had used one of the alternative realization functors of [19], then $B\{e\}$ would have been a large contractible space. This would have complicated certain proofs (e.g. Proposition 3.8, Lemma 5.6).)

LEMMA 3.1. *Suppose that $\mathcal{N}$ acts on Z on the right, and that, for each $n\in\mathcal{N}$, the map $z\mapsto zn$ of Z to itself is a weak* (*resp. an abelian homology*) *equivalence. Then*

$$Z\to Z/\!/\mathcal{N}\to */\!/\mathcal{N}$$

is a homotopy (*resp. an abelian homology*) *fibration sequence.*

(There are, of course, corresponding results for the other kinds of homology equivalences.)

Proof. When $\mathcal{N}$ acts on Z by weak equivalences, it follows by applying [2] (1.3) (1.4) (1.5) that the map $Z/\!/\mathcal{N}\to */\!/\mathcal{N}$ is a quasifibration with fibre Z. (Quillen proves this result in [17] p. 90 in the case when $\mathcal{N}$ is discrete.) When $\mathcal{N}$ acts on Z by homology equivalences, the result is Proposition 2 of [9]. D. B. A. Epstein pointed out that there are two errors in the proof given there, namely:

(i) Since the path space P in Proposition 5 need not be paracompact even if B is, one cannot assume that B is paracompact in Proposition 6.

(ii) Since $\mathscr{B}$ is not necessarily closed under finite intersections, it is not clear why $|\mathscr{B}|\simeq B$ in Proposition 6.

In fact, both these errors may be corrected by appealing to the covering lemma [21] Proposition A.5, which is quoted as Lemma 4.2 below. For, in that lemma take $X=B$, let $\mathscr{C}$ be the discrete category formed by the sets in $\mathscr{B}$ and their inclusions, and let $\tilde{F}$ be the inclusion functor. Then $|\mathscr{B}|$ is just B_F. Since the sets in $\mathscr{B}$ form a basis for the topology of B, one can, given sets $U_1, \ldots, U_k$ in $\mathscr{B}$ and a point $x \in U_1 \cap \cdots \cap U_k$, find a set U in $\mathscr{B}$ such that $x \in U \subset U_1 \cap \cdots \cap U_k$. It follows easily that each category $\mathscr{C}_x$ is filtering. Therefore, each space $\|\mathscr{C}_x\|$ is contractible, and $B_F \xrightarrow{\alpha} B$ as required. $\square$

Notice that if $\mathcal{N}$ is grouplike (that is, if $\pi_0\mathcal{N}$ is a group) then any action of $\mathcal{N}$ on a space is by homotopy equivalences. Hence $Z \to Z/\!\!/\mathcal{N} \to * /\!\!/\mathcal{N}$ is always a homotopy fibration sequence in this case. As an example, suppose that $Z=\mathcal{N}$. Since $\mathcal{N}/\!\!/\mathcal{N}$ is contractible (see [18]), one obtains the following well known result of Dold–Lashof [2].

LEMMA 3.2. *$\mathcal{N} \simeq \Omega B\mathcal{N}$ if and only if $\pi_0\mathcal{N}$ is a group.*

Now, consider a continuous monoid homomorphism β of $\mathcal{M}$ onto $\mathcal{N}$. Then $\mathcal{M}$ acts on the left of $\mathcal{N}$ by the action $n \mapsto m \cdot n = \beta(m)n$, which commutes with the right action of $\mathcal{N}$ on itself by multiplication, $n \mapsto nn'$. Therefore, we may form $\mathcal{M}\backslash\!\!\backslash\mathcal{N}$ and $(\mathcal{M}\backslash\!\!\backslash\mathcal{N})/\!\!/\mathcal{N} \cong \mathcal{M}\backslash\!\!\backslash(\mathcal{N}/\!\!/\mathcal{N})$. Since $\mathcal{N}/\!\!/\mathcal{N}$ is contractible, $\mathcal{M}\backslash\!\!\backslash\mathcal{N}/\!\!/\mathcal{N}$ is weakly equivalent to $B\mathcal{M}$, and by taking Z equal to $\mathcal{M}\backslash\!\!\backslash\mathcal{N}$ in Lemma 3.1, we obtain:

LEMMA 3.3. *If, for each $n \in \mathcal{N}$, right multiplication by n induces a weak* (*resp. an abelian homology*) *equivalence $\mathcal{M}\backslash\!\!\backslash\mathcal{N} \to \mathcal{M}\backslash\!\!\backslash\mathcal{N}$, then the sequence*

$$\mathcal{M}\backslash\!\!\backslash\mathcal{N} \to B\mathcal{M} \to B\mathcal{N}$$

is a homotopy (*resp. an abelian homology*) *fibration sequence.*

Note. Lemmas 3.1 and 3.3 remain true if right and left actions are interchanged.

Let us apply this first to the map $\beta: M \to \mathcal{M}$, where $\mathcal{M}$ is a topological monoid which is called M when it is considered with the discrete topology. As usual, $\bar{B}\mathcal{M}$ denotes the homotopy fibre of $BM \to B\mathcal{M}$ at the base point $B\{e\}$.

LEMMA 3.4. *If $\pi_0\mathcal{M}$ is a group, then $M\backslash\!\!\backslash\mathcal{M} \simeq \bar{B}\mathcal{M}$.*

Proof. Since $\mathscr{M}$ is grouplike, this follows immediately from Lemma 3.3. □

Let $\mathscr{M}_0$ be the identity component of $\mathscr{M}$.

LEMMA 3.5. *If each component of $\mathscr{M}$ contains an invertible element, then the inclusion $M_0\backslash\!\backslash\mathscr{M}_0 \hookrightarrow M\backslash\!\backslash\mathscr{M}$ is an equivalence. Thus $\bar{B}\mathscr{M}_0 \simeq \bar{B}\mathscr{M}$, and so $\bar{B}\mathscr{M}$ is connected.*

Proof. Let $\tilde{I}:\mathscr{C}(M_0\backslash\!\backslash\mathscr{M}_0)\to\mathscr{C}(M\backslash\!\backslash\mathscr{M})$ be the inclusion functor. We will define a functor $\tilde{F}:\mathscr{C}(M\backslash\!\backslash\mathscr{M})\to\mathscr{C}(M_0\backslash\!\backslash\mathscr{M}_0)$ which is inverse to $\tilde{I}$ in the sense that $\tilde{F}\circ\tilde{I}=\tilde{Id}$ and there is a natural transformation $\tilde{T}:\tilde{I}\circ\tilde{F}\to\tilde{Id}$. Thus, $\tilde{F}$ is right adjoint to $\tilde{I}$. It then follows from [18] that the maps of classifying spaces induced by $\tilde{I}$ and $\tilde{F}$ are mutual homotopy inverses.

Choose an invertible element in each component of $\mathscr{M}$, and, for each $m\in\mathscr{M}$, write $\theta(m)$ for the chosen element in the component of m. Let $\tilde{F}$ take the object m to the object $\theta(m)^{-1}m$ and the morphism $m':m\to m'm$ to the morphism $\theta(m'm)^{-1}m'\theta(m):\theta(m)^{-1}m\to\theta(m'm)^{-1}m'm$. It is easy to check that $\tilde{F}$ is a (continuous) functor. (Observe that $\mathscr{M}$ is locally connected because it is good.) Further, because the following diagram commutes

$$\begin{tikzcd}
\theta(m)^{-1}m \arrow[r, "\theta(m)"] \arrow[d] & m \arrow[d] \\
\theta(m'm)^{-1}m'm \arrow[r, "\theta(m'm)"] & m'm,
\end{tikzcd}$$

we may define the natural transformation $\tilde{T}$ by $\tilde{T}(m)=\theta(m)$.

To see that $\bar{B}\mathscr{M}_0$ is connected, use the exact sequence $\pi_1 B\mathscr{M}_0\to\pi_0\bar{B}\mathscr{M}_0\to\pi_0 BM_0$. Since $\pi_1 B\mathscr{M}_0\cong\pi_0\mathscr{M}_0$ by Lemma 3.2, both $\pi_1 B\mathscr{M}_0$ and $\pi_0 BM_0$ vanish. □

Let us now consider a surjection of discrete monoids $\beta:M\to N$, with kernel $K=\beta^{-1}(e)$. Unless additional conditions are imposed, there need be no particular relation between BK, BM and BN. (For instance, if $M=\{(i,j):0\leq i\leq j\}\subset\mathbf{Z}\oplus\mathbf{Z}$, $N=\{j\in\mathbf{Z};\ j\geq 0\}$ and $\beta(i,j)=j$, then $K=\beta^{-1}(0)=(0,0)$, while the fibre of $BM\simeq B(\mathbf{Z}\oplus\mathbf{Z})\to BN\simeq B\mathbf{Z}$ is $B\mathbf{Z}\simeq S^1$.) Segal proves the following proposition in [21], §2.

PROPOSITION 3.6. *Suppose that $1\to K\to M\xrightarrow{\beta} N\to 1$ is an exact sequence of discrete monoids, and that*

(i) *there is a section $s:N\to M$ such that $\beta\circ s=id.$, and, for each $n\in N$, the map $k\to s(n)k$ is a bijection $K\to\beta^{-1}(n)$; and*

(ii) *for each* $n \in N$ *the endomorphism* $c_n : K \to K$ *defined by* $ks(n) = s(n)c_n(k)$ *is a weak (resp. an abelian homology) equivalence.*

Then $BK \to BM \to BN$ *is a homotopy (resp. an abelian homology) fibration sequence.*

(This follows from Lemma 3.3. For condition (i) implies that the inclusion $K \backslash\!\backslash * \hookrightarrow M \backslash\!\backslash N$ is an equivalence, and it then follows from (ii) that the map induced on $K \backslash\!\backslash *$ by right multiplication by n is the appropriate kind of equivalence.)

Here is a sufficient condition for each c_n to be an abelian homology equivalence.

LEMMA 3.7. *Suppose that for each* $n \in N$ *and finite subset* $\{k_1, \ldots, k_p\} \subseteq K$, *there is an invertible element* $k \in K$ *such that* $c_n(k_i) = k^{-1}k_ik$ *for* $i = 1, \ldots, p$. *Then* c_n *acts as the identity on* $H_*(BK, \mathcal{A})$, *for all abelian systems* $\mathcal{A}$ *of local coefficients on* BK.

Proof. (See [21] (2.10).) If the map $f: BK \to BK$ is induced by conjugation by $k \in K$, then, for all abelian systems $\mathcal{A}$ of local coefficients, $f^*\mathcal{A} \cong \mathcal{A}$ and f induces the identity map $H_*(BK, \mathcal{A}) \to H_*(BK, \mathcal{A})$. Therefore, because the homology of a monoid is the direct limit of the homology of its finitely generated submonoids, the same is true for c_n. $\square$

PROPOSITION 3.8. *Suppose that*

$$\begin{array}{ccccc} K & \to & M & \to & N \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}' & \to & \mathcal{M}' & \to & \mathcal{N}' \end{array}$$

is a commutative diagram of monoids and monoid homomorphisms such that K *and* $\mathcal{K}'$ *are contained in the kernels of the homomorphisms* $M \to N$ *and* $\mathcal{M}' \to \mathcal{N}'$. *Suppose further that the bottom row is a homotopy fibration sequence in which* $\mathcal{M}'$ *is connected, and that* $BK \to BM \to BN$ *is an abelian homology fibration sequence. Then* $\pi_0\mathcal{K}'$ *is an abelian group and*

$$\bar{B}\mathcal{K}' \to \bar{B}\mathcal{M}' \to \bar{B}\mathcal{N}'$$

is an integer homology fibration sequence, where $\bar{B}\mathcal{K}'$ *denotes the homotopy fibre of* $BK \to B\mathcal{K}'$, *etc.*

Proof. The first step is to show that $B\mathscr{K}' \to B\mathscr{M}' \to B\mathscr{N}'$ is a fibration sequence. This is essentially well known (for instance, May proves similar results in [14]), but I know of no convenient reference and so give a proof here. Notice first that because $\mathscr{M}'$ is assumed connected, $\pi_0\mathscr{K}'$ is a quotient of the abelian group $\pi_1\mathscr{N}'$ and so is itself an abelian group. "Therefore $\mathscr{K}'$ is grouplike. Since the monoids $\mathscr{M}'$ and $\mathscr{N}'$ are connected, they are also grouplike. Hence, by Lemma 3.2, the sequence $\Omega B\mathscr{K}' \to \Omega B\mathscr{M}' \to \Omega B\mathscr{N}'$ is equivalent to $\mathscr{K}' \to \mathscr{M}' \to \mathscr{N}'$ and so is a fibration sequence.

Now let F be the homotopy fibre of the map $B\mathscr{M}' \to B\mathscr{N}'$ at the point $B\{e\}$, and let α be the natural inclusion $B\mathscr{K}' \hookrightarrow F$. Then ΩF is the homotopy fibre of the map $\Omega B\mathscr{M}' \to \Omega B\mathscr{N}'$ so that $\Omega\alpha : \Omega B\mathscr{K}' \to \Omega F$ is an equivalence. But $B\mathscr{K}'$ and F are both connected. Hence α is an equivalence, as required."

Now, consider the commutative diagram

$$
\begin{tikzcd}
\Omega B\mathscr{K}' \arrow[rr] \arrow[dr, hook, "\alpha_0"'] \arrow[dd] & & \Omega B\mathscr{M}' \arrow[r] \arrow[dd] & \Omega B\mathscr{N}' \arrow[dd] \\
& F_0 \arrow[ur] \arrow[dd] & & \\
\bar{B}\mathscr{K}' \arrow[rr, dashed] \arrow[dr, hook, "\alpha_1"'] \arrow[dd] & & \bar{B}\mathscr{M}' \arrow[r] \arrow[dd] & \bar{B}\mathscr{N}' \arrow[dd] \\
& F_1 \arrow[ur] \arrow[dd] & & \\
BK \arrow[rr, dashed] \arrow[dr, hook, "\alpha_2"'] \arrow[dd] & & BM \arrow[r] \arrow[dd] & BN \arrow[dd] \\
& F_2 \arrow[ur] \arrow[dd] & & \\
B\mathscr{K}' \arrow[rr, dashed] \arrow[dr, hook, "\alpha_3"'] & & B\mathscr{M}' \arrow[r] & B\mathscr{N}' \\
& F_3 \arrow[ur] & &
\end{tikzcd}
$$

where F_0 is the homotopy fibre of $\Omega B\mathscr{M}' \to \Omega B\mathscr{N}'$, and similarly for the other rows. By the previous remarks, both α_0 and α_3 are weak equivalences. We have assumed that α_2 is an abelian homology equivalence, and want to deduce that α_1 is an integer homology equivalence.

Since $\bar{B}\mathscr{K}'$ is the homotopy fibre of $BK \to B\mathscr{K}'$, the map $\bar{B}\mathscr{K}' \to BK$ is a fibration with fibre $\Omega B\mathscr{K}'$. Therefore, $\pi_1 BK$ acts on $H_*(\Omega B\mathscr{K}')$ via the group $\pi_1 B\mathscr{K}' \cong \pi_0\mathscr{K}'$, which as remarked above, is abelian. Similarly, $\Omega B\mathscr{M}' \to \bar{B}\mathscr{M}' \to BM$ and $\Omega B\mathscr{N}' \to \bar{B}\mathscr{N}' \to BN$ are fibrations, from which it follows that $F_0 \to F_1 \to$

F_2 is, also. Moreover, it is easy to check that F_1 is the homotopy fibre of $F_2 \to F_3$. Therefore, $\pi_1 F_2$ acts on $H_*(F_0)$ via the abelian group $\pi_1 F_3 \cong \pi_1 B\mathcal{K}'$. The result now follows by comparing the homology spectral sequences for the fibrations $\Omega B\mathcal{K}' \to \bar{B}\mathcal{K}' \to BK$ and $F_0 \to F_1 \to F_2$. $\square$

Proof of Proposition 2.2. Let M be $\mathrm{Hom}_0(X, Y, \mathrm{rel}\, A)$ and N be $\mathrm{Emb}_0^X(Y, \mathrm{rel}\, A)$, and let K be the kernel of the restriction $\rho: M \to N$. Then it is easy to see that any section s for which $\beta \circ s = id$ satisfies the first condition of Proposition 3.6. The second condition is also satisfied. For, if $\{k_1, \ldots, k_p\}$ is a finite subset of K as in Lemma 3.7, there is a neighbourhood U of $Y \cup A$ in X which is disjoint from the support of each k_i. By the isotopy extension theorem [3], each embedding $n \in N$ extends to a homeomorphism $h \in M$ with support in U. Thus $s(n)c_n(k_i) = k_i s(n) = hk_i h^{-1} s(n) = s(n)k^{-1}k_i k$, where $k = h^{-1}s(n) \in K$. Therefore $BK \to BM \to BN$ is an abelian homology fibration sequence, as required.

Now, let $\mathcal{M}'$ be $\mathcal{H}om_0(X, Y, \mathrm{rel}\, A)$, and $\mathcal{N}'$ be $\mathcal{E}mb_0^X(Y, \mathrm{rel}\, A)$, and let $\mathcal{K}'$ be the kernel of the restriction $\mathcal{M}' \to \mathcal{N}'$. Then $\mathcal{M}'$ is connected and $\mathcal{K}' \to \mathcal{M}' \to \mathcal{N}'$ is a homotopy fibration sequence by [3]. It remains to identify $\bar{B}\mathcal{K}'$ with $\bar{B}\mathcal{H}om_0 \times (X, \mathrm{rel}\, Y \cup A)$. But clearly $\mathcal{K}'$ is weakly equivalent to $\mathcal{H}om(X, \mathrm{rel}\, Y \cup A) \cap \mathcal{H}om_0(X, Y, \mathrm{rel}\, A)$, as we see by using a collar on the frontier of Y. Let us denote the monoid $\mathcal{H}om(X, \mathrm{rel}\, Y \cup A) \cap \mathcal{H}om_0(X, Y, \mathrm{rel}\, A)$ by $\mathcal{K}$. Thus $\mathcal{K}$ is a union of components of the group $\mathcal{H}om(X, \mathrm{rel}\, Y \cup A)$. Because $\pi_0\mathcal{K} \cong \pi_0\mathcal{K}'$ is a group, $\mathcal{K}$ must in fact be a group. Notice also that K is just $\mathcal{K}$ with the discrete topology. It is easy to see that $B\mathcal{K} \simeq B\mathcal{K}'$. It follows that $\bar{B}\mathcal{K} \simeq \bar{B}\mathcal{K}'$. By Lemma 3.5, which we may apply because $\mathcal{K}$ is a group, $\bar{B}\mathcal{K} \simeq \bar{B}\mathcal{K}_0$. Since $\mathcal{K}_0 = \mathcal{H}om_0(X, \mathrm{rel}\, Y \cup A)$, the result follows. $\square$

§4. On monoids of homeomorphisms which embed a submanifold in itself

In this section we will prove Proposition 2.1. Since $\bar{B}\mathcal{H}om_0(X, \mathrm{rel}\, A) \simeq \bar{B}\mathcal{H}om(X, \mathrm{rel}\, A)$ by Lemma 3.5, it suffices to show that if (X, Y) is a nice pair and if $(\mathrm{Fr}\, Y) - A$ is relatively compact in X then the inclusion $\bar{B}\mathcal{H}om_0(X, Y, \mathrm{rel}\, A) \hookrightarrow \bar{B}\mathcal{H}om_0(X, \mathrm{rel}\, A)$ is a weak equivalence. Our argument uses the compactness condition and the fact that Y has the same dimension as X. For purposes of comparison, consider the situation when Y is a single point $\{y_0\}$ and $X = [0, 1]$. Then $\mathcal{H}om_0(X, \{y_0\})$ is a contractible group, and by comparing the fibrations $\mathcal{H}om_0 \to \bar{B}\mathcal{H}om_0 \to B\,\mathrm{Hom}_0$ for X and $(X, \{y_0\})$ one sees that $\pi_1 \bar{B}\mathcal{H}om_0(X, \{y_0\} \not\cong \pi_1 \bar{B}\mathcal{H}om_0\, X$.

Proposition 2.1 is a consequence of the following lemma.

LEMMA 4.1. *Let $\mathcal{M}$ be a connected submonoid of a topological group $\mathscr{G}$ which is contained in an open subset $\mathscr{U}$ of $\mathscr{G}$, and suppose that the following conditions are satisfied:*

(i) *the inclusion $\mathcal{M} \to \mathscr{U}$ is a weak equivalence;*

(ii) *for all $g \in \mathscr{G}$, $g\mathscr{U} \subseteq \mathscr{U}$ if and only if $g \in \mathcal{M}$;*

(iii) *if $z \in g\mathscr{U} \cap h\mathscr{U}$ for some $z, g, h \in \mathscr{G}$, then there is $f \in \mathscr{G}$ such that $z \in f\mathscr{U} \subseteq g\mathscr{U} \cap h\mathscr{U}$.*

Then, the inclusion $\bar{B}\mathcal{M} \hookrightarrow \bar{B}\mathscr{G}$ is a weak equivalence.

Before proving this, let us apply it to our situation. So, $\mathscr{G} = \mathscr{H}\!omo_0\,(X, \text{rel } A)$ and $\mathcal{M} = \mathscr{H}\!omo_0\,(X, Y \text{ rel } A)$. In order to define $\mathscr{U}$, recall that, because Y is cleanly embedded in X, the frontier, Fr Y, of Y in X is bicollared in X. Choose a bicollar $\text{Fr } Y \times [-1, 2]$, with $\text{Fr } Y \times [0, 2] \subseteq Y$ and $\text{Fr } Y \times \{0\}$ identified with $\text{Fr } Y \subseteq X$, and for each $\lambda \in [0, 2]$ let Y_λ be the submanifold $Y - (\text{Fr } Y \times [0, \lambda))$. Without loss of generality, we may assume that $A \cap \text{Fr } Y \times [-1, 2]$ is a product. We now define $\mathscr{U}$ to be the component of the set $\{g \in \mathscr{G} : g(Y_1) \subset \mathring{Y}\}$ which contains the identity element, where $\mathring{Y}$ denotes the relative interior $Y - \text{Fr } Y$ of Y. Then $\mathscr{U}$ is open because $(\text{Fr } Y_1) - A$ is relatively compact. Also, $\mathcal{M} \subseteq \mathscr{U}$, and it is not difficult to see that conditions (i) and (ii) above are satisfied. Thus it remains to prove (iii). Now the hypothesis of (iii) implies that $z(Y_1) \subseteq g(\mathring{Y}) \cap h(\mathring{Y})$. Since $(\text{Fr } Y_1) - A$ is relatively compact and $\mathring{Y}$ is open, there is $\lambda \in [0, 1)$ such that $z(Y_\lambda) \subseteq g(\mathring{Y}) \cap h(\mathring{Y})$. Also, because z, g and h are the identity near A, there is a continuous function $\psi : \text{Fr } Y \to [0, \lambda]$ such that $A \cap \text{Fr } Y \subseteq \text{Int } \psi^{-1}(0)$, and z, g, and h are all equal to the identity on $\{\psi^{-1}([0, \lambda))\} \times [0, 1]$. Put $Y_\psi = \mathring{Y} - \{y \times [0, \psi(y)) : y \in \text{Fr } Y\}$. (See Figure 4.1). Then $Y_\lambda \subseteq Y_\psi$, and $z(Y_\psi) \subseteq g(\mathring{Y}) \cap h(\mathring{Y})$. Moreover, there is a shrinking map w, which is homotopic to the

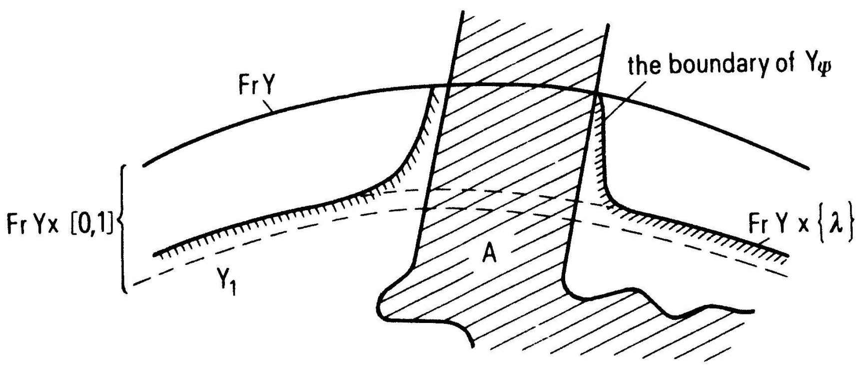

Figure 4.1

identity and is such that $w = id$ on $Y_1 \cup A$ and $w(\mathring{Y}) \subseteq Y_\psi$. Then $w^{-1} \in \mathcal{U}$, so that $z \in zw\mathcal{U}$. Clearly, $zw\mathcal{U} \subseteq g\mathcal{U} \cap h\mathcal{U}$. Thus all the conditions of Lemma 4.1 are satisfied, and so it follows that $\bar{B}\,\mathcal{H}om_0(X, Y, \text{rel } A) \simeq \bar{B}\,\mathcal{H}om_0(X, \text{rel } A)$.

The proof of Lemma 4.1 has several steps. Notice first that, by condition (ii), M (which, as usual, means $\mathcal{M}$ taken with the discrete topology) acts on $\mathcal{U}$ by multiplication on the left, so that we may form $M \backslash\!\!\backslash \mathcal{U}$ as in §3. Since $\mathcal{M} \simeq \mathcal{U}$, it follows from [19] Prop A.1(ii) that the inclusion $M \backslash\!\!\backslash \mathcal{M} \hookrightarrow M \backslash\!\!\backslash \mathcal{U}$ is a weak equivalence. Therefore, by Lemma 3.4, it suffices to prove that the inclusion $M \backslash\!\!\backslash \mathcal{U} \hookrightarrow G \backslash\!\!\backslash \mathcal{G}$ is a weak equivalence. Now recall from the proof of Lemma 3.3 that, in order to understand the map $\beta_* : B\mathcal{M} \to B\mathcal{N}$, we enlarged $B\mathcal{M} = \mathcal{M} \backslash\!\!\backslash *$ to $(\mathcal{M} \backslash\!\!\backslash \mathcal{N}) /\!\!/ \mathcal{N}$, and looked at the projection $(\mathcal{M} \backslash\!\!\backslash \mathcal{N}) /\!\!/ \mathcal{N} \to * /\!\!/ \mathcal{N}$ instead of β_*. In exactly the same way, we here enlarge $M \backslash\!\!\backslash \mathcal{U}$, replacing it by the equivalent space $(M \backslash\!\!\backslash \mathcal{U} \times G) /\!\!/ G$, where we suppose that M acts on $\mathcal{U} \times G$ diagonally by $m : (u, g) \mapsto (mu, mg)$, and that G acts on the right by $h : (u, g) \mapsto (u, gh)$. Notice that these actions of M and G commute, so that $(M \backslash\!\!\backslash \mathcal{U} \times G) /\!\!/ G \cong M \backslash\!\!\backslash (\mathcal{U} \times G /\!\!/ G)$ can be formed as in §3. Now, $M \backslash\!\!\backslash \mathcal{U} \times G$ lies over $\mathcal{G}$. In other words, the map $\mathcal{U} \times G \to \mathcal{G}$ given by $(u, g) \mapsto g^{-1}u$ takes each M-orbit $\{(mu, mg) : m \in M\}$ in $\mathcal{U} \times G$ to the single point $g^{-1}u$, and so extends naturally to a projection $\pi : M \backslash\!\!\backslash \mathcal{U} \times G \to \mathcal{G}$. Further, if we suppose that G acts on $\mathcal{G}$ by $h : g \mapsto h^{-1}g$, then the map π is G-equivariant and so gives rise to a map $(M \backslash\!\!\backslash \mathcal{U} \times G) /\!\!/ G \to \mathcal{G} /\!\!/ G$.

Consider the following diagram:

$$
\begin{array}{ccccccc}
M \backslash\!\!\backslash \mathcal{U} & \xleftarrow{\simeq} & M \backslash\!\!\backslash (\mathcal{U} \times G /\!\!/ G) & \cong & (M \backslash\!\!\backslash \mathcal{U} \times G) /\!\!/ G & \longrightarrow & \mathcal{G} /\!\!/ G \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{=} \\
G \backslash\!\!\backslash \mathcal{G} & \xleftarrow{\simeq} & G \backslash\!\!\backslash (\mathcal{G} \times G /\!\!/ G) & \cong & (G \backslash\!\!\backslash \mathcal{G} \times G) /\!\!/ G & \longrightarrow & \mathcal{G} /\!\!/ G.
\end{array}
$$

(Here, the bottom row is formed just as the top, and all vertical maps are inclusions.) Clearly, the conclusion $M \backslash\!\!\backslash \mathcal{U} \simeq G \backslash\!\!\backslash \mathcal{G}$ will follow, once we show that the projections $\pi : M \backslash\!\!\backslash \mathcal{U} \times G \to \mathcal{G}$ and $\pi' : G \backslash\!\!\backslash \mathcal{G} \times G \to \mathcal{G}$ are weak equivalences. To do this, we need the following covering lemma, which is proved by Segal in [21], Prop. (A.5).(2) We will use the thick realization $\|\cdot\|$ here to be consistent with [21]. Since all realizations of $\mathcal{C}(M \backslash\!\!\backslash \mathcal{U} \times G)$ are equivalent, this is permissible.

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2 Philip Trauber has also given a proof of this (unpublished), in a rather different spirit. Readers of Segal's proof should note that the definition of the sets V_σ given in §5 is not quite correct. One could define them as follows. For each simplex σ in $\langle C \rangle$ let $\bar{V}_\sigma$ be the union of the (closed) stars of the vertices of σ in the first barycentric subdivision of the simplicial complex $\langle C \rangle$. Put $V_\sigma = \text{Int } \bar{V}_\sigma$. Then $V_\sigma \cap V_\tau = V_{\sigma \cap \tau}$, and the V_σ form an open covering of $\langle C \rangle$. It is not too hard to show that the space $Y = \bigcup_{\sigma \in \langle C \rangle} F_\sigma \times V_\sigma$ has the desired properties.

LEMMA 4.2. *Let $\mathscr{C}$ be a category and $\tilde{F}$ a functor from $\mathscr{C}$ to the discrete category of open subsets of a space X and their inclusions. For each $x \in X$, define $\mathscr{C}_x$ as the full subcategory of $\mathscr{C}$ spanned by the objects α such that $x \in \tilde{F}(\alpha)$. Let X_F be the realization of the topological category whose objects are pairs (α, x) with $\alpha \in \mathrm{Obj}\,\mathscr{C}$, $x \in \tilde{F}(\alpha)$, and where there are morphisms $(\alpha, x) \to (\beta, y)$ only if $x = y$, in which case they correspond to the morphisms $\alpha \to \beta$ in $\mathscr{C}$. A projection $\pi : X_F \to X$ is defined by $(\alpha, x) \mapsto x$. If $\|\mathscr{C}_x\|$ is contractible for all $x \in X$, then π is a weak homotopy equivalence.*

To apply this, let $\mathscr{C}$ be the category $\mathscr{C}(M \backslash\!\backslash G)$, where M acts on G by multiplication on the left, and let $\tilde{F}$ be the contravariant functor, which takes the object $g \in G$ to the open set $g^{-1}\mathscr{U}$ in $\mathscr{G}$, and takes the morphism $m : g \to mg$ to the inclusion $g^{-1}\mathscr{U} \to g^{-1}m^{-1}\mathscr{U}$. Clearly X_F is just $M \backslash\!\backslash \mathscr{U} \times G$. Therefore, in order to prove that $M \backslash\!\backslash \mathscr{U} \times G \to \mathscr{G}$ is a weak equivalence, we need only check that each $\|\mathscr{C}_x\|$ is contractible. We may think of $\mathscr{C}_x$ as the category of the partially ordered set G_x, whose objects are $\{g \in G : x \in g^{-1}\mathscr{U}\}$ and where $g \leq h$ if and only if $g^{-1}\mathscr{U} \subseteq h^{-1}\mathscr{U}$, or equivalently, if and only if $hg^{-1} \in M$. (These are the same by condition (ii) of Lemma 4.1.) But condition (iii) of Lemma 4.1 implies that every pair of elements of G_x has a common lower bound. Hence $\mathscr{C}_x$ is filtering and so has contractible realization: see [17] p. 85. The result follows.

Exactly the same argument applies to show that $G \backslash\!\backslash \mathscr{G} \times G \to \mathscr{G}$ is an equivalence. (This may be proved more easily by noticing that the categories $\mathscr{C}(G \backslash\!\backslash \mathscr{G} \times G)$ and $\mathscr{C}(\{e\} \backslash\!\backslash \mathscr{G})$ are equivalent, so that $G \backslash\!\backslash \mathscr{G} \times G \simeq \{e\} \backslash\!\backslash \mathscr{G} \simeq \mathscr{G}$. Here, $\{e\}$ denotes the trivial group.) This completes the proof of Proposition 2.1. □

§5. On monoids of embeddings

In this section we prove the versions of Propositions 2.1 and 2.2 which are used in [8] §3. That paper deals with monoids of C^r-embeddings rather than groups of homeomorphisms, and, because the proofs given above use the fact that $\mathscr{H}om(X, \mathrm{rel}\, A)$ is a group instead of just a monoid, they do not immediately extend to this case.

Before going further, it is convenient to change our notation slightly. We will work in the C^r-category for some r in the range $0 \leq r \leq \infty$. For any C^r-manifold X, and closed subsets $Y_1, \ldots, Y_k$, A of X, we denote by $\mathscr{E}mb(X, Y_1, \ldots, Y_k, \mathrm{rel}\, A)$ the monoid of all C^r-self-embeddings of X which take each Y_i into itself and are the identity near A, with the C^r-version of the topology defined in §2. Its identity component is $\mathscr{E}mb_0(X, Y_1, \ldots, Y_k, \mathrm{rel}\, A)$, and, as usual, we write

$\mathrm{Emb}_0(X, Y_1, \ldots, Y_k, \mathrm{rel}\, A)$ for the corresponding discrete monoid. Let X^* denote the union of X with a collar neighbourhood of ∂X. Thus $X^* = X \amalg_{\partial X} \partial X \times [0, 1]$. Then we define $\mathrm{Emb}^g(X, Y_1, \ldots, Y_k, \mathrm{rel}\ A)$ to be the quotient $\mathrm{Emb}(X^*, X, Y_1, \ldots, Y_k, \mathrm{rel}\, A)/\sim$, where $h \sim h'$ if and only if $h = h'$ near X. (The label "g" indicates that $\mathrm{Emb}^g X$ consists of *germs* of self-embeddings.) The monoid $\mathrm{Emb}_0^g(X, Y_1, \ldots, Y_k, \mathrm{rel}\, A)$ is defined similarly. Notice that, if X is compact, $\mathrm{Emb}_0^g X$ is just the C^r-analogue of the monoid $\mathrm{Emb}_0^{X^*} X$ defined in §2.

We write $\bar{B}\,\mathscr{E}\mathit{mb}_0^g(X, Y_1, \ldots, Y_k, \mathrm{rel}\, A)$ for the homotopy fibre of the map $B\,\mathrm{Emb}_0^g(X, Y_1, \ldots, Y_k, \mathrm{rel}\, A) \to B\,\mathscr{E}\mathit{mb}_0(X, Y_1, \ldots, Y_k, \mathrm{rel}\, A)$. Notice that if X is compact, the obvious restriction maps $\bar{B}\,\mathscr{E}\mathit{mb}_0^g(X, \mathrm{rel}\, A) \to \bar{B}\,\mathscr{E}\mathit{mb}_0(X, \mathrm{rel}\, A) \to \bar{B}\,\mathscr{E}\mathit{mb}_0(\mathrm{Int}\, X, \mathrm{rel}\, A)$ are weak equivalences. (This is proved in Proposition 5.7 at the end of this section.) It is convenient to use $\bar{B}\,\mathscr{E}\mathit{mb}_0^g$ rather than $\bar{B}\,\mathscr{E}\mathit{mb}_0$ because the conditions of Lemma 3.7 hold for Emb_0^g but not for Emb_0: see the proof of Proposition 5.1 below. As in §2, we will say that (X, Y) is a nice manifold pair if X is a C^r-manifold, possibly with boundary, and if Y is a C^r-cleanly embedded submanifold of X, which may have codimension 2 corners if $r \geq 1$, and which has the same dimension as X. Thus we again make no compactness assumptions, although in practice X will usually be compact. Finally we remark that, because monoids of C^r-embeddings have the homotopy type of CW complexes when $r \geq 1$ [15], all weak equivalences involving such monoids are in fact homotopy equivalences.

We prove the following versions of Propositions 2.1 and 2.2.

PROPOSITION 5.1. *If (X, Y) is a nice manifold pair such that X is compact, and if A is any closed subset of X, then the inclusion*

$$\bar{B}\,\mathscr{E}\mathit{mb}_0^g(X, Y, \mathrm{rel}\, A) \hookrightarrow \bar{B}\,\mathscr{E}\mathit{mb}_0^g(X, \mathrm{rel}\, A)$$

is a weak equivalence.

PROPOSITION 5.2. *If X, Y, and A are as in Proposition 5.1, then*

$$\bar{B}\,\mathscr{E}\mathit{mb}_0^g(X, \mathrm{rel}\, Y \cup A) \to \bar{B}\,\mathscr{E}\mathit{mb}_0^g(X, Y, \mathrm{rel}\, A) \to \bar{B}\,\mathscr{E}\mathit{mb}_0^g(Y, \mathrm{rel}\, A)$$

is an integer homology fibration sequence. Moreover, $\pi_1 \bar{B}\,\mathscr{E}\mathit{mb}_0^g(Y, \mathrm{rel}\, A)$ acts trivially on $H_*(\bar{B}\,\mathscr{E}\mathit{mb}_0^g(X, \mathrm{rel}\, Y \cup A); \mathbf{Z})$.

In order to prove these propositions we need some information about $\pi_1 B\,\mathrm{Emb}_0^g(X, Y, \mathrm{rel}\, A)$. Let A' be the union of A with all components of $X - A$ which do not intersect ∂X and let $A'' = (X - A') \cup A$. Then $A' \cup A'' = X$ and $A' \cap A'' = A$. Also $\mathrm{Emb}_0^g(X, \mathrm{rel}\, A) \cong \mathrm{Emb}_0^g(X, \mathrm{rel}\, A') \times \mathrm{Emb}_0^g(X, \mathrm{rel}\, A'')$, where, because $\partial X \subseteq A''$, the monoid $\mathrm{Emb}_0^g(X, \mathrm{rel}\, A'')$ is, in fact, the group of diffeomorphisms $\mathrm{Diff}_0(X, \mathrm{rel}\, A'')$.

LEMMA 5.3. *If $X - A$ is relatively compact, then* $\pi_1 B\, \mathrm{Emb}_0^g(X, \mathrm{rel}\, A) \cong \mathrm{Diff}_0(X, \mathrm{rel}\, A'')$.

Proof. We must show that $\pi_1 B\, \mathrm{Emb}_0^g(X, \mathrm{rel}\, A') = 0$. Since for any discrete monoid M the image of M in $\pi_1 BM$ generates the group $\pi_1 BM$, it suffices to show that if θ is any homomorphism of $M = \mathrm{Emb}_0^g(X, \mathrm{rel}\, A')$ to a group, then θ maps M onto the identity element, $\{e\}$.

Let $\mathcal{V}$ be a covering of $X - A'$ by open sets whose closures are each diffeomorphic to the closed half-disc D_+^n (where $n = \dim X$ and $D_+^n = \{x \in D^n : x_n \geq 0\}$). Then, by [3] if $r = 0$ and [16] if $r \geq 1$, M is generated by embeddings with support in some $V \in \mathcal{V}$. Moreover, because each component of $X - A'$ intersects ∂X, there is, for each $V \in \mathcal{V}$, an embedding $m_V \in \mathrm{Emb}_0^g(X, \mathrm{rel}\, A')$ such that $V \cap m_V(X) = \varnothing$. (If γ is a C^r-path in $X - A'$ connecting ∂X to V, we may take m_V to be an embedding which is the identity outside a neighbourhood of $V \cup \gamma$ and which is the result of poking X along γ.) Therefore, for any element $n \in M$ with support in V, we have $nm_V = m_V$, so that $\theta(n)\theta(m_V) = \theta(m_V)$. Since θ is a homomorphism into a group, $\theta(n) = e$. Thus, because these elements n generate M, θ maps M onto $\{e\}$. □

LEMMA 5.4. *If (X, Y) is a nice pair and $X - A$ is relatively compact, then* $\pi_1 B\, \mathrm{Emb}_0^g(X, Y, \mathrm{rel}\, A') = 0$ *and*

$$\pi_1 B\, \mathrm{Emb}_0^g(X, Y, \mathrm{rel}\, A) \cong \pi_1 B\, \mathrm{Diff}_0(X, Y, \mathrm{rel}\, A'').$$

Note. It is not necessarily true that $\pi_1 B\, \mathrm{Diff}_0(X, Y, \mathrm{rel}\, A'') \cong \mathrm{Diff}_0(X, \mathrm{rel}\, A'')$. For example, if X is the circle S^1 and Y is a contractible arc in S^1, there is an exact sequence

$$0 \to \mathbf{Z} \to \pi_1 B\, \mathrm{Diff}_0(S^1, Y) \to \mathrm{Diff}_0\, S^1 \to 0.$$

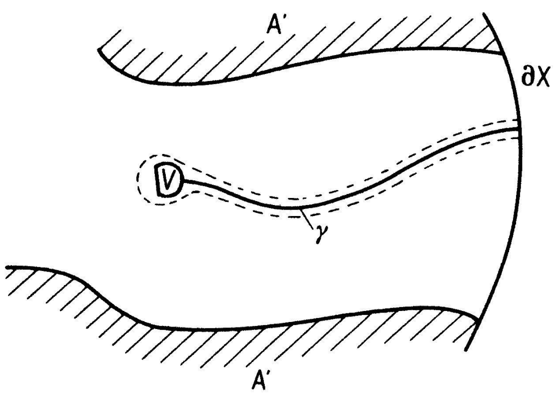

Figure 5.1

In fact, $\pi_1 B\,\mathrm{Diff}_0(S^1, Y)$ is isomorphic to the group of orientation preserving periodic diffeomorphisms of $\mathbf{R}$. This follows because, by the C^∞-version of Proposition 2.1, the map $B\,\mathrm{Diff}_0(S^1, Y) \to B\,\mathrm{Diff}_0 S^1$ has the same fibre as $B\,\mathcal{D}iff_0(S^1, Y) \to B\,\mathcal{D}iff_0 S^1$. Since $\mathcal{D}iff_0(S^1, Y)$ is contractible, while $\mathcal{D}iff_0 S^1 \simeq S^1$, this fibre is just S^1.

Proof of Lemma 5.4. Again, we must show that $\pi_1 BM = 0$, where $M = \mathrm{Emb}_0^g(X, Y, \mathrm{rel}\, A')$. The proof is similar to that of Lemma 5.3, except that, if γ crosses ∂Y, the m_V constructed there need not take Y into Y. To get around this difficulty, let us suppose that the closure of each $V \in \mathcal{V}$ intersects at most one component of ∂Y, and for each V choose γ so that it crosses ∂Y a finite number of times and ends at a point of $V \cap Y$, if $V \cap Y \neq \emptyset$. Then, let j_V be the number of times γ crosses ∂Y if $\bar{V}$ lies entirely in the same component of $X - \partial Y$ as the end point of γ, and let it be this number plus 1 otherwise. We will prove that $\theta(n) = e$, for all $n \in M$ with support in V, by induction on j_V. For, if $j_V \leq 1$, the embeddings m_V with $V \cap m_V(X) = \emptyset$ may be constructed to lie in M. Thus $nm_V = m_V$ for all $n \in M$ with support in V, which implies, as before, that $\theta(n) = e$. Suppose this statement has been proved for all V with $j_V < j_0$ and let V_0 have $j_{V_0} = j_0$. Then there is a half-disc V' slightly further down γ with $j_{V'} = j_0 - 1$. (See Fig. 5.2.) We may assume that $\bar{V}'$ lies either entirely inside or entirely outside Y. The arc γ crosses ∂Y at most once in between V_0 and V'. If $V' \subseteq Y$, there is a diffeomorphism $m \in M$ such that $m(V_0) = V'$. Therefore, if $n_0 \in M$ has support in V_0, there is $n' \in M$ with support in V' such that $mn_0 = n'm$. Since $\theta(n') = e$ by the inductive hypothesis, $\theta(n_0) = e$ as well. In the case where $V' \cap Y = \emptyset$, there is a diffeomorphism $m \in M$ which takes V' onto V_0, and the argument proceeds similarly. This completes the inductive step. □

COROLLARY 5.5. *If $X - A$ is relatively compact, then*

$$\pi_1 \bar{B}\mathcal{E}mb_0^g(X, Y, \mathrm{rel}\, A')$$

is abelian.

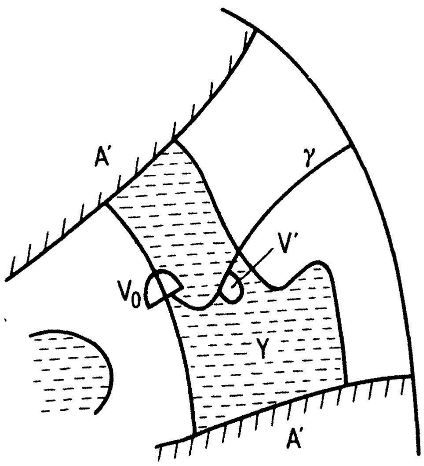

Figure 5.2

Proof. By Lemma 5.4 it is a quotient of the abelian group $\pi_2 B\,\mathscr{E}mb_0(X, Y, \mathrm{rel}\, A')$. □

We can now prove a special case of Proposition 5.2.

LEMMA 5.6. *Suppose that (X, Y) is a nice manifold pair such that Y is a compact submanifold in* $\mathrm{Int}\, X$, *and let Z be any closed subset of Y. Then*

$$\bar{B}\,\mathscr{E}mb_0^g(X, \mathrm{rel}\, Y\cup A) \to \bar{B}\,\mathscr{E}mb_0^g(X, Y, Z, \mathrm{rel}\, A) \to \bar{B}\,\mathscr{E}mb_0^g(Y, Z, \mathrm{rel}\, A)$$

is an integer homology fibration sequence. Moreover, $\pi_1(\bar{B}\,\mathscr{E}mb_0^g(Y, Z, \mathrm{rel}\, A))$ acts trivially on $H_*(\bar{B}\,\mathscr{E}mb_0^g(X, \mathrm{rel}\, Y\cup A); \mathbf{Z})$.

Proof. Let us adopt the notation of Proposition 3.8. Thus, we write M for $\mathrm{Emb}_0^g(X, Y, Z, \mathrm{rel}\, A)$, N for $\mathrm{Emb}_0^g(Y, Z, \mathrm{rel}\, A)$, and K for the kernel of $M \to N$. Similarly, $\mathcal{M}'$ is $\mathscr{E}mb_0(X, Y, Z, \mathrm{rel}\, A)$, $\mathcal{N}'$ is $\mathscr{E}mb_0(Y, Z, \mathrm{rel}\, Y\cap A)$ and $\mathscr{K}'$ is the kernel of $\mathcal{M}' \to \mathcal{N}'$. Then, $\mathscr{K}' \to \mathcal{M}' \to \mathcal{N}'$ is a homotopy fibration sequence. Also, because $Y \subseteq \mathrm{Int}\, X$, every embedding in $\mathrm{Emb}_0^g(Y, Z, \mathrm{rel}\, A)$ extends to a diffeomorphism of X. It is easy to check that if, for each $n \in N$, we choose $s(n)$ to be some diffeomorphism in $\beta^{-1}(n)$, then condition (i) of Proposition 3.5 is satisfied. Condition (ii) is also satisfied (see the proof of Proposition 2.2 given at the end of §3). Therefore, by Propositions 3.6 and 3.8, the sequence $\bar{B}\mathscr{K}' \to \bar{B}\mathcal{M}' \to \bar{B}\mathcal{N}'$ is an integer homology fibration sequence. We now have to identity $\bar{B}\mathscr{K}'$ with $\bar{B}\,\mathscr{E}mb_0^g(X, \mathrm{rel}\, Y\cup A)$ as was done in the proof of Proposition 2.2 mentioned above.

Let $\mathscr{K} = \mathscr{E}mb(X, \mathrm{rel}\, Y\cup A) \cap \mathscr{E}mb_0(X, Y, \mathrm{rel}\, A)$. Then, by a collaring argument, we have as before that $\mathscr{K}' \simeq \mathscr{K}$. Also $\pi_0\mathscr{K} \cong \pi_0\mathscr{K}'$ is a group. However $\mathscr{K}$ itself will not be a group in general, and so, in order to apply Lemma 3.5, we must check that each component of $\mathscr{K}$ contains an invertible element. To see this, let g be in $\mathscr{K}$. By definition of $\mathscr{K}$, there is an isotopy g_t, $0 \leq t \leq 1$, in $\mathscr{E}mb_0(X, Y, \mathrm{rel}\, A)$ with $g_0 = id$ and $g_1 = g$. Since Y is a compact submanifold in the interior of X, we may extend $g_t \mid Y$, $0 \leq t \leq 1$, to an isotopy $\tilde{g}_t$ in $\mathscr{D}iff_0(X, \mathrm{rel}\, A)$. Then $\tilde{g}_1$ is in $\mathscr{K}$, and is isotopic to g by the isotopy $\tilde{g}_1\tilde{g}_t^{-1}g_t$, $0 \leq t \leq 1$. Because $\tilde{g}_t^{-1}g_t = id$ on Y, the elements g and $\tilde{g}_1$ lie in the same component of $\mathscr{K}$. Therefore, we may apply Lemma 3.5 to $\mathscr{K}$. Hence $\bar{B}\mathscr{K}' \simeq \bar{B}\,\mathscr{E}mb_0^g(X, \mathrm{rel}\, Y\cup A)$ as required. This completes the proof of the first half of the lemma.

It remains to show that $\pi_1\bar{B}\mathcal{N}'$ acts trivially on $H_*(\bar{B}\mathscr{K}; \mathbf{Z})$. Let A' be the union of A with all components of $Y - A$ which do not intersect ∂Y and let $A'' = (X - A') \cup A$. Then $X = A' \cup A''$ and $A' \cap A'' = A$. Since $\overline{X - Y} \subseteq A''$, the spaces $\bar{B}\mathcal{M}'$ and $\bar{B}\mathcal{N}'$ split as products: $\bar{B}\mathcal{M}' \cong \bar{B}\,\mathscr{E}mb_0^g(X, Y, Z, \mathrm{rel}\, A') \times \bar{B}\,\mathscr{D}iff_0(Y, Z, \mathrm{rel}\, A'')$ and $\bar{B}\mathcal{N}' \cong \bar{B}\,\mathscr{E}mb_0^g(Y, Z, \mathrm{rel}\, A') \times \bar{B}\,\mathscr{D}iff_0(Y, Z, \mathrm{rel}\, A'')$.

Therefore it suffices to consider the case when $A = A'$. By Lemma 5.4, $\pi_1 B\,\mathrm{Emb}_0^g(Y, Z, \mathrm{rel}\ A') = 0$, and so $\pi_2 B\mathcal{N}'$ surjects onto $\pi_1 \bar{B}\mathcal{N}'$. Notice that $\pi_2 B\mathcal{N}' \cong \pi_1\mathcal{N}'$, because $\mathcal{N}'$ is connected (see Lemma 3.2). Since $\pi_0\mathcal{K}$ is a group, it follows from Lemma 3.4 that $\bar{B}\mathcal{K} \simeq K\backslash\backslash\mathcal{K}$. Similarly, because both $\mathcal{M}'$ and $\mathcal{N}'$ are connected, $\bar{B}\mathcal{M}' \simeq M\backslash\backslash\mathcal{M}'$ and $\bar{B}\mathcal{N}' \simeq N\backslash\backslash\mathcal{N}'$. Therefore, we have a diagram

$$\begin{tikzcd}
K\backslash\backslash\mathcal{K} \arrow[r] \arrow[dr, hook, "\alpha"'] & M\backslash\backslash\mathcal{M}' \arrow[r, "\rho"] & N\backslash\backslash\mathcal{N}', \\
 & F \arrow[u] &
\end{tikzcd}$$

where F is the homotopy fibre of $M\backslash\backslash\mathcal{M}' \to N\backslash\backslash\mathcal{N}'$ and α is an inclusion, and we must show that $\pi_1\mathcal{N}'$ acts trivially on $H_*(K\backslash\backslash\mathcal{K}) \cong H_*(F)$.

We first claim that the action of $\pi_1\mathcal{N}'$ on $H_*(K\backslash\backslash\mathcal{K})$ may be described as follows. Let $\ell(t)$, $0 \le t \le 1$, be a loop in $\mathcal{N}'$ based at e which represents the element λ of $\pi_1\mathcal{N}'$, and let $\tilde{\ell}$ be a lifting of ℓ to $\mathcal{M}'$ with $\tilde{\ell}(0) = e$, chosen so that $\tilde{\ell}(1)$ is the identity near Y. Then $\tilde{\ell}(1)$ is in K, and so acts on $K\backslash\backslash\mathcal{K}$ by multiplication on the right. We claim that this induces the action of λ on $H_*(K\backslash\backslash\mathcal{K})$. To justify this, observe that the elements of F are pairs (z, p), where $z \in M\backslash\backslash\mathcal{M}'$ and $p(t)$, $0 \le t \le 1$, is a path in $N\backslash\backslash\mathcal{N}'$ with $p(0) = e$ and $p(1) = \rho(z)$. The loop ℓ acts on F by $(z, p) \mapsto (z, \ell^{-1} \circ p)$, where $\ell^{-1} \circ p$ is the path obtained by going first round ℓ^{-1} and then along p. It is easy to check that this action is homotopic to $(z, p) \mapsto (z \cdot \tilde{\ell}(1), (\ell^{-1} \circ p) \cdot \ell)$. Here $(\ell^{-1} \circ p) \cdot \ell$ is the product of the paths $\ell^{-1} \circ p$ and ℓ, which is defined because $\mathcal{N}'$ acts on $N\backslash\backslash\mathcal{N}'$ by right multiplication. If p is the constant path then $(\ell^{-1} \circ p) \cdot \ell$ is a loop which canonically contracts to the constant loop. Therefore, because $\alpha(K\backslash\backslash\mathcal{K}) \subset \{(z, p) \in F : p \text{ is constant}\}$, there is a homotopy commutative diagram

$$\begin{tikzcd}
K\backslash\backslash\mathcal{K} \arrow[r, hook, "\alpha"] \arrow[d, "\text{mult by } \tilde{\ell}(1)"'] & F \arrow[d, "\text{action of } \ell"] \\
K\backslash\backslash\mathcal{K} \arrow[r, hook, "\alpha"] & F
\end{tikzcd}$$

as claimed.

It remains to show that multiplication by $\tilde{\ell}(1)$ induces the identity map on $H_*(K\backslash\backslash\mathcal{K})$. Consider, for each open neighbourhood U of $Y \cup A$, the subgroups $K_U = \{k \in K : k = id \text{ on } U\}$ and $\mathcal{K}_U = \{k \in \mathcal{K} : k = id \text{ on } U\}$ of K and $\mathcal{K}$ respectively. Clearly, it suffices to show that right multiplication by $\tilde{\ell}(1) : K_U\backslash\backslash\mathcal{K}_U \to K\backslash\backslash\mathcal{K}$ is homotopic to the inclusion map. It follows from the isotopy extension theorem that we may choose the lifting $\tilde{\ell}$ so that $\tilde{\ell}(1)$ has support in $U-(Y\cup A)$. Then $\tilde{\ell}(1)$ *commutes* with $\mathcal{K}_U$. Therefore, there is a natural transformation $\tilde{T}$, from the inclusion functor $\tilde{I}:(K_U\backslash\backslash\mathcal{K}_U)\hookrightarrow(K\backslash\backslash\mathcal{K})$ to the right multiplication functor $\tilde{R}(\tilde{\ell}(1))$, which is defined by $\tilde{T}(k)=\tilde{\ell}(1)$, for all $k\in\mathcal{K}_U$. It follows as in Lemma 3.5 that the maps induced by $\tilde{I}$ and $\tilde{R}(\tilde{\ell}(1))$ are homotopic. This completes the proof of Lemma 5.6. □

Proof of Proposition 5.1. If $\partial X=\varnothing$, this is just the C^r-version of Proposition 2.2. Therefore we may suppose that X is connected and has $\partial X\neq\varnothing$. Further if A' and A'' are as in Lemma 5.3, then $\bar{B}\,\mathscr{E}mb_0^g\,(X,\operatorname{rel} A)$ is the product $\bar{B}\,\mathscr{E}mb_0^g\,(X,\operatorname{rel} A')\times\bar{B}\,\mathscr{E}mb_0^g\,(X,\operatorname{rel} A'')$, and similarly for $\bar{B}\,\mathscr{E}mb_0^g\,(X,Y,\operatorname{rel} A)$. But the inclusion $\bar{B}\,\mathscr{E}mb_0^g\,(X,Y,\operatorname{rel} A'')\hookrightarrow\bar{B}\,\mathscr{E}mb_0^g\,(X,\operatorname{rel} A'')$ is a weak equivalence by the C^r-version of Proposition 2.1, because $\partial X\subseteq A''$, so that $\operatorname{Emb}_0^g\,(X,\operatorname{rel} A'')$ is a group. Therefore it suffices to prove that the inclusion $\bar{B}\,\mathscr{E}mb_0^g\,(X,Y,\operatorname{rel} A')\to\bar{B}\,\mathscr{E}mb_0^g\,(X,\operatorname{rel} A')$ is a weak equivalence.

Let S be the double of X, and consider the diagram:

$$\begin{tikzcd}
\bar{B}\,\mathscr{D}iff_0\,(S,\operatorname{rel} X) \arrow[r] \arrow[d, equal] & \bar{B}\,\mathscr{D}iff_0\,(S,X,Y,\operatorname{rel} A') \arrow[r] \arrow[d, hook, "j_1", "\simeq"'] & \bar{B}\,\mathscr{E}mb_0^g\,(X,Y,\operatorname{rel} A') \arrow[d, hook, "j_2"] \\
\bar{B}\,\mathscr{D}iff_0\,(S,\operatorname{rel} X) \arrow[r] & \bar{B}\,\mathscr{D}iff_0\,(S,X,\operatorname{rel} A') \arrow[r] & \bar{B}\,\mathscr{E}mb_0^g\,(X,\operatorname{rel} A').
\end{tikzcd}$$

Here j_1 is a (weak) equivalence. To see this, let i be the inclusion $\bar{B}\,\mathscr{D}iff_0\,(S,X,\operatorname{rel} A')\hookrightarrow\bar{B}\,\mathscr{D}iff_0\,(S,\operatorname{rel} A')$. This is an equivalence by the C^r-version of Proposition 2.1. The proof of this proposition given in §4 may be easily adapted to show that the composite $i\circ j_1$ is an equivalence as well. In fact, in order to apply Lemma 4.1 in this case one only has to find a suitable open neighbourhood $\mathscr{U}$ of $\mathcal{M}=\mathscr{D}iff_0\,(S,X,Y,\operatorname{rel} A')$ in $\mathscr{G}=\mathscr{D}iff_0\,(S,\operatorname{rel} A')$. Because (X,Y) is a nice pair, one can define sets X_λ and Y_λ much as before and then put $\mathscr{U}$ equal to the set of elements g such that $g(X_1)\subset\mathring{X}$ and $g(Y_1)\subset\mathring{Y}$. Thus j_1 is an equivalence. Our aim is to show that j_2 is also an equivalence.

By Lemma 5.6 both rows in the diagram are integer homology fibration sequences in which π_1(base) acts trivially on H_*(fibre). Moreover both the groups π_1(base) are abelian by Corollary 5.5. By comparing the exact sequences of terms of low degree in the spectral sequences associated to the rows, one sees that j_2 induces an isomorphism on $H_1(\,\cdot\,;\mathbf{Z})$. Therefore, it induces an isomorphism on π_1. By considering the corresponding fibrations over the universal covers of the base spaces and comparing spectral sequences there (cf. [10] XI), one finds that j_2 induces an isomorphism on the integer homology of the universal covers. Hence it is a (weak) equivalence, as required. □

Proof of Proposition 5.2. Let X^* be the union of X with a collar neighbourhood of ∂X as before, and consider the following commutative diagram:

$$\begin{tikzcd}
\bar B\,\mathscr{E}mb_0^g(X^*, \mathrm{rel}\ Y\cup A) \arrow[r] & \bar B\,\mathscr{E}mb_0^g(X^*, Y, \mathrm{rel}\ A) \arrow[r] & \bar B\,\mathscr{E}mb_0^g(Y, \mathrm{rel}\ A) \\
\bar B\,\mathscr{E}mb_0^g(X^*, X, \mathrm{rel}\ Y\cup A) \arrow[r] \arrow[u, hook, "\simeq"', "j_1"] \arrow[d, "\simeq"', "p_1"] & \bar B\,\mathscr{E}mb_0^g(X^*, X, Y, \mathrm{rel}\ A) \arrow[r] \arrow[u, hook, "\simeq"', "j_2"] \arrow[d, "\simeq"', "p_2"] & \bar B\,\mathscr{E}mb_0^g(Y, \mathrm{rel}\ A) \arrow[u, equal] \arrow[d, equal] \\
\bar B\,\mathscr{E}mb_0^g(X, \mathrm{rel}\ Y\cup A) \arrow[r] & \bar B\,\mathscr{E}mb_0^g(X, Y, \mathrm{rel}\ A) \arrow[r] & \bar B\,\mathscr{E}mb_0^g(Y, \mathrm{rel}\ A).
\end{tikzcd}$$

Now j_1 is an equivalence by Proposition 5.1 which we have just proved. A similar argument shows that j_2 is an equivalence. We will see in a moment that both the restriction maps p_1, p_2 are equivalences. Proposition 5.2 will then follow because Lemma 5.6 applies to the top row.

To prove that p_1 is an equivalence, it suffices to show that the corresponding maps for $\mathscr{E}mb_0$ and $B\,\mathrm{Emb}_0^g$ are equivalences. But

$$\mathscr{E}mb_0(X^*, X, \mathrm{rel}\ Y\cup A)\to\mathscr{E}mb_0(X, \mathrm{rel}\ Y\cup A)$$

is a weak fibration with fibre $\mathscr{E}mb_0(X^*, \mathrm{rel}\ X)$, and so is an equivalence because the monoid $\mathscr{E}mb_0(X^*, \mathrm{rel}\ X)$ is contractible. The space $B\,\mathrm{Emb}_0^g(X^*, \mathrm{rel}\ X)$ is also contractible, and, by applying Proposition 3.6, it follows that the map on $B\,\mathrm{Emb}_0^g$ is also an equivalence. (Details of this argument may be found in [21] (2.7).) Thus p_1 is an equivalence. Similarly, p_2 is an equivalence. $\square$

This completes the proof of the main propositions used in [8]. We end this section by looking at the effect of "boundary conditions" on $\bar B\,\mathscr{E}mb_0$.

PROPOSITION 5.7. *If A is a closed subset of the compact manifold X, then the restriction maps*

$$\bar B\,\mathscr{E}mb_0^g(X, \mathrm{rel}\ A)\to\bar B\,\mathscr{E}mb_0(X, \mathrm{rel}\ A)\to\bar B\,\mathscr{E}mb_0(\mathrm{Int}\ X, \mathrm{rel}\ A)$$

are (weak) equivalences.

Proof. Segal proves in [21] Proposition 2.8 that the restriction maps

$$B\,\mathrm{Emb}_0^g\,X\to B\,\mathrm{Emb}_0\,X\to B\,\mathrm{Emb}_0(\mathrm{Int}\ X)$$

are equivalences. His argument uses the manifold X^*, which is the union of X

with a collar on ∂X. If we assume that $A = A^* \cap X$, where A^* is a closed subset of X^* which is a product near the collar $\overline{X^* - X}$, then his argument adapts easily to show that

$$B\,\mathrm{Emb}_0^g\,(X, \mathrm{rel}\, A) \xrightarrow{\simeq} B\,\mathrm{Emb}_0\,(X, \mathrm{rel}\, A) \xrightarrow{\simeq} B\,\mathrm{Emb}_0\,(\mathrm{Int}\, X, \mathrm{rel}\, A).$$

However, any A which is a product near ∂X has this form. Therefore, by an obvious limiting argument, the above equivalences hold for all A.

To complete the proof it remains to show that the restriction $\mathscr{E}mb_0\,(X, \mathrm{rel}\, A) \to \mathscr{E}mb_0\,(\mathrm{Int}\, X, \mathrm{rel}\, A)$ is a weak equivalence. Again, we may assume that A is a product near $\overline{X^* - X}$. Consider the diagram

$$\begin{tikzcd}
\mathscr{E}mb_0\,(\mathrm{Int}\, X^*, \mathrm{rel}\, A) & \mathscr{E}mb_0\,(\mathrm{Int}\, X^*, X, \mathrm{rel}\, A) \arrow[l, hook', "i"', "\simeq"] \arrow[r, "\rho_2", "\simeq"'] \arrow[d, "\rho_1"] & \mathscr{E}mb_0\,(X, \mathrm{rel}\, A) \arrow[dl, "\rho_3"] \\
& \mathscr{E}mb_0\,(\mathrm{Int}\, X, \mathrm{rel}\, A), \arrow[ul, "\bar{\varphi}"', "\cong"] &
\end{tikzcd}$$

where $\bar{\varphi}$ is the isomorphism induced by a diffeomorphism $\varphi : (\mathrm{Int}\, X^*, A^*) \xrightarrow{\cong} (\mathrm{Int}\, X, A)$ which pushes the collar $\overline{X^* - X}$ into X. As in the proof of Proposition 5.2 above, the restriction ρ_2 is a weak equivalence. Also, i is a weak equivalence because any compact subset of $\mathscr{E}mb_0\,(\mathrm{Int}\, X^*, \mathrm{rel}\, A)$ may be deformation retracted into $\mathscr{E}mb_0\,(\mathrm{Int}\, X^*, X, \mathrm{rel}\, A)$. (In fact, if $A \cap \overline{(X^* - X)} = \varnothing$, it is not difficult to define a homotopy inverse for i.) Finally, note that the restrictions of i and $\bar{\varphi} \circ \rho_1$ to any compact subset of $\mathscr{E}mb_0\,(\mathrm{Int}\, X^*, X, \mathrm{rel}\, A)$ are homotopic. When $A \cap \overline{(X^* - X)} = \varnothing$, an explicit homotopy ψ_t from i to $\bar{\varphi} \circ \rho_1$ may be defined as follows. Let φ_λ be an isotopy in $\mathscr{E}mb\, X^*$, with $\varphi_1 = id$ and $\varphi_0 = \varphi$, which takes X^* onto $X \cup \partial X \times [0, \lambda]$ for all $0 \leq \lambda \leq 1$, and then put

$$\begin{aligned}
\psi_t(f) &= f \circ \varphi_{1-2t} \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{2}, \\
&= \varphi_{2-2t}^{-1} \circ f \circ \varphi_0 \quad \text{for} \quad \tfrac{1}{2} \leq t \leq 1.
\end{aligned}$$

It follows that ρ_1 is a weak equivalence. Hence ρ_3 is too. □

Note. Taking $X = D^n$ in Proposition 5.7 we find that $\bar{B}\,\mathscr{E}mb_0^g\, D^n \simeq \bar{B}\,\mathscr{E}mb_0\, D^n \simeq \bar{B}\,\mathscr{E}mb_0\, \mathbf{R}^n$. It is easy to adapt the proof of Proposition 1.1 given at the end of §2 to show that $\bar{B}\,\mathscr{E}mb_0\, \mathbf{R}^n \simeq \bar{B}\Gamma_n^r$ (notation as in [8]). Hence $\bar{B}\,\mathscr{E}mb_0^g\, D^n \simeq \bar{B}\Gamma_n^r$. This result is the starting point of the inductive argument of [8].

Appendix: Erratum to [8]

The following two mistakes occur in [8].

(i) The thin realization of $B\Gamma$ was used. This is incorrect because, as was noted at the beginning of §3, the thin realization may well have the wrong homotopy type.

(ii) Lemmas 1 and 2 rely on Proposition 2.8 of [20], whose proof contains a small error.

There are also some misprints, namely:

(a) p. 431 line -3 and p. 433 line 6: replace f_x by f_X;
(b) p. 434 line 9: replace B Emb by $B\ \mathrm{Emb}_0^g$;
(c) p. 435 line 9 should read "morphism $m: s \to ms$'";
(d) p. 439 line 7: replace $E(X\times Y)\backslash\!\backslash\mathscr{E}(X\times Y)\times Y$ by

$$\mathscr{C}(E(X\times Y)\backslash\!\backslash\mathscr{E}(X\times Y)\times Y)$$

and similarly on lines 8 and 11;
(e) p. 443 line 3: replace $\mathrm{Emb}_0(Y, \mathrm{rel}\, A)$ by $\mathscr{E}\!\mathit{mb}_0(Y, \mathrm{rel}\, A)$;
(f) p. 444 line 8: $A = Y_0$ not $A + Y_0$.

The errors mentioned in (i) and (ii) above may be corrected in the following way.

(i) If we use the thick realization $\|\cdot\|$ of [19] Appendix A instead of the thin realization then, because all the monoids considered in [8] are good, the only thing which will be affected is the construction of the commutative diagram (†) in Proposition 1. As before, we start from the diagram $(**)$ of categories and functors:

$$\begin{tikzcd}
\mathscr{C}(E\backslash\!\backslash\mathscr{E}\times X) \arrow[r, "\bar{F}"] \arrow[d] & \Gamma(X) \arrow[d, "\bar{\nu}"] \\
\mathscr{C}(e\backslash\!\backslash X) \arrow[ur, hook, "\bar{\gamma}"] \arrow[r, "\bar{\tau}"] & \mathscr{G}\ell(X) \arrow[loop right, "\bar{H}"]
\end{tikzcd}$$

We will realize this diagram using the *thick* realization for the top row and the *thin* realization for the bottom row. Notice that $\|\mathscr{C}(E\backslash\!\backslash\mathscr{E}\times X)\|$ is homeomorphic to $\|\mathscr{C}(E\backslash\!\backslash\mathscr{E})\|\times X$. We therefore get a diagram:

$$\begin{tikzcd}
\|\mathscr{C}(E\backslash\!\backslash\mathscr{E})\|\times X \arrow[r, "\|F\|"] \arrow[d, "\pi"'] & \|\Gamma(X)\| \arrow[d, "\bar{\nu}"] \\
X \arrow[ur, hook, "\gamma"] \arrow[r, hook, "\tau"] & |\mathscr{G}\ell(X)| \arrow[loop right, "H"]
\end{tikzcd} \qquad (\#)$$

We will see in (ii) below that [20] Proposition 2.8 holds for the thick realization. Therefore Lemma 1 holds for this realization. Also, Lemma 2 holds for the thin realization because all the simplicial spaces involved are good. Therefore, $\bar{\nu}:\|\Gamma(X)\|\to|\mathscr{G}\ell(X)|$ is a model for $B\Gamma\to BGL$. Let L be the pull-back over τ of the Hurewicz fibration associated to $\bar{\nu}$, and consider the commutative diagram

$$\begin{tikzcd}
\|\mathscr{C}(e\setminus\!\!\setminus X)\| \arrow[r, "\|\gamma\|"] \arrow[d, "p"'] & \|\Gamma(X)\| \arrow[d, "\bar{\nu}"] \\
X \arrow[ur, hook, "\gamma"] \arrow[r, hook, "\tau"] & |\mathscr{G}\ell(X)|,
\end{tikzcd}$$

where p is the obvious projection. As before, the inclusion γ gives rise to a canonical section s_0 of L. Observe that $p:\|\mathscr{C}(e\setminus\!\!\setminus X)\|\to X$ is a product bundle with contractible fibre $\|\mathscr{C}(e\setminus\!\!\setminus e)\|$. We will denote its image in L by D. Then $s_0(X)\subset D$, and D contracts fibrewise onto $s_0(X)$.

We now define $\mathscr{S}(X)$ to be the space of continuous sections of L over X with the compact-open topology. If $U\subset X$ is open, let $\mathscr{S}(X,\mathrm{rel}_D\,U)$ (resp. $\mathscr{S}(X,\mathrm{rel}\,U)$) be the subspace of $\mathscr{S}(X)$ consisting of sections which take values in D (resp. equal s_0) over U. Finally, if $A\subset X$ is closed, define $\mathscr{S}(X,\mathrm{rel}_D\,A)$ to be the direct limit $\varinjlim \mathscr{S}(X,\mathrm{rel}_D\,U)$, where U runs over all open neighbourhoods of A in X, with the direct limit topology. Define $\mathscr{S}(X,\mathrm{rel}\,A)$ similarly. Then it is not hard to check that the inclusion $\mathscr{S}(X,\mathrm{rel}\,A)\hookrightarrow\mathscr{S}(X,\mathrm{rel}_D\,A)$ is a (weak) equivalence. (This follows because $\mathscr{S}(X,\mathrm{rel}_D\,U)$ deformation retracts into $\mathscr{S}(X,\mathrm{rel}\,V)$ whenever U and V are open sets such that $\bar{V}\subset U$.)

Observe that diagram (#) gives rise to a map $f_X:E\setminus\!\!\setminus\mathscr{E}\to\mathscr{S}(X)$ just as before. This map behaves well under restriction to a submanifold Y. Also, one can check that f_X takes $E(X,\mathrm{rel}\,A)\setminus\!\!\setminus\mathscr{E}(X,\mathrm{rel}\,A)$ to $\mathscr{S}(X,\mathrm{rel}_D\,A)$. Therefore, given Y, Z, and A as in Proposition 1, we have a strictly commutative diagram:

$$\begin{tikzcd}
\bar{B}\,\mathscr{E}mb_0^{g}(Y,\mathrm{rel}\,Z\cup A) \arrow[r] \arrow[d] & \bar{B}\,\mathscr{E}mb_0^{g}(Y,Z,\mathrm{rel}\,A) \arrow[r] \arrow[d] & \bar{B}\,\mathscr{E}mb_0^{g}(Z,\mathrm{rel}\,A) \arrow[d] \\
\mathscr{S}(Y,\mathrm{rel}_D\,Z\cup A) \arrow[r] & \mathscr{S}(Y,\mathrm{rel}_D\,A) \arrow[r] & \mathscr{S}(Z,\mathrm{rel}_D\,A) \\
\mathscr{S}(Y,\mathrm{rel}\,Z\cup A) \arrow[r] \arrow[u, hook, "\simeq"'] & \mathscr{S}(Y,\mathrm{rel}\,A) \arrow[r] \arrow[u, hook, "\simeq"'] & \mathscr{S}(Z,\mathrm{rel}\,A), \arrow[u, hook, "\simeq"']
\end{tikzcd}$$

where the top row is formed using thick realizations. The bottom row is a homotopy fibration sequence by [8] Lemma 3. Therefore, the middle row is, too. We may now apply all the arguments of [8] using this diagram instead of diagram (†), and hence prove all the theorems in [8].

(ii) We will prove [20] Proposition 2.8 using the thick realization $\|\cdot\|$ and assuming that the spaces of objects and morphisms of the category C are in $\mathcal{HG}$. Then the spaces $\|\tilde{C}\|$ and $\|E\tilde{C}_0\| \times_{\|EC_0\|} \|C\|$ are homeomorphic, and it is enough to show that $\pi : \|E\tilde{C}_0\| \to \|EC_0\|$ is shrinkable. (The notation is as in [20].) The original proof fails at this point because the sets $\{EU_\alpha\}_{\alpha \in A}$ do not form a covering of EC_0. In fact, one can construct a section s of π as follows. Choose a well-ordering of A and a partition of unity φ_α subordinate to the covering U. If $y = (x_0, \ldots, x_k; t_0, \ldots, t_k)$ is an element of $\|EC_0\|$, where the x_i are in C_0 and $\sum_i t_i = 1$, set

$$s(v) = ((x_0, \alpha(0, 1)), \ldots, (x_0, \alpha(0, p_0)), (x_1, \alpha(1, 1)), \ldots, (x_k, \alpha(k, p_k));$$
$$t_0 \cdot \varphi_{\alpha(0,1)}(x_0), \ldots, t_k \cdot \varphi_{\alpha(k,p_k)}(x_k)),$$

where, for each i, the $\alpha(i, j)$, $1 \le j \le p_i$, are chosen so that $\alpha(i, 1) < \cdots < \alpha(i, p_i)$ and $\sum_j \varphi_{\alpha(i,j)}(x_i) = 1$. Notice that s is well defined since degeneracies are not collapsed when the thick realization is formed. (Thus in $\|EC_0\|$ the elements $(x, x, y; t_0, t_1, t_2)$ and $(x, y; t_0 + t_1, t_2)$ are not the same unless either t_0 or t_1 equals 0.) It is now easy to see that $\|E\tilde{C}_0\|$ contracts fibrewise onto the image of s. Hence π is shrinkable, as claimed.

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